A Fibrational Approach to (Multiplicative Additive) Indexed Linear Logic

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Refining/decorating type systems : Church and Curry knitted together

À la Church decoration on Curry typed terms

A typed term $\vdash t : A$ may be "decorated" by a more precise type

$$\vdash_{X} t : A^{❀} \triangleleft A$$

X is a decoration context, later called "locus"

Same proof if any

$$\frac{\pi^{❀}}{\Gamma^{❀} \vdash_{X} t : A^{❀}} \implies \frac{\pi}{\Gamma \vdash t : A}$$

Example : size types

$$\vdash_{i,j} concat : [A]_i * [A]_j \to [A]_{i+j} \triangleleft [A] * [A] \to [A]$$

A logical counterpart : step-preserving forgetfulness

Every decorated formula $A^{❀}$ refines a unique formula A

Same goes for proofs.

This preserves cut-elimination

if $\dfrac{\pi^{❀}}{\Gamma^{❀} \vdash_X A^{❀}}$ and if $\pi \rightsquigarrow \rho$ then there is a decoration $\pi^{❀} \rightsquigarrow \rho^{❀}$

Curry and Howard's blind-spot

$\dfrac{\pi}{\Gamma \vdash A} \quad \Longrightarrow$ for any given $\Gamma^{❀}$, $A^{❀}$ and X, at most one proof $\dfrac{\pi^{❀}}{\Gamma^{❀} \vdash_X A^{❀}}$

Difficult... at least make it true in your models...

Sub-decoration : more or less refined information

Two proofs with different decorations are OK and can even represent level of precision

$$\frac{\pi^{❁}}{\Gamma^{❁} \vdash_X A^{❁}} \lhd \frac{\pi^{❀}}{\Gamma^{❀} \vdash_Y A^{❀}} \lhd \frac{\pi}{\Gamma \vdash A}$$

Proofs can even be transported by rewriting / base-change

$$f\left(\frac{\pi^{❀}}{\Gamma^{❀} \vdash_Y A^{❀}}\right) = \frac{f\left(\pi^{❀}\right)}{f\left(\Gamma^{❀}\right) \vdash f\left(A^{❀}\right)}$$

f traverses the term (like LL's negation), only modifying decoration.

Base-changes are crucial in a lot of advanced decoration systems

A paradigmatic example : Indexed linear logic (IndLL)

Sequents are sets of intersection types

$A^{❀} \vdash_X B^{❀}$ is an X-indexed family of intersection types refining $A \multimap B$

$$\frac{a \in \mathtt{atom}(X)}{a \vdash_X a} \qquad \frac{\Gamma \vdash_X A \qquad A \vdash_X \Delta}{\Gamma \vdash_X \Delta} \qquad \frac{}{\vdash_X \mathbf{1}} \qquad \frac{\Gamma \vdash_X A \qquad \Delta \vdash_X B}{\Gamma, \Delta \vdash_X A \otimes B}$$

$$\frac{\Gamma \vdash_X A, B}{\Gamma \vdash_X A \parr B} \qquad \frac{0 \Vdash \Gamma}{\Gamma \vdash_0 \top} \qquad \frac{\iota_1(\Gamma) \vdash_X A \qquad \iota_2(\Gamma) \vdash_Y B}{\Gamma \vdash_{X+Y} A \,\&\, B} \qquad \frac{\Gamma \vdash_X A \qquad 0 \Vdash B}{\Gamma \vdash_X A_X \oplus_0 B}$$

$$\frac{\Gamma \vdash_X B \qquad 0 \Vdash A}{\Gamma \vdash_X A_0 \oplus_X B} \qquad \frac{\Gamma \vdash_X ?_u \iota_1(A), ?_v \iota_2(A)}{\Gamma \vdash_X ?_{\langle u,v \rangle} A} \qquad \frac{\Gamma \vdash_X \qquad 0 \Vdash B}{\Gamma \vdash_X ?_{\mathtt{term}} B} \qquad \frac{\Gamma \vdash_X B}{\Gamma \vdash_X ?_{id} B}$$

$$\frac{!_{u_1} A_1, \dots, !_{u_n} A_n \vdash_Y B \qquad v : X \to Y}{!_{v;u_1} A_1, \dots, !_{v;u_n} A_n \vdash_X !_v B} \qquad \frac{\Gamma \vdash_X ?_u f(B) \qquad v \leq u; f}{\Gamma \vdash_X ?_v B}$$

And ever more base-changes in the cut elimination procedural.

My long term objectives

Transform models into decorated logics

I am convinced that any model of LL can be fully characterised by a well suited decoration-system of LL.

Should extends outside of LL.

Study continuum between syntax and model

Somewhere between the model decoration and the absence of decoration should be a precise but commutable one !

A MLL per locus

$$\frac{\Gamma \vdash_X A, \Delta}{\Gamma, A^\perp \vdash_X \Delta} \qquad \frac{\Gamma, A \vdash_X \Delta}{\Gamma \vdash_X A^\perp, \Delta} \qquad \frac{\Gamma \vdash_X A, B, \Delta}{\Gamma \vdash_X B, A, \Delta} \qquad \frac{\Gamma \vdash_X A \qquad A \vdash_X \Delta}{\Gamma \vdash_X \Delta}$$

$$\frac{}{\vdash_X \mathbf{1}} \qquad \frac{\Gamma \vdash_X A \qquad \Delta \vdash_X B}{\Gamma, \Delta \vdash_X A \otimes B} \qquad \frac{\Gamma, A \vdash_X B}{\Gamma \vdash_X A \multimap B}$$

Different atoms/constants and constants in each locus

$$\frac{a \in \mathtt{atom}(X)}{a \vdash_X a} \qquad \frac{c \in \mathtt{constant}(X)}{\vdash_X c}$$

Base change : rewriting that goes throw the formula

If $f : X \to Y$, and $\dfrac{\pi}{A \vdash_{\color{red}Y} B}$ then $\dfrac{f(\pi)}{f(A) \vdash_{\color{red}X} f(B)}$

$$f(\mathbf{1}) := \mathbf{1} \quad f(A \otimes B) := f(A) \otimes f(B) \quad f(A \multimap B) := f(A) \multimap f(B)$$

Like negation, f only acts on atoms

A category $\mathbb{Loci}$ of loci and a functor as model

$$
\begin{array}{llll}
\texttt{Model}: & \mathbb{Loci}^{op} & \to & \texttt{StarAutonomousCat} \\
 & X & \mapsto & (\vdash_{X})\text{-system} \\
 & \downarrow f & \mapsto & f() \\
 & Y & \mapsto & (\vdash_{Y})\text{-system}
\end{array}
$$

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IndLL (co-)products will be Cartesian (co-)product in the (op-)fibration !

The coproduct is a transversal operator

$$\frac{0 \Vdash \Gamma}{\mathbb{0} \vdash_{0} \Gamma} \qquad \frac{A \vdash_{X} \iota_1(\Gamma) \qquad B \vdash_{Y} \iota_2(\Gamma)}{A_X \oplus_Y B \vdash_{X+Y} \Gamma} \qquad \frac{\Gamma \vdash_{X} A \qquad 0 \Vdash B}{\Gamma \vdash_{X} A_X \oplus_0 B}$$

We use a co-product structure on $\mathbb{Loci}$

WARNING : the products & also use the co-product of $\mathbb{Loci}$

The coproduct is a transversal operator who lives in the fibration

$$\frac{0 \Vdash \Gamma}{\mathbb{0} \vdash_{0} \Gamma} \qquad \frac{A \vdash_{X} \iota_1(\Gamma) \qquad B \vdash_{Y} \iota_2(\Gamma)}{A\, {}_{X}\!\oplus_{Y} B \vdash_{X+Y} \Gamma} \qquad \frac{\Gamma \vdash_{X} A \qquad 0 \Vdash B}{\Gamma \vdash_{X} A\, {}_{X}\!\oplus_{0} B}$$

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In $\int$Model we have

$$(X, A) \oplus (Y, B) \quad := \quad (X + Y, A\ {}_{X}\!\oplus_{Y}\ B)$$

that is a Cartesian co-product !

Syntactical constrains : Extensiveness

Recall that base change should traverse the term

if $f : X \to Y + Z$

$$f\left(\frac{\dfrac{\pi_1}{A \vdash_Y \iota_1(\Gamma)} \qquad \dfrac{\pi_2}{B \vdash_Z \iota_2(\Gamma)}}{A\ {}_X\!\oplus_Y B \vdash_{Y+Z} \Gamma}\right) := \frac{\dfrac{f_{|Y}(\pi_1)}{f_{|Y}(A) \vdash_{f^{-1}Y} \iota_1(f(\Gamma))} \qquad \dfrac{f_{|Z}(\pi_2)}{f_{|Z}(B) \vdash_{f^{-1}Z} \iota_2(f(\Gamma))}}{f_{|Y}(A)\ {}_{f^{-1}Y}\!\oplus_{f^{-1}Z} f_{|Z}(B) \vdash_X f(\Gamma)}$$

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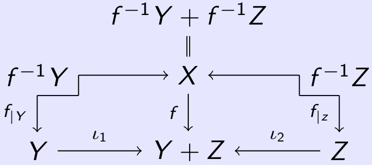
Extensivity : pullback preserves co-product injections



In a coherent way (choice up-to equiv.)

High level analyse of the extensivity condition

Extensiveness is syntactical (geometric ?)

The model structure has absolutely no use of extensivity !
Mathematicians class the extensivity as a geometric property... any GeoCal link here ?

We use it in order to define a nomalizable rewriting process

We will use similar properties to define $f(!_u A)$ or any $f\left(\text{operator}(\vec{A})\right)$.

Semi-Cartesian (co) products ?

(extended version only)

Remember : the logic lives in fibers

We only “see” the image of the co-products from the fibration.

We can cheat in the fibration

We only need to have semi-Cartesianness plus a shadow of a co-diagonal.

Consequence : $\mathbb{Loci}$ only need semi-Cartesian co-products

Not even the shadow of co-diagonal is needed in the syntax.

Remark : if $\mathbb{Loci}$ has Cartesian co-product, then the “shadow of co-diagonal” is externalised into a real one

General comment on fiber/fibration

The internalisation (fibration$\rightarrow$fibers) and the externalisation (fiber$\rightarrow$fibration) are essential properties !

Exponentials : an indexed transversal functor

$dom(!_v A) = source(v)$ so that $\dfrac{!_{u_1} A_1, \dots, !_{u_n} A_n \vdash_Y B \qquad v : X \to Y}{!_{v;u_1} A_1, \dots, !_{v;u_n} A_n \vdash_X !_v B}$

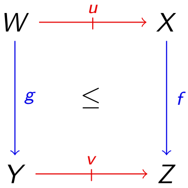
First issue : v in $!_v$ are not locus morphisms

(unclear in the original IndLL where everything are functions with different properties...)

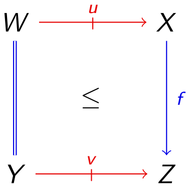
Second issue : product structure on those v's

$$\dfrac{!_u \iota_1(A), !_v \iota_2(A) \vdash_X \Gamma}{!_{\langle u,v \rangle} A \vdash_X \Gamma} \text{ where } dom(A) = Y + Z$$

A (thin) double category $\mathbb{L}\mathrm{oci} * \mathbb{E}\mathrm{xpo}$



A (thin) double category $\mathbb{L}\mathrm{oci} * \mathbb{E}\mathrm{xpo}$



gives

$$\frac{\Gamma \vdash_X ?_u f(B) \qquad v \leq u; f}{\Gamma \vdash_X ?_v B}$$

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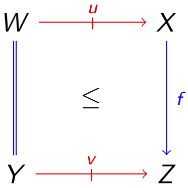


gives

$$\frac{\Gamma \vdash_X ?_u f(B) \qquad v \leq u; f}{\Gamma \vdash_X ?_v B}$$

Remark : Sub-decoration rules are identity rules when un-decorated, we should include them in real rules...

Semantics issue : toward thin double fibration

$\texttt{Model}$:	$\mathbb{Loci}^{op} * \mathbb{Expo}^{op}$	$\xrightarrow{lax}$	$\texttt{StAut} * \texttt{Cat}$
	$W \xrightarrow{u} X$	$\mapsto$	$(\vdash_W) \xrightarrow{!_u} (\vdash_X)$
	$\downarrow f \quad \downarrow$	$\mapsto$	$f() \qquad g()$
	$Y \xrightarrow{v} Z$	$\mapsto$	$(\vdash_Y) \xrightarrow{!_v} (\vdash_Z)$

The Graal of Seely isomorphism

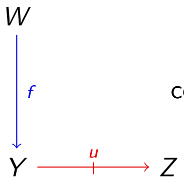
$$?_{\langle u,v\rangle}(A \oplus B) \quad \simeq \quad !_u A \parr !_v B$$

It's fibrational interpretation

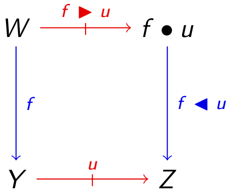
The $\oplus$ is the externalisation (through the horiz. lax fibration) of the $\parr$

Matching pair of action : Relaxing requirements

We where requiring that $\mathbb{Loci} \subseteq \mathbb{Expo}$, but it is not necessary, only :



completes “canonically” to



$$f(!_u A) \quad := \quad !_{f \blacktriangleright u}(f \blacktriangleleft u)(A)$$

Model : match the square to an equality

Correspond to a Beck-Chevalley condition

Conclusion

You can construct an indexed linear logic from

- A monoidal double category,
- whose vertical category is a semi-Extensive,
- whose horizontal one is Cartesian,
- with a matching pair of actions :
  $\_ \square \_ : \mathbb{Loci}(X, Y) \times \mathbb{Expo}(Y, Z) \to \mathtt{morph}(\mathtt{arrow})$

Remains to do :

- find and study many examples,
- extract them from models,
- a (2-)category of such a $\mathbb{Loci} * \mathbb{Expo}$ double categories ?
- other operators (fixpoints, quantifiers...)
- what about other logics ? (e.g., BLL or separation logic)
- write the exponential part...