The Composition of Combinatorial Flows

Giti Omidvar and Lutz Straßburger

Inria Saclay
Ecole Polytechnique

GT Scalp, Marseille, 15 February 2023

Preliminaries: Open Deduction

$$\frac{\mathsf{t}}{a \vee \bar{a}}\,\mathsf{ai}{\downarrow} \qquad \frac{(A \vee B) \wedge C}{A \vee (B \wedge C)}\,\mathsf{s} \qquad \frac{\bar{a} \wedge a}{\mathsf{f}}\,\mathsf{ai}{\uparrow}$$

$$\frac{a \vee a}{a}\,\mathsf{ac}{\downarrow} \qquad \frac{(A \wedge C) \vee (B \wedge D)}{(A \vee B) \wedge (C \vee D)}\,\mathsf{m} \qquad \frac{a}{a \wedge a}\,\mathsf{ac}{\uparrow}$$

$$\frac{\mathsf{f}}{a}\,\mathsf{aw}{\downarrow} \qquad \frac{\mathsf{f}}{\mathsf{t}}\,\mathsf{mix} \qquad \frac{a}{\mathsf{t}}\,\mathsf{aw}{\uparrow}$$

Preliminaries: Open Deduction

$$\cfrac{\cfrac{a \wedge \boxed{\cfrac{\bar{a}}{\bar{a} \wedge \bar{a}}\,\mathsf{ac}\!\uparrow}}{\boxed{\cfrac{\boxed{\cfrac{a \wedge \bar{a}}{a \wedge \left[\boxed{\cfrac{\boxed{\cfrac{\mathsf{f}}{\mathsf{t}}\,\mathsf{mix}}}{a \vee \bar{a}}\,\mathsf{ai}\!\downarrow} \vee \bar{a}\right]}\,\equiv}}{\boxed{\cfrac{a \wedge \boxed{\cfrac{\bar{a} \vee \bar{a}}{\bar{a}}\,\mathsf{ac}\!\downarrow}}{\mathsf{f}}\,\mathsf{ai}\!\uparrow} \vee \boxed{\cfrac{a}{a \wedge a}\,\mathsf{ac}\!\uparrow}}\,\mathsf{s}} \wedge \bar{a}}\,\equiv}{a \wedge \boxed{\cfrac{a \wedge \bar{a}}{\mathsf{f}}\,\mathsf{ai}\!\uparrow}}\,\equiv$$

$$\frac{\mathsf{t}}{a \vee \bar{a}}\,\mathsf{ai}\!\downarrow \qquad \frac{(A \vee B) \wedge C}{A \vee (B \wedge C)}\,\mathsf{s} \qquad \frac{\bar{a} \wedge a}{\mathsf{f}}\,\mathsf{ai}\!\uparrow$$

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Preliminaries: Open Deduction

$$
\begin{array}{c}
a \wedge \boxed{\dfrac{\bar a}{\bar a \wedge \bar a}\,\mathsf{ac}\uparrow} \\
\hline
\boxed{\dfrac{\boxed{\dfrac{a \wedge \bar a}{a \wedge \left[\boxed{\dfrac{\boxed{\dfrac{\mathsf{f}}{\mathsf{t}}\,\mathsf{mix}}}{a \vee \bar a}\,\mathsf{ai}\downarrow} \vee \bar a\right]}\,\equiv}}{\boxed{\dfrac{a \wedge \boxed{\dfrac{\bar a \vee \bar a}{\bar a}\,\mathsf{ac}\downarrow}}{\mathsf{f}}\,\mathsf{ai}\uparrow} \vee \boxed{\dfrac{a}{a \wedge a}\,\mathsf{ac}\uparrow}}\,\mathsf{s}} \wedge \bar a \\
\hline
a \wedge \boxed{\dfrac{a \wedge \bar a}{\mathsf{f}}\,\mathsf{ai}\uparrow}
\end{array}
\begin{array}{l} \\ \equiv \\ \\ \equiv \\ \end{array}
$$

$$
\dfrac{\mathsf{t}}{a \vee \bar a}\,\mathsf{ai}\downarrow \qquad \dfrac{(A \vee B) \wedge C}{A \vee (B \wedge C)}\,\mathsf{s} \qquad \dfrac{\bar a \wedge a}{\mathsf{f}}\,\mathsf{ai}\uparrow
$$

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$$

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\dfrac{\mathsf{f}}{a}\,\mathsf{aw}\downarrow \qquad \dfrac{\mathsf{f}}{\mathsf{t}}\,\mathsf{mix} \qquad \dfrac{a}{\mathsf{t}}\,\mathsf{aw}\uparrow
$$

Preliminaries: Open Deduction and Atomic Flows

$$
\dfrac{a \wedge \boxed{\dfrac{\bar{a}}{\bar{a} \wedge \bar{a}}\,\mathsf{ac}{\uparrow}}}{\boxed{\dfrac{\boxed{\dfrac{a \wedge \bar{a}}{a \wedge \left[\boxed{\dfrac{\boxed{\dfrac{\mathsf{f}}{\mathsf{t}}\,\mathsf{mix}}}{a \vee \bar{a}}\,\mathsf{ai}{\downarrow}} \vee \bar{a}\right]}\,\equiv}}{\boxed{\dfrac{a \wedge \boxed{\dfrac{\bar{a} \vee \bar{a}}{\bar{a}}\,\mathsf{ac}{\downarrow}}}{\mathsf{f}}\,\mathsf{ai}{\uparrow}} \vee \boxed{\dfrac{a}{a \wedge a}\,\mathsf{ac}{\uparrow}}}\,\mathsf{s}} \wedge \bar{a}}\,\equiv
$$

$$
\dfrac{\boxed{\dfrac{\boxed{\dfrac{a \wedge \bar{a}}{a \wedge \left[\boxed{\dfrac{\boxed{\dfrac{\mathsf{f}}{\mathsf{t}}\,\mathsf{mix}}}{a \vee \bar{a}}\,\mathsf{ai}{\downarrow}} \vee \bar{a}\right]}\,\equiv}}{\boxed{\dfrac{a \wedge \boxed{\dfrac{\bar{a} \vee \bar{a}}{\bar{a}}\,\mathsf{ac}{\downarrow}}}{\mathsf{f}}\,\mathsf{ai}{\uparrow}} \vee \boxed{\dfrac{a}{a \wedge a}\,\mathsf{ac}{\uparrow}}}\,\mathsf{s}} \wedge \bar{a}}{a \wedge \boxed{\dfrac{a \wedge \bar{a}}{\mathsf{f}}\,\mathsf{ai}{\uparrow}}}\,\equiv
$$

$$
\dfrac{\mathsf{t}}{a \vee \bar{a}}\,\mathsf{ai}{\downarrow} \qquad \dfrac{(A \vee B) \wedge C}{A \vee (B \wedge C)}\,\mathsf{s} \qquad \dfrac{\bar{a} \wedge a}{\mathsf{f}}\,\mathsf{ai}{\uparrow}
$$

$$
\dfrac{a \vee a}{a}\,\mathsf{ac}{\downarrow} \qquad \dfrac{(A \wedge C) \vee (B \wedge D)}{(A \vee B) \wedge (C \vee D)}\,\mathsf{m} \qquad \dfrac{a}{a \wedge a}\,\mathsf{ac}{\uparrow}
$$

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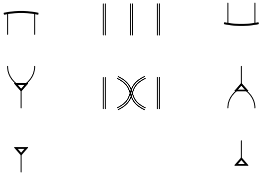
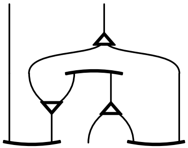
\dfrac{\mathsf{f}}{a}\,\mathsf{aw}{\downarrow} \qquad \dfrac{\mathsf{f}}{\mathsf{t}}\,\mathsf{mix} \qquad \dfrac{a}{\mathsf{t}}\,\mathsf{aw}{\uparrow}
$$



Alessio Guglielmi and Tom Gundersen, LMCS 2008

$$
\dfrac{a \wedge \boxed{\dfrac{\bar{a}}{\bar{a} \wedge \bar{a}}\,\mathsf{ac}\!\uparrow}}{\dfrac{\boxed{\dfrac{\boxed{\dfrac{a \wedge \bar{a}}{a \wedge \left[\boxed{\dfrac{\boxed{\dfrac{\mathsf{f}}{\mathsf{t}}\,\mathsf{mix}}}{a \vee \bar{a}}\,\mathsf{ai}\!\downarrow} \vee \bar{a}\right]}\,\equiv}}{\boxed{\dfrac{a \wedge \boxed{\dfrac{\bar{a} \vee \bar{a}}{\bar{a}}\,\mathsf{ac}\!\downarrow}}{\mathsf{f}}\,\mathsf{ai}\!\uparrow} \vee \boxed{\dfrac{a}{a \wedge a}\,\mathsf{ac}\!\uparrow}}\,\mathsf{s}} \wedge \bar{a}}{a \wedge \boxed{\dfrac{a \wedge \bar{a}}{\mathsf{f}}\,\mathsf{ai}\!\uparrow}}\,\equiv}\,\equiv
$$



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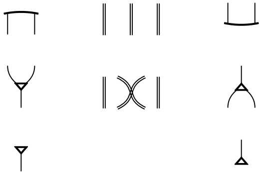
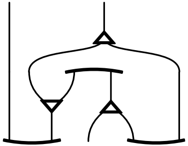
$$



Alessio Guglielmi and Tom Gundersen, LMCS 2008

$$
\dfrac{a \wedge \boxed{\dfrac{\bar a}{\bar a \wedge \bar a}\,\mathsf{ac}\!\uparrow}}{\boxed{\dfrac{\boxed{\dfrac{a \wedge \bar a}{a \wedge \left[\boxed{\dfrac{\boxed{\dfrac{\mathsf{f}}{\mathsf{t}}\,\mathsf{mix}}}{a \vee \bar a}\,\mathsf{ai}\!\downarrow} \vee \bar a\right]}\,\equiv}}{\boxed{\dfrac{a \wedge \boxed{\dfrac{\bar a \vee \bar a}{\bar a}\,\mathsf{ac}\!\downarrow}}{\mathsf{f}}\,\mathsf{ai}\!\uparrow} \vee \boxed{\dfrac{a}{a \wedge a}\,\mathsf{ac}\!\uparrow}}\,\mathsf{s}} \wedge \bar a}\,\equiv
$$

$$
\equiv\quad a \wedge \boxed{\dfrac{a \wedge \bar a}{\mathsf{f}}\,\mathsf{ai}\!\uparrow}
$$

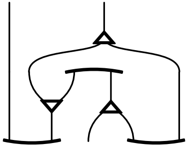


1. We cannot read back a proof from atomic flows

Anupam Das, RTA 2013

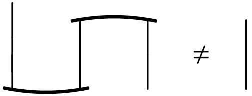
Alessio Guglielmi and Tom Gundersen, LMCS 2008

$$\cfrac{\cfrac{a \wedge \boxed{\cfrac{\bar a}{\bar a \wedge \bar a}\,\mathsf{ac}\!\uparrow}}{\boxed{\cfrac{\boxed{\cfrac{a \wedge \bar a}{a \wedge \left[\boxed{\cfrac{\boxed{\cfrac{\mathsf{f}}{\mathsf{t}}\,\mathsf{mix}}}{a \vee \bar a}\,\mathsf{ai}\!\downarrow} \vee \bar a\right]}\,\equiv}}{\boxed{\cfrac{a \wedge \boxed{\cfrac{\bar a \vee \bar a}{\bar a}\,\mathsf{ac}\!\downarrow}}{\mathsf{f}}\,\mathsf{ai}\!\uparrow} \vee \boxed{\cfrac{a}{a \wedge a}\,\mathsf{ac}\!\uparrow}}\,\mathsf{s}} \wedge \bar a}\,\equiv}{a \wedge \boxed{\cfrac{a \wedge \bar a}{\mathsf{f}}\,\mathsf{ai}\!\uparrow}}\,\equiv$$



1. We cannot read back a proof from atomic flows

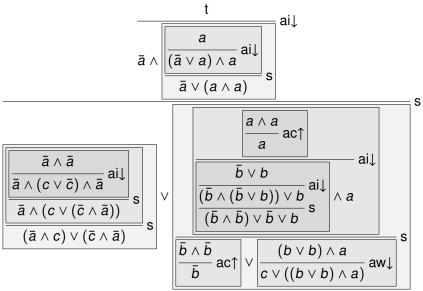
2. yanking is not possible

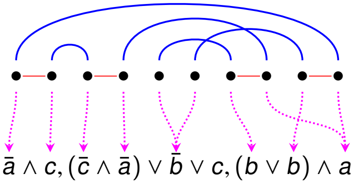


Anupam Das, RTA 2013

Alessio Guglielmi and Tom Gundersen, LMCS 2008

Preliminaries: Combinatorial Proofs





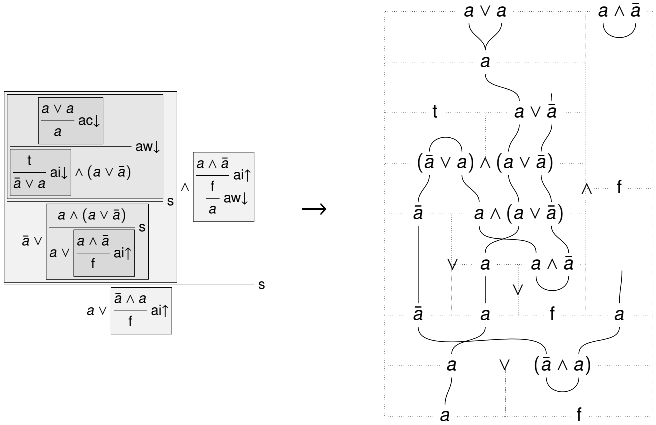
- Total seperation of linear part and resource management of the proof → size explosion

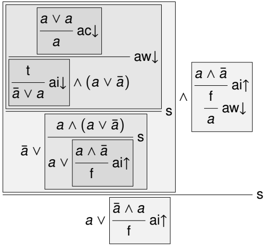
Dominic Hughes, Annals of Mathematics 2006

From Open Deduction to Preflows

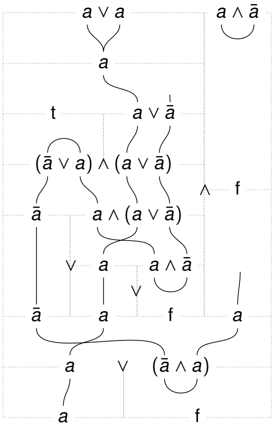
$$
\dfrac{
\boxed{
\dfrac{
\boxed{
\dfrac{
\boxed{\dfrac{a \vee a}{a}\,\mathsf{ac}{\downarrow}}
}{
\boxed{\dfrac{\mathsf{t}}{\bar{a} \vee a}\,\mathsf{ai}{\downarrow}} \wedge (a \vee \bar{a})
}\,\mathsf{aw}{\downarrow}
}
}{
\bar{a} \vee \boxed{\dfrac{a \wedge (a \vee \bar{a})}{a \vee \boxed{\dfrac{a \wedge \bar{a}}{\mathsf{f}}\,\mathsf{ai}{\uparrow}}}\,\mathsf{s}}
}\,\mathsf{s}
}
\wedge
\boxed{\dfrac{\dfrac{a \wedge \bar{a}}{\mathsf{f}}\,\mathsf{ai}{\uparrow}}{a}\,\mathsf{aw}{\downarrow}}
}{
a \vee \boxed{\dfrac{\bar{a} \wedge a}{\mathsf{f}}\,\mathsf{ai}{\uparrow}}
}\,\mathsf{s}
$$

From Open Deduction to Preflows

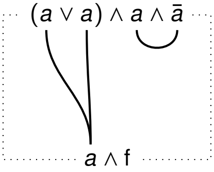




$\longrightarrow$



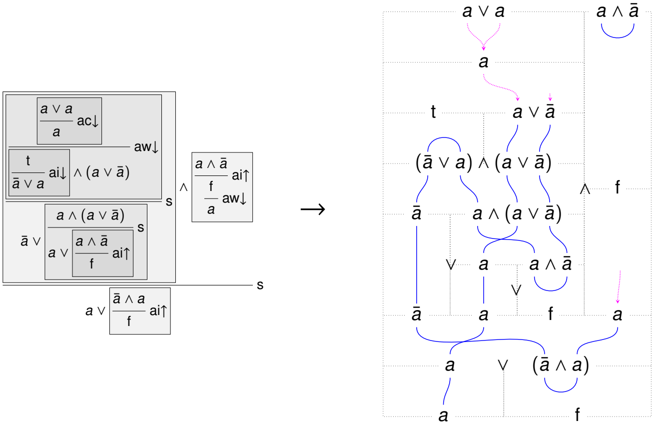
$\longrightarrow$



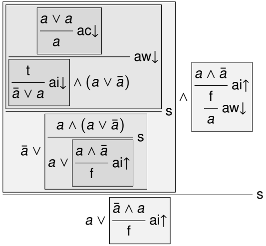
Corresponds to B-nets

B-nets: François Lamarche and Lutz Straßburger, TLCA 2005

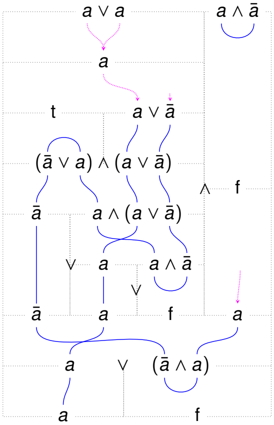
From Open Deduction to Combinatorial Flows



From Open Deduction to Combinatorial Flows



→



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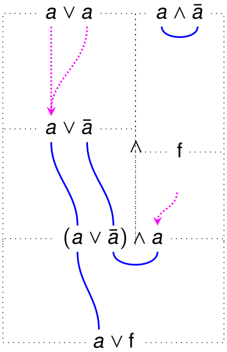

Formulas:

$$A, B := \ \mathsf{t} \mid \mathsf{f} \mid a \mid \overline{a} \mid A \vee B \mid A \wedge B$$

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$$A \wedge B \equiv B \wedge A \qquad (A \wedge B) \wedge C \equiv A \wedge (B \wedge C) \qquad A \wedge \mathsf{t} \equiv A \qquad \mathsf{t} \vee \mathsf{t} \equiv \mathsf{t}$$

$$A \vee B \equiv B \vee A \qquad (A \vee B) \vee C \equiv A \vee (B \vee C) \qquad A \vee \mathsf{f} \equiv A \qquad \mathsf{f} \wedge \mathsf{f} \equiv \mathsf{f}$$

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$$A, B := \ \mathsf{t} \mid \mathsf{f} \mid a \mid \overline{a} \mid A \vee B \mid A \wedge B$$

$$\begin{array}{llll} A \wedge B \equiv B \wedge A & (A \wedge B) \wedge C \equiv A \wedge (B \wedge C) & A \wedge \mathsf{t} \equiv A & \mathsf{t} \vee \mathsf{t} \equiv \mathsf{t} \\ A \vee B \equiv B \vee A & (A \vee B) \vee C \equiv A \vee (B \vee C) & A \vee \mathsf{f} \equiv A & \mathsf{f} \wedge \mathsf{f} \equiv \mathsf{f} \end{array}$$

Unit-free Formulas:

$$A, B := \ a \mid \overline{a} \mid A \vee B \mid A \wedge B$$

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Unit-free Formulas:

$$A, B := \ a \mid \overline{a} \mid A \vee B \mid A \wedge B$$

Pure Formulas: $A \equiv \mathsf{t}$ or $A \equiv \mathsf{f}$ or A is equivalent to a unit-free formula.

Graph of a formula

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- $\mathcal{G}(\mathsf{t}) = \mathcal{G}(\mathsf{f})$: empty graph

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- $\mathcal{G}(a)$: $\bullet_{a}$
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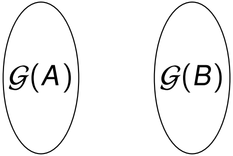
- $\mathcal{G}(A \vee B)$:



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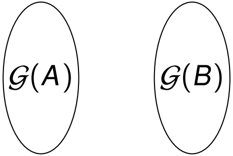
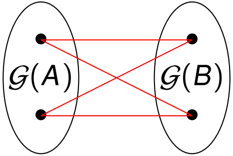
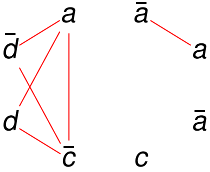


- $\mathcal{G}(A \wedge B)$:



Graph of a formula

$$((d \vee \bar{d}) \wedge (\bar{c} \wedge a)) \vee ((c \vee \bar{a}) \vee (a \wedge \bar{a}))$$



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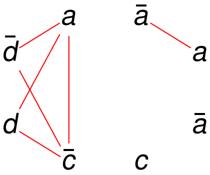
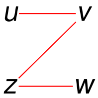


A **cograph** is a graph without $\mathcal{P}_4$:



Graph of a formula

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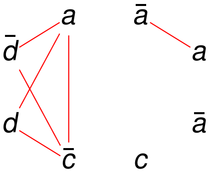

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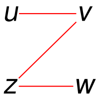


Theorem

A graph $\mathcal{G}$ is graph of a formula A if and only if $\mathcal{G}$ is a cograph.

Multiplicative Flows

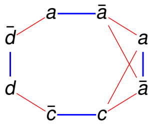
A triple $\phi = \langle A, B, \mathbb{B}_\phi \rangle$ is an **m-flow** if A and B are pure formulas, $\mathbb{B}_\phi$ is a perfect matching on the atom occurences of $\bar{A} \vee B$

Christian Retore, Theoretical Computer Science 2003 (Handsome Proof Nets)

Multiplicative Flows

A triple $\phi = \langle A, B, \mathbb{B}_\phi \rangle$ is an **m-flow** if A and B are pure formulas, $\mathbb{B}_\phi$ is a perfect matching on the atom occurences of $\bar{A} \vee B$ such that the underlying RB-cograph $\mathcal{G}(\phi)$ is æ-acyclic.

correctness criterion:



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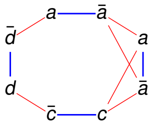

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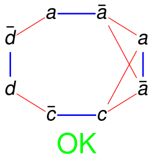

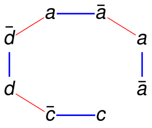

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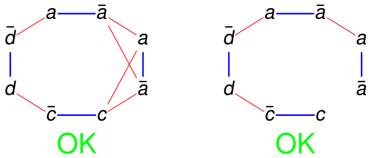
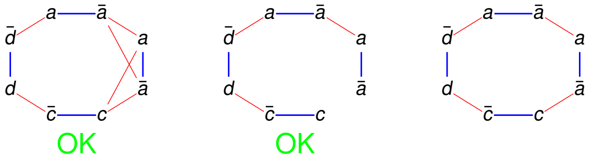


Christian Retore, Theoretical Computer Science 2003 (Handsome Proof Nets)

Multiplicative Flows

A triple $\phi = \langle A, B, \mathbb{B}_\phi \rangle$ is an **m-flow** if A and B are pure formulas, $\mathbb{B}_\phi$ is a perfect matching on the atom occurences of $\bar{A} \vee B$ such that the underlying RB-cograph $\mathcal{G}(\phi)$ is æ-acyclic.

correctness criterion:



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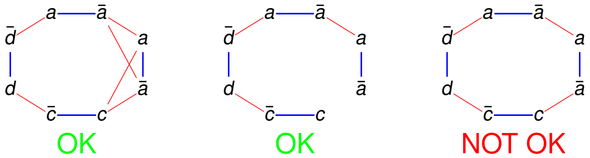

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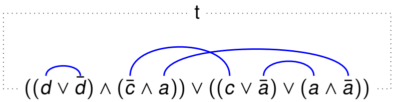
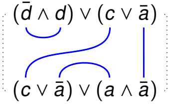
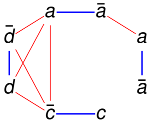
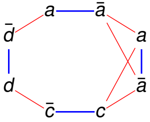


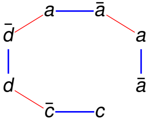




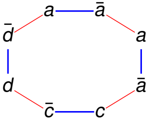
correctness criterion:



OK



OK



NOT OK

Christian Retore, Theoretical Computer Science 2003 (Handsome Proof Nets)

Theorem

Let $\begin{array}{c} A \\ \mathcal{D} \big\Vert \{\mathsf{ai}{\downarrow}, \mathsf{ai}{\uparrow}, \mathsf{s}, \mathsf{mix}\} \\ B \end{array}$ be a derivation. If A and B are pure, then the translation of $\mathcal{D}$ is an m-flow.

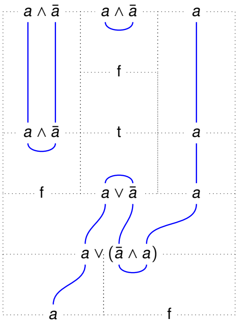
Theorem

Let $\phi = \langle A, B, \mathbb{B}_\phi \rangle$ be an m-flow. Then there is a derivation $\begin{array}{c} A \\ \mathcal{D} \big\Vert \{\mathsf{ai}{\downarrow}, \mathsf{ai}{\uparrow}, \mathsf{s}, \mathsf{mix}\} \\ B \end{array}$ whose translation is ϕ.

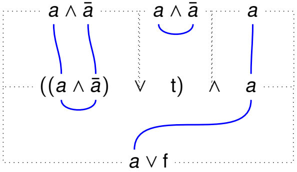
$$\frac{\mathsf{t}}{a \vee \bar{a}}\,\mathsf{ai}{\downarrow} \qquad \frac{(A \vee B) \wedge C}{A \vee (B \wedge C)}\,\mathsf{s} \qquad \frac{\bar{a} \wedge a}{\mathsf{f}}\,\mathsf{ai}{\uparrow} \qquad \frac{\mathsf{f}}{\mathsf{t}}\,\mathsf{mix}$$

$$
\dfrac{\dfrac{[(a \wedge \bar{a}) \vee \boxed{\dfrac{\dfrac{a \wedge \bar{a}}{\mathsf{f}}\,\mathsf{ai}\!\uparrow}{\mathsf{t}}\,\mathsf{mix}}\,] \wedge a}{[\boxed{\dfrac{a \wedge \bar{a}}{\mathsf{f}}\,\mathsf{ai}\!\uparrow} \vee \boxed{\dfrac{\mathsf{t}}{a \vee a}\,\mathsf{ai}\!\downarrow}\,] \wedge a}\,\equiv}{a \vee \boxed{\dfrac{a \wedge \bar{a}}{\mathsf{f}}\,\mathsf{ai}\!\uparrow}}\,\mathsf{s}
$$

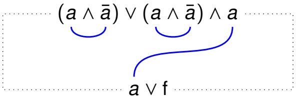
$\rightarrow$



$\downarrow$



$\leftarrow$



Additive Flows

- A triple $\phi = \langle A, B, f_\phi^\downarrow \rangle$ is an **a$^\downarrow$-flow** if A and B are pure, and $A \neq$ t, and $f_\phi^\downarrow$ is a skew fibration $f_\phi^\downarrow : \mathcal{G}(A) \to \mathcal{G}(B)$.

- A triple $\phi = \langle A, B, f^{\downarrow}_{\phi} \rangle$ is an **a$^{\downarrow}$-flow** if A and B are pure, and $A \neq \mathsf{t}$, and $f^{\downarrow}_{\phi}$ is a skew fibration $f^{\downarrow}_{\phi} \colon \mathcal{G}(A) \to \mathcal{G}(B)$.

A **skew fibration** is a graph homomorphism $f \colon \mathcal{G} \to \mathcal{H}$ such that

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A **skew fibration** is a graph homomorphism $f \colon \mathcal{G} \to \mathcal{H}$ such that for every $v \in V_G$ and $w \in V_H$, with $f(v)w \in \mathcal{E}_H$,

$$\begin{tikzcd} v \arrow[d, dashed] & \\ f(v) \arrow[r, dash] & w \end{tikzcd}$$

- A triple $\phi = \langle A, B, f^{\downarrow}_{\phi} \rangle$ is an **a$^{\downarrow}$-flow** if A and B are pure, and $A \neq$ t, and $f^{\downarrow}_{\phi}$ is a skew fibration $f^{\downarrow}_{\phi} \colon \mathcal{G}(A) \to \mathcal{G}(B)$.

A **skew fibration** is a graph homomorphism $f \colon \mathcal{G} \to \mathcal{H}$ such that for every $v \in V_G$ and $w \in V_H$, with $f(v)w \in \mathcal{E}_H$, there exists $z \in \mathcal{G}$ with the edge $vz \in \mathcal{E}_G$

$$\begin{tikzcd} v \arrow[rr, dash] \arrow[d, dotted] & & z \\ f(v) \arrow[dr, dash] & & \\ & w & \end{tikzcd}$$

Additive Flows

- A triple $\phi = \langle A, B, f^{\downarrow}_{\phi} \rangle$ is an **a$^{\downarrow}$-flow** if A and B are pure, and $A \neq \mathrm{t}$, and $f^{\downarrow}_{\phi}$ is a skew fibration $f^{\downarrow}_{\phi} \colon \mathcal{G}(A) \to \mathcal{G}(B)$.

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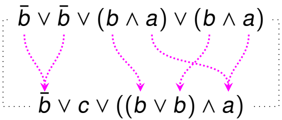
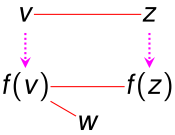
$$
\begin{tikzcd}
v \arrow[r, dash] \arrow[d, dotted] & z \arrow[d, dotted] \\
f(v) \arrow[r, dash] \arrow[dr, dash] & f(z) \\
& w
\end{tikzcd}
$$

- A triple $\phi = \langle A, B, f_\phi^{\downarrow} \rangle$ is an **a$^{\downarrow}$-flow** if A and B are pure, and $A \neq \mathrm{t}$, and $f_\phi^{\downarrow}$ is a skew fibration $f_\phi^{\downarrow} \colon \mathcal{G}(A) \to \mathcal{G}(B)$.



a$^{\downarrow}$-flow

A **skew fibration** is a graph homomorphism $f \colon \mathcal{G} \to \mathcal{H}$ such that for every $v \in V_G$ and $w \in V_H$, with $f(v)w \in \mathcal{E}_H$, there exists $z \in \mathcal{G}$ with the edge $vz \in \mathcal{E}_G$ such that the edge $f(z)w$ does not exist in $\mathcal{H}$.



Additive Flows

- A triple $\phi = \langle A, B, f_\phi^\downarrow \rangle$ is an **a$^\downarrow$-flow** if A and B are pure, and $A \neq$ t, and $f_\phi^\downarrow$ is a skew fibration $f_\phi^\downarrow \colon \mathcal{G}(A) \to \mathcal{G}(B)$.

$$\begin{array}{cc}
\bar{b} \vee \bar{b} \vee (b \wedge a) \vee (b \wedge a) & \bar{b} \vee \bar{b} \vee (b \wedge a) \vee (b \wedge a) \\
\downarrow & \downarrow \\
\bar{b} \vee c \vee ((b \vee b) \wedge a) & (\bar{b} \wedge c) \vee ((b \vee b) \wedge a)
\end{array}$$

a$^\downarrow$-flow — not a skew fibration

A **skew fibration** is a graph homomorphism $f \colon \mathcal{G} \to \mathcal{H}$ such that for every $v \in V_G$ and $w \in V_H$, with $f(v)w \in \mathcal{E}_H$, there exists $z \in \mathcal{G}$ with the edge $vz \in \mathcal{E}_G$ such that the edge $f(z)w$ does not exist in $\mathcal{H}$.

$$\begin{tikzcd}
v \arrow[r, dash] \arrow[d, dotted] & z \arrow[d, dotted] \\
f(v) \arrow[r, dash] \arrow[dr, dash] & f(z) \\
& w
\end{tikzcd}$$

Additive Flows

- A triple $\phi = \langle A, B, f_\phi^\downarrow \rangle$ is an **a$^\downarrow$-flow** if A and B are pure, and $A \neq \mathsf{t}$, and $f_\phi^\downarrow$ is a skew fibration $f_\phi^\downarrow \colon \mathcal{G}(A) \to \mathcal{G}(B)$.
- A triple $\phi = \langle C, D, f_\phi^\uparrow \rangle$ is an **a$^\uparrow$-flow** if C and D are pure, and $D \neq \mathsf{f}$, and $f_\phi^\uparrow$ is a skew fibration $f_\phi^\uparrow \colon \mathcal{G}(\overline{D}) \to \mathcal{G}(\overline{C})$.

$$(a \wedge (b \vee c)) \vee (b \wedge c)$$
$$(a \vee b) \wedge (a \vee b) \wedge (a \vee c)$$

a$^\uparrow$-flow

$$\bar{b} \vee \bar{b} \vee (b \wedge a) \vee (b \wedge a)$$
$$\bar{b} \vee c \vee ((b \vee b) \wedge a)$$

a$^\downarrow$-flow

$$\bar{b} \vee \bar{b} \vee (b \wedge a) \vee (b \wedge a)$$
$$(\bar{b} \wedge c) \vee ((b \vee b) \wedge a)$$

not a skew fibration

A **skew fibration** is a graph homomorphism $f \colon \mathcal{G} \to \mathcal{H}$ such that for every $v \in V_G$ and $w \in V_H$, with $f(v)w \in \mathcal{E}_H$, there exists $z \in \mathcal{G}$ with the edge $vz \in \mathcal{E}_G$ such that the edge $f(z)w$ does not exist in $\mathcal{H}$.

$$\begin{tikzcd}
v \arrow[r, dash] \arrow[d, dotted] & z \arrow[d, dotted] \\
f(v) \arrow[r, dash] \arrow[dr, dash] & f(z) \\
& w
\end{tikzcd}$$

Additive Flows

Theorem

Let $\begin{array}{c} A \\ \mathcal{D} \Vert \{\mathsf{aw}\downarrow, \mathsf{ac}\downarrow, \mathsf{m}\} \\ B \end{array}$ be a derivation. If A and B are pure, then translation of $\mathcal{D}$ is an $\mathsf{a}^{\downarrow}$-flow. Dually, if A and B are pure in $\begin{array}{c} A \\ \mathcal{D} \Vert \{\mathsf{aw}\uparrow, \mathsf{ac}\uparrow, \mathsf{m}\} \\ B \end{array}$ then translation of $\mathcal{D}$ is an $\mathsf{a}^{\uparrow}$-flow.

Theorem

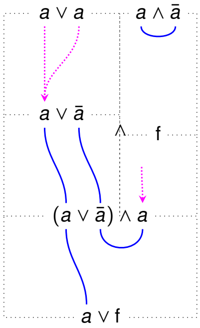
Let $\phi = \langle A, B, f^{\downarrow}_{\phi} \rangle$ be an $\mathsf{a}^{\downarrow}$-flow. Then there is a derivation $\begin{array}{c} A \\ \mathcal{D} \Vert \{\mathsf{aw}\downarrow, \mathsf{ac}\downarrow, \mathsf{m}\} \\ B \end{array}$ whose translation is ϕ. For every $\mathsf{a}^{\uparrow}$-flow ψ we have $\begin{array}{c} A \\ \mathcal{D} \Vert \{\mathsf{aw}\uparrow, \mathsf{ac}\uparrow, \mathsf{m}\} \\ B \end{array}$ whose translation is ψ.

$$\frac{a \vee a}{a}\,\mathsf{ac}\downarrow \qquad \frac{\mathsf{f}}{a}\,\mathsf{aw}\downarrow \qquad \frac{(A \wedge C) \vee (B \wedge D)}{(A \vee B) \wedge (C \vee D)}\,\mathsf{m} \qquad \frac{a}{\mathsf{t}}\,\mathsf{aw}\uparrow \qquad \frac{a}{a \wedge a}\,\mathsf{ac}\uparrow$$

Purification

Pure Formulas: $A \equiv \mathsf{t}$ or $A \equiv \mathsf{f}$ or A is equivalent to a unit-free formula.

Slice of a Combinatorial flow:



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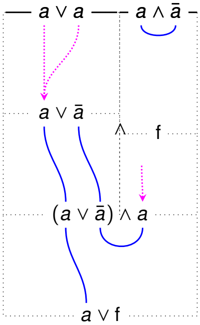

$(a \vee a) \wedge (a \wedge \bar{a})$ pure

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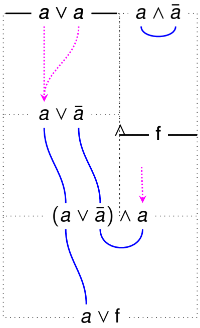
Slice of a Combinatorial flow:

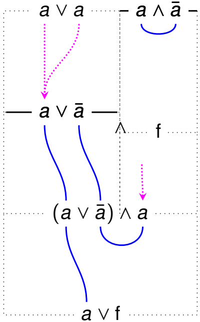


$(a \vee a) \wedge \mathsf{f}$ not pure

Purification

Pure Formulas: $A \equiv \mathsf{t}$ or $A \equiv \mathsf{f}$ or A is equivalent to a unit-free formula.

Slice of a Combinatorial flow:



$(a \vee \bar{a}) \wedge (a \wedge \bar{a})$ pure

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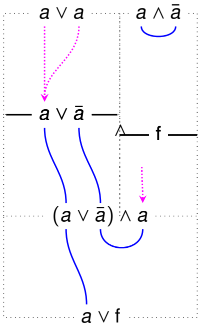
Slice of a Combinatorial flow:



$(a \vee \bar{a}) \wedge \mathsf{f}$ not pure

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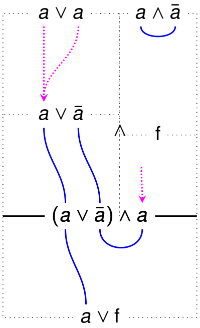
Slice of a Combinatorial flow:

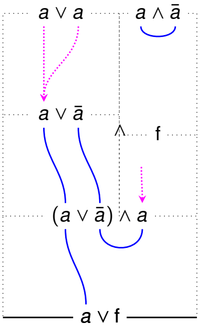


$(a \vee \bar{a}) \wedge a$ pure

Purification

Pure Formulas: $A \equiv \mathsf{t}$ or $A \equiv \mathsf{f}$ or A is equivalent to a unit-free formula.

Slice of a Combinatorial flow:

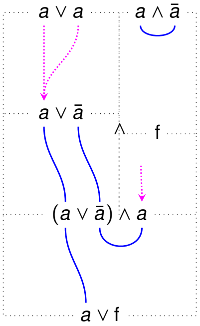


$a \vee \mathsf{f}$ pure

Purification of a formula:

$$
\begin{array}{llll}
A \wedge \mathsf{t} \rightsquigarrow A & \mathsf{t} \wedge A \rightsquigarrow A & A \vee \mathsf{t} \rightsquigarrow \mathsf{t} & \mathsf{t} \vee A \rightsquigarrow \mathsf{t} \\
A \vee \mathsf{f} \rightsquigarrow A & \mathsf{f} \vee A \rightsquigarrow A & A \wedge \mathsf{f} \rightsquigarrow \mathsf{f} & \mathsf{f} \wedge A \rightsquigarrow \mathsf{f}
\end{array}
$$

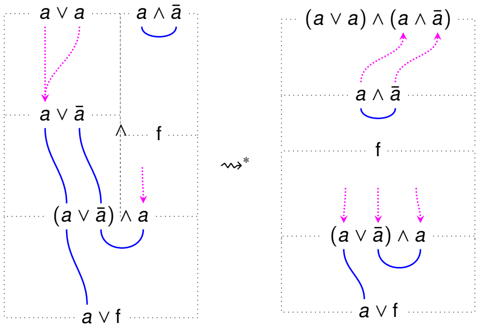
Purification of combinatorial flows:



Purification of a formula:

$$A \wedge \mathsf{t} \rightsquigarrow A \qquad \mathsf{t} \wedge A \rightsquigarrow A \qquad A \vee \mathsf{t} \rightsquigarrow \mathsf{t} \qquad \mathsf{t} \vee A \rightsquigarrow \mathsf{t}$$
$$A \vee \mathsf{f} \rightsquigarrow A \qquad \mathsf{f} \vee A \rightsquigarrow A \qquad A \wedge \mathsf{f} \rightsquigarrow \mathsf{f} \qquad \mathsf{f} \wedge A \rightsquigarrow \mathsf{f}$$

Purification of combinatorial flows:

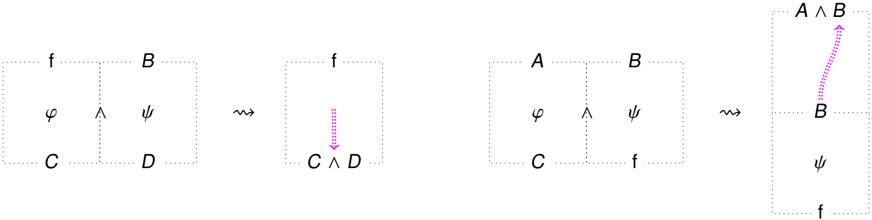


purification

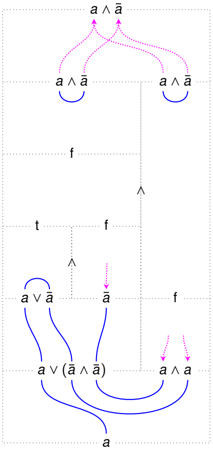
Purification of a formula:

$$
\begin{array}{cccc}
A \wedge \mathsf{t} \rightsquigarrow A & \mathsf{t} \wedge A \rightsquigarrow A & A \vee \mathsf{t} \rightsquigarrow \mathsf{t} & \mathsf{t} \vee A \rightsquigarrow \mathsf{t} \\
A \vee \mathsf{f} \rightsquigarrow A & \mathsf{f} \vee A \rightsquigarrow A & A \wedge \mathsf{f} \rightsquigarrow \mathsf{f} & \mathsf{f} \wedge A \rightsquigarrow \mathsf{f}
\end{array}
$$

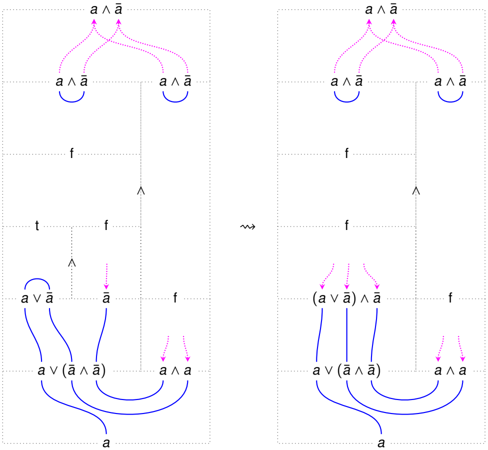
Purification of combinatorial flows:



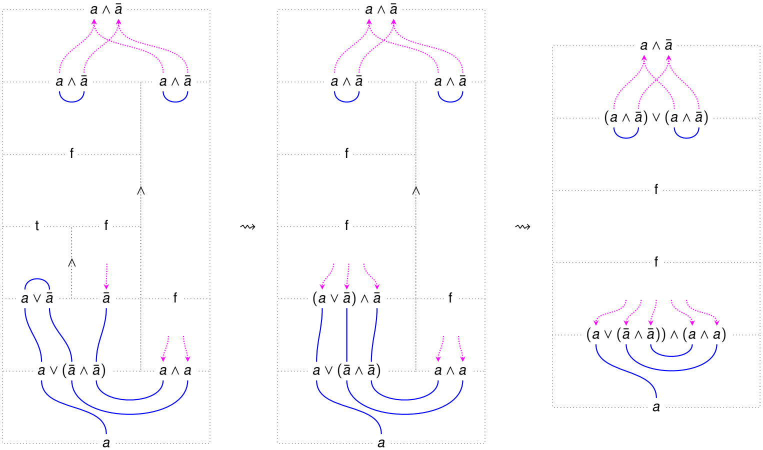
Purification Example



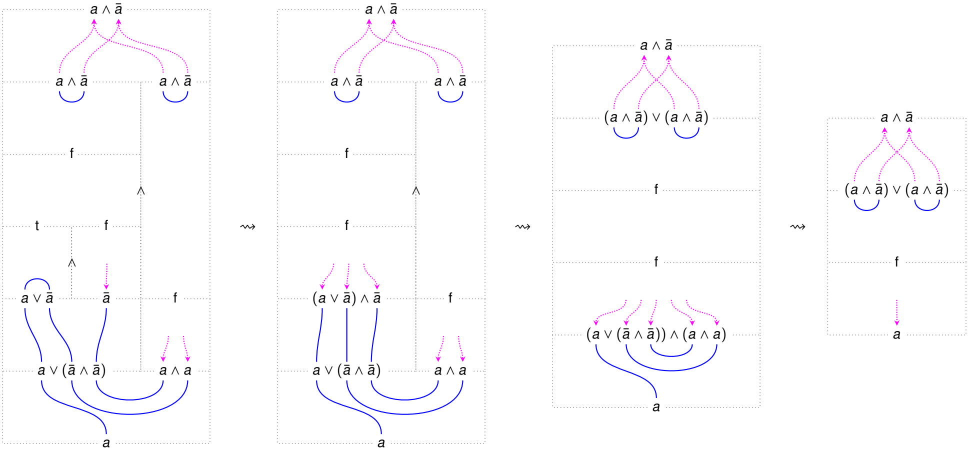
Purification Example



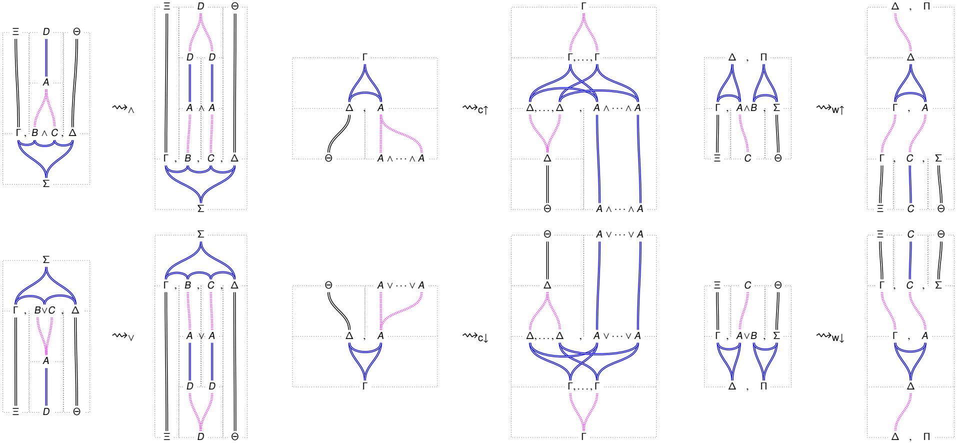
Purification Example



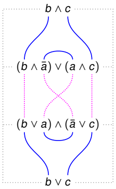
Purification Example



Normalization (Work in Progress)



Normalization Example



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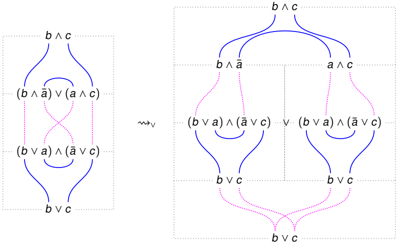

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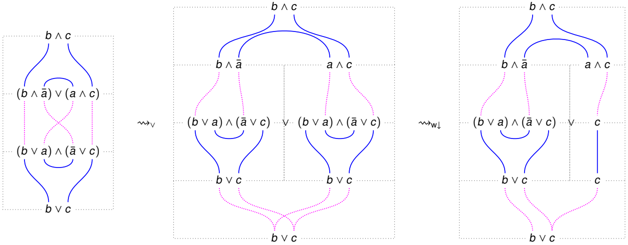
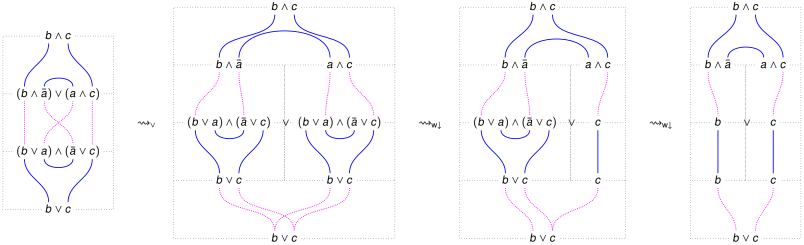
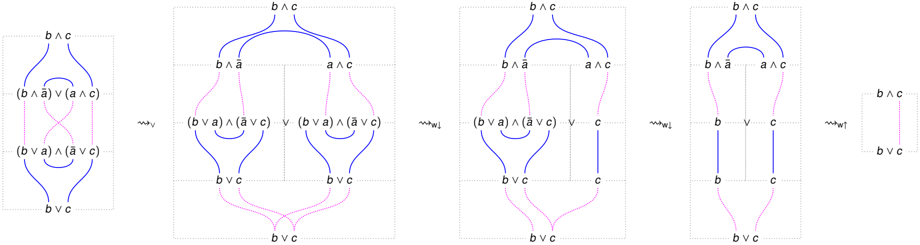


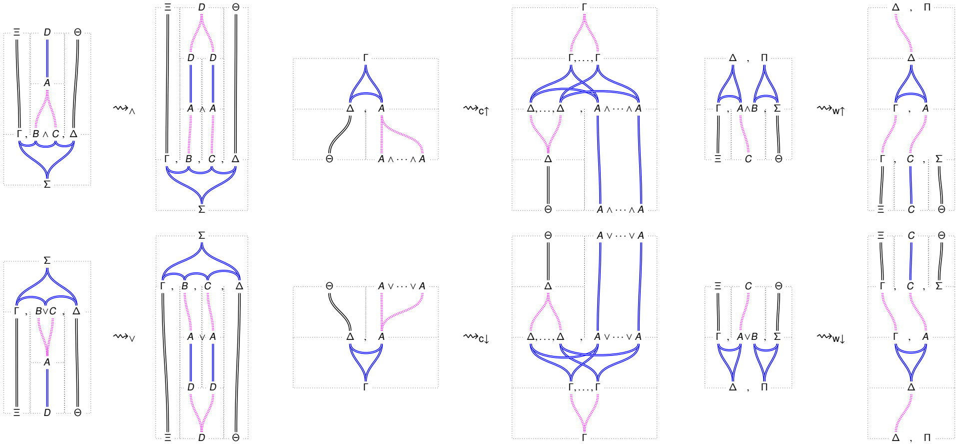
Normalization Example



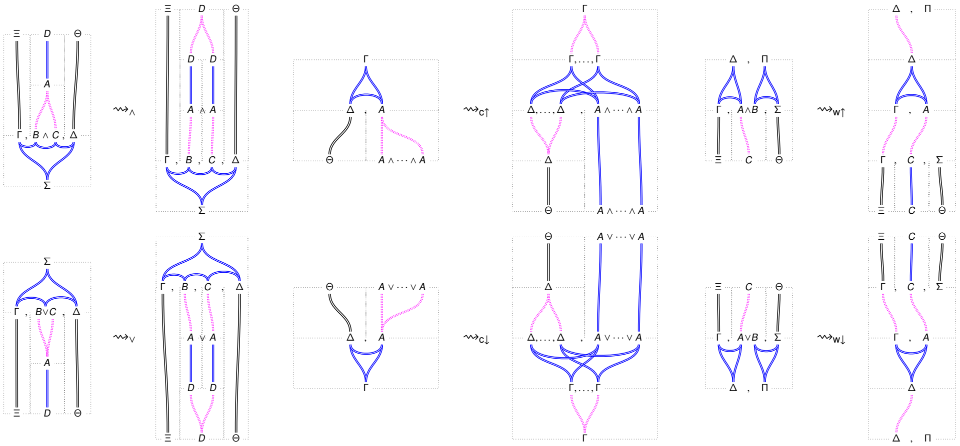
Normalization Example



Normalization (Work in Progress)

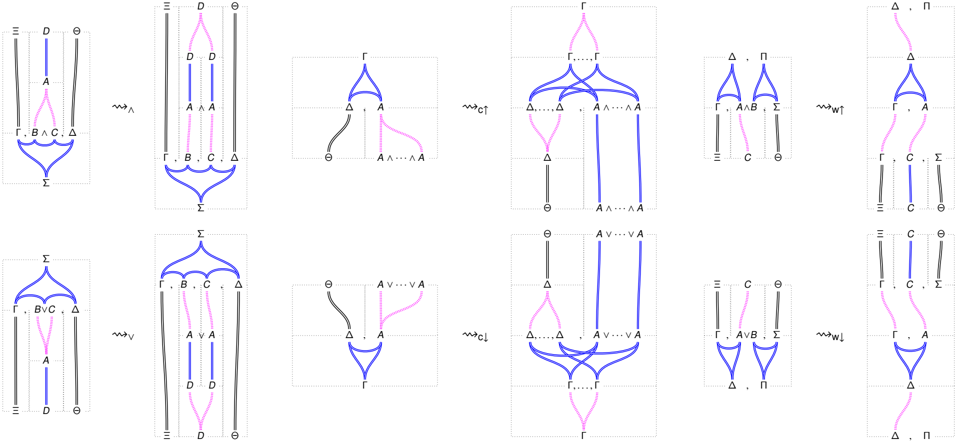


Normalization (Work in Progress)



NOT confluent

Normalization (Work in Progress)



NOT confluent and NOT terminating

What to remember from this talk?

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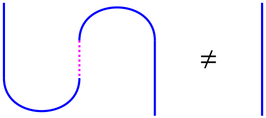

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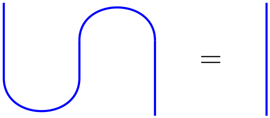
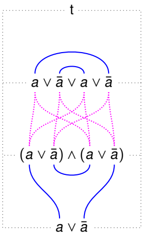


Future Work

- Normalization Termination
- Proof identity
- Other Logics (Forexample: Modal Logic and Intuitionistic Logic)

What to remember from this talk?

Normalization is not Confluent and not Terminating



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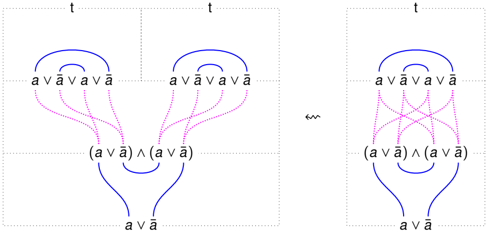

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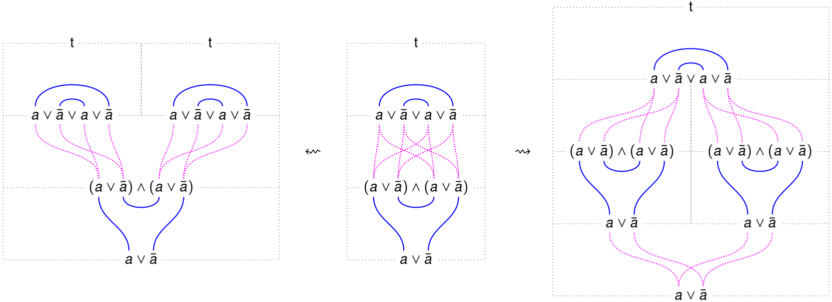
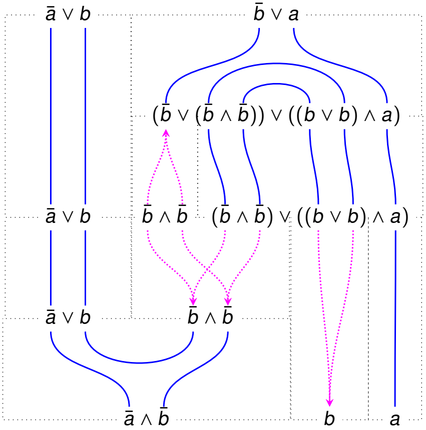


Yanking example5



A combinatorial proof with cuts for the sequent Γ is a combinatorial proof for the sequent $\Gamma, C_1 \wedge \bar{C}_1, \ldots, C_n \wedge \bar{C}_n$ where $C_1, \ldots C_n$ are cut formulas. (Everything is unit-free)

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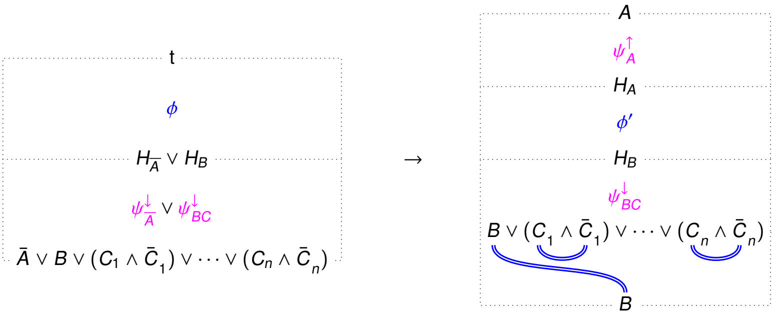
Translating a combinatorial proof with cuts of $\bar{A}, B$ to a combinatorial flow from A to B:



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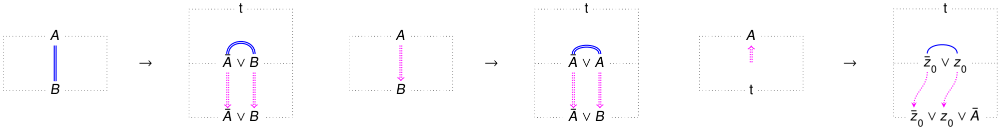
Translating a combinatorial flow with premise A and conclusion B to a combinatorial proof with cuts for $\bar{A}, B$:



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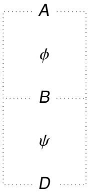

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Translating a combinatorial flow with premise A and conclusion B to a combinatorial proof with cuts for $\bar{A}, B$:

$$\begin{array}{c} A \\ \phi \\ B \\ \psi \\ D \end{array} \quad \rightarrow \quad \begin{array}{c} \mathsf{t} \\ \phi \\ H_{\bar{A}} \vee H_B \\ \phi^{\downarrow}_{\bar{A}} \vee \phi^{\downarrow}_{BC} \\ \bar{A} \vee B \vee (C_1 \wedge \bar{C}_1) \vee \ldots \end{array} \qquad \begin{array}{c} \mathsf{t} \\ \psi \\ H_{\bar{B}} \vee H_D \\ \psi^{\downarrow}_{\bar{B}} \vee \psi^{\downarrow}_{DC'} \\ \bar{B} \vee D \vee (C'_1 \wedge \bar{C}'_1) \vee \ldots \end{array}$$

Combinatorial Flows vs. Combinatorial Proofs with cuts

A combinatorial proof with cuts for the sequent Γ is a combinatorial proof for the sequent $\Gamma, C_1 \wedge \bar{C}_1, \ldots, C_n \wedge \bar{C}_n$ where $C_1, \ldots C_n$ are cut formulas. (Everything is unit-free)

Translating a combinatorial flow with premise A and conclusion B to a combinatorial proof with cuts for $\bar{A}, B$:

$$
\begin{array}{ccccccc}
A & & \mathsf{t} & & \mathsf{t} & & \mathsf{t} \\
\phi & & \phi & & \psi & & \phi \wedge \psi \\
B & \to & H_{\overline{A}} \vee H_B & & H_{\overline{B}} \vee H_D & \to & H_{\overline{A}} \vee (H_B \wedge H_{\overline{B}}) \vee H_D \\
\psi & & \phi^{\downarrow}_{\overline{A}} \vee \phi^{\downarrow}_{BC} & & \psi^{\downarrow}_{\overline{B}} \vee \psi^{\downarrow}_{DC'} & & \phi^{\downarrow}_{\overline{A}} \vee (\phi^{\downarrow}_{BC} \wedge \psi^{\downarrow}_{\overline{B}}) \vee \psi^{\downarrow}_{DC'} \\
D & & \bar{A} \vee B \vee (C_1 \wedge \bar{C}_1) \vee \ldots & & \bar{B} \vee D \vee (C'_1 \wedge \bar{C}'_1) \vee \ldots & & \bar{A} \vee (B \wedge \bar{B}) \vee D \vee (C_1 \wedge \bar{C}_1) \vee (C'_1 \wedge \bar{C}'_1) \vee \ldots
\end{array}
$$