

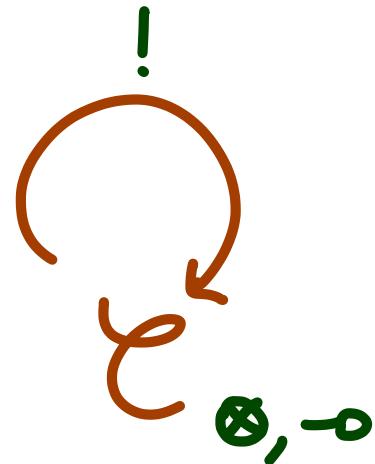
Call-by-value in Bicategories of Games

Hugo Paquet
LiPN

joint work with Philip Saville

Resources and effects

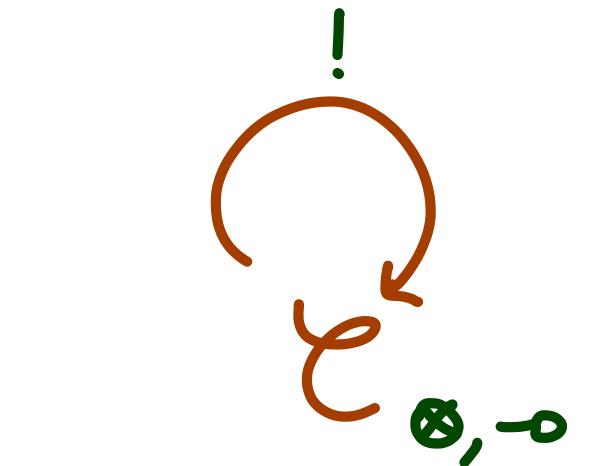
Models of Linear logic



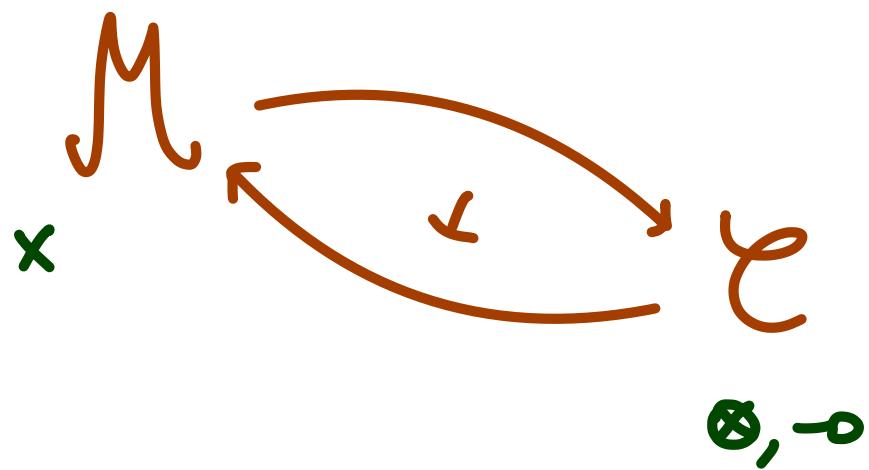
exponential
comonad

Resources and effects

Models of Linear logic



exponential
comonad



linear / non-linear
adjunction

Resources and effects

Moggi, Levy, ...

Models of effectful languages

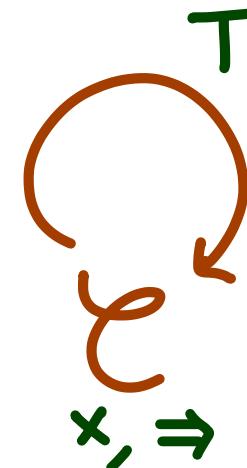


Resources and effects

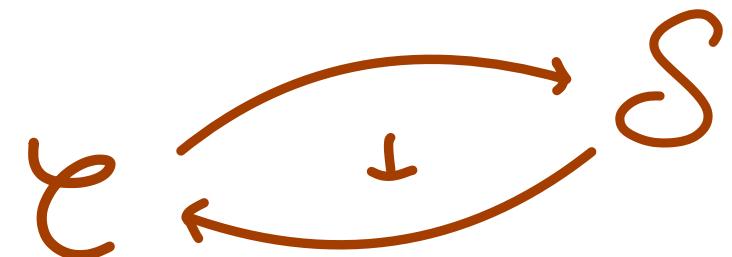
Moggi, Levy, ...

Models of effectful languages

Strong
monad



Strong
adjunction



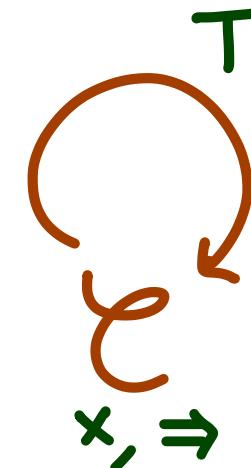
Resources and effects

Moggi, Levy, ...

Models of effectful languages

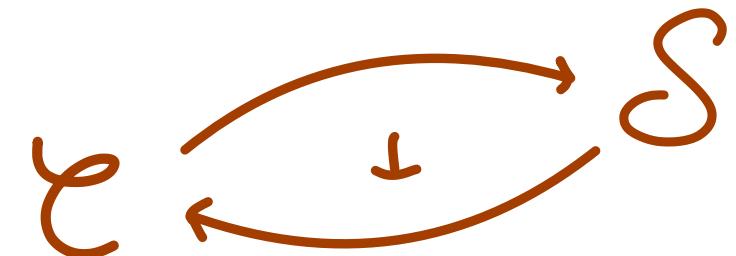
(CBV)

Strong monad



(CBPV)

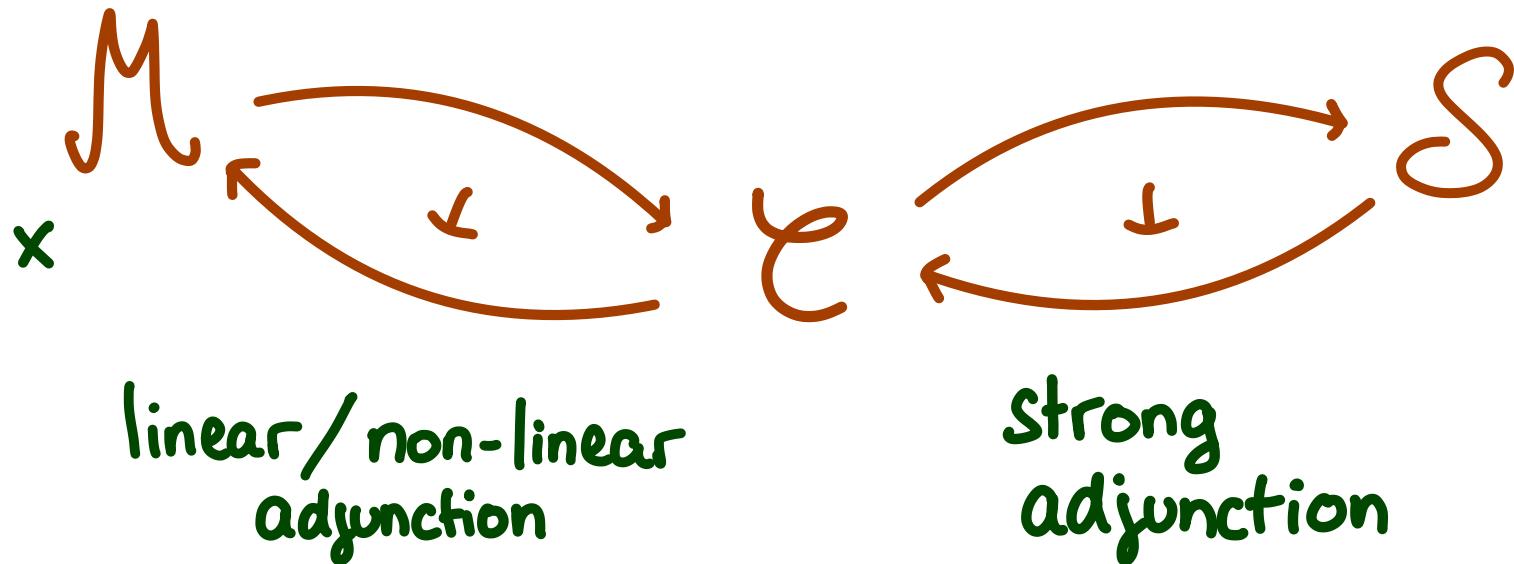
Strong adjunction



Resources and effects

Munch-Maccagnoni, Fiore

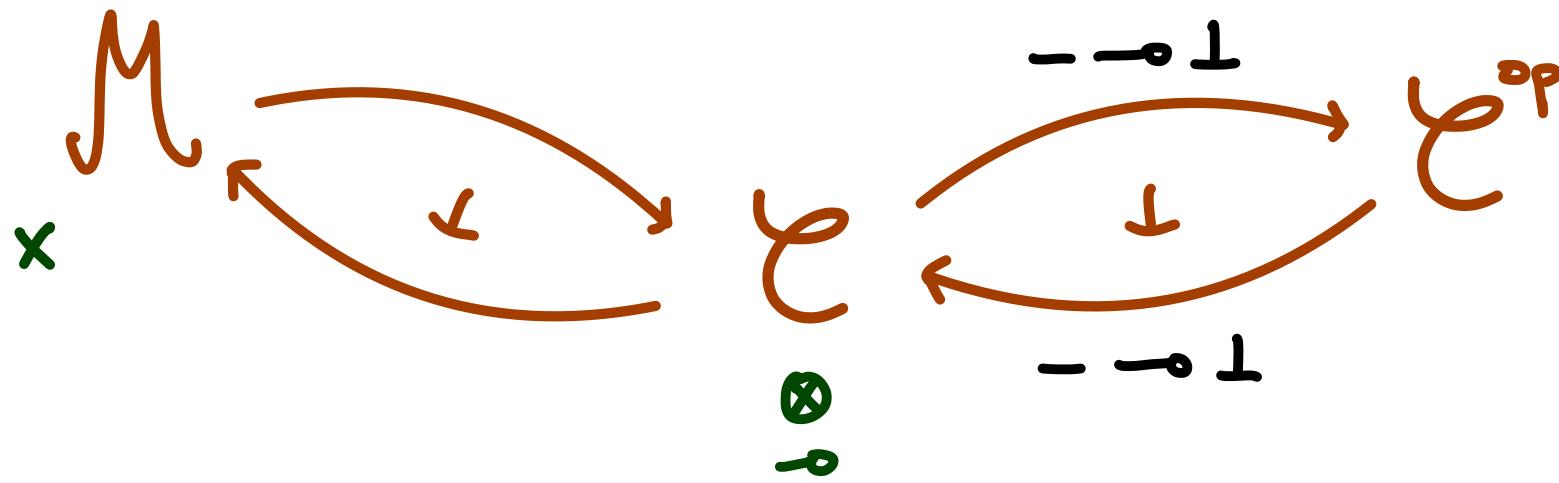
- Can combine the two viewpoints ("Linear CBPV")



- Several canonical examples.

Resources and effects

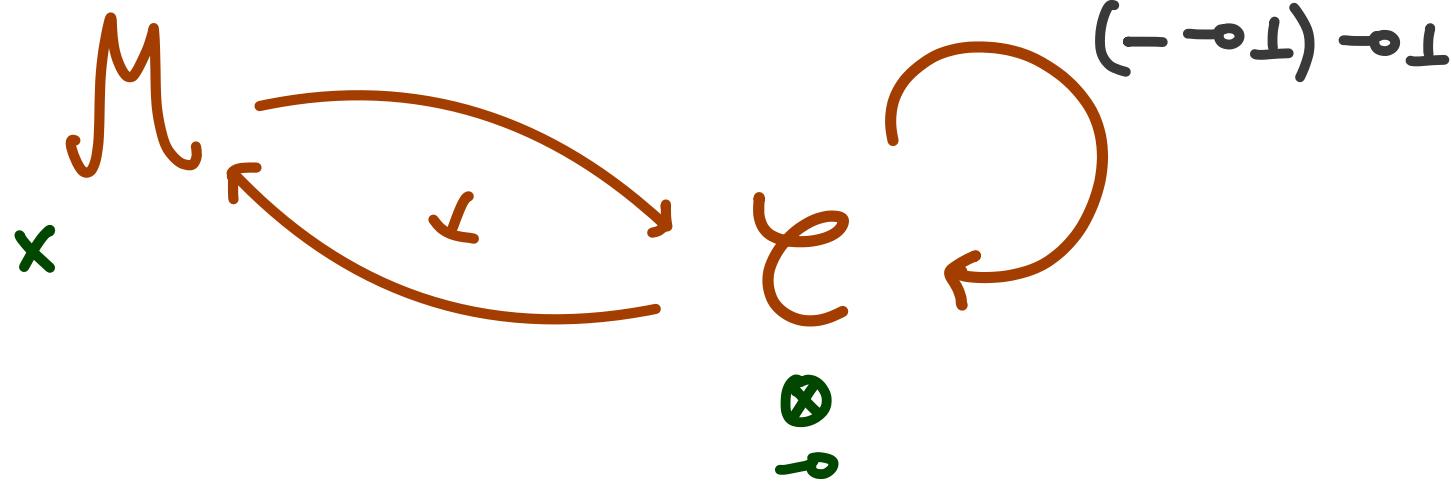
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- e.g.



linear / non-linear
adjunction

Resources and effects

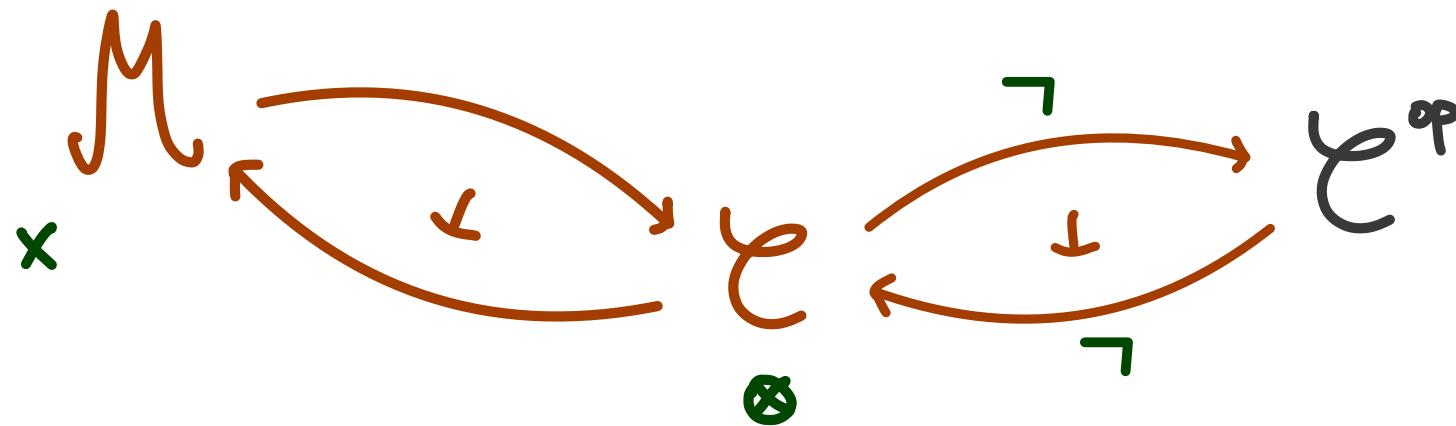
- Can combine the two viewpoints ("Linear CBPV")
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linear / non-linear
adjunction

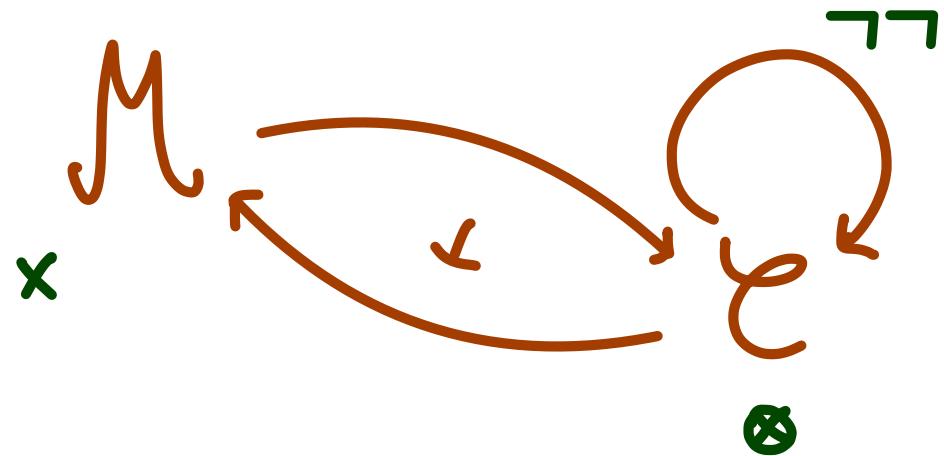
Resources and effects

Gane Semantics :



Resources and effects

Game Semantics :

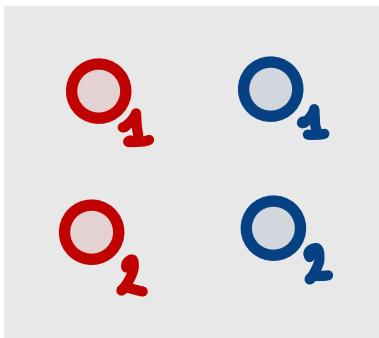


Game Semantics

Two players

- player
- opponent

A simple game :



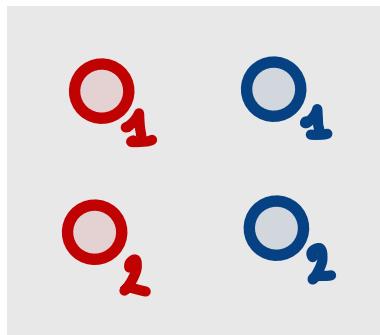
A

Game Semantics

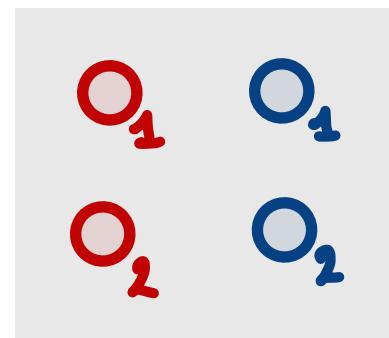
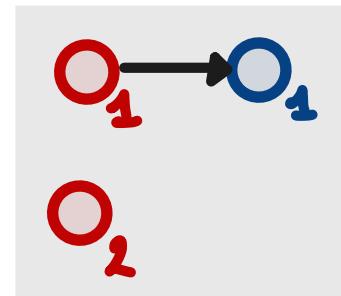
Two players

- player
- opponent

A simple game:



Strategies over the game:



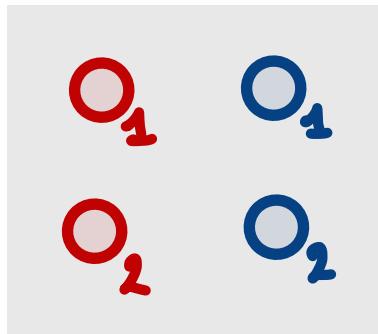
A

Game Semantics

Two players

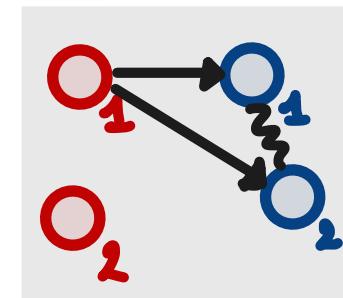
- player
- opponent

A simple game:

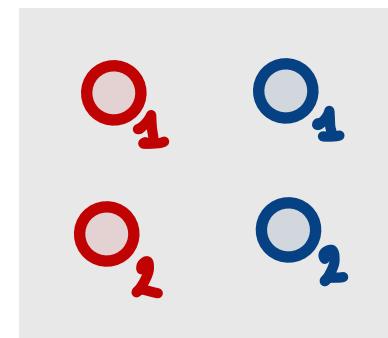


A

Strategies over the game:



A



A

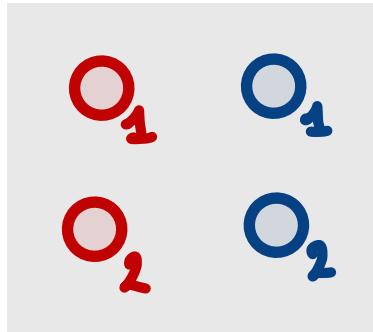
Game Semantics

use event structures
[Winskel, Clairambault,
Castellan, ...]

Two players

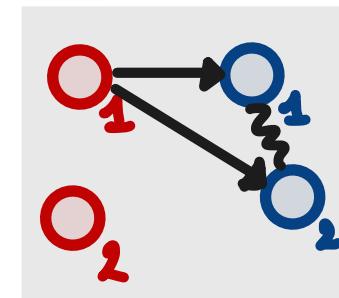
- player
- opponent

A simple game:

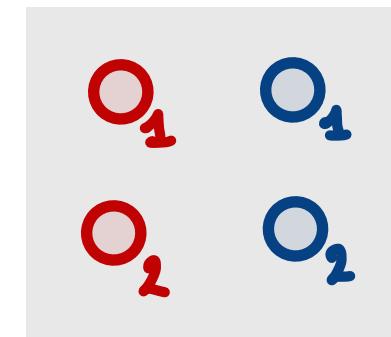


A

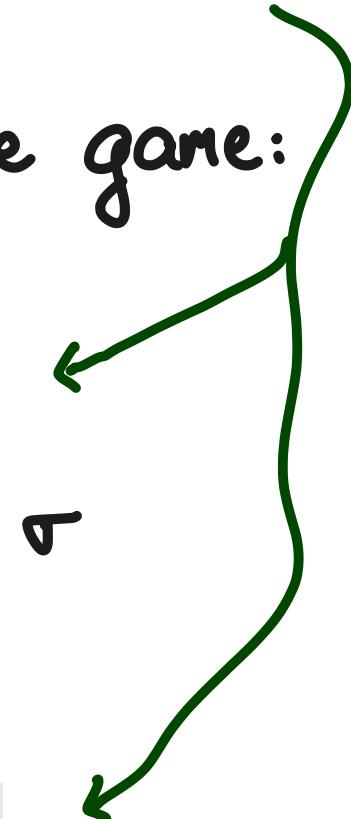
Strategies over the game:



A



A



Strategies $\mathbb{N} \rightarrow \mathbb{N}$

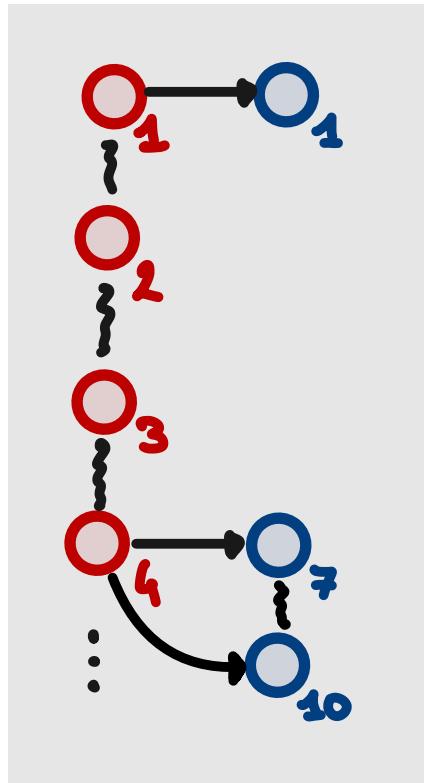
Distinguish between values & computations

to make sense of $(\lambda x.M)V = M[V/x]$ in CBV

Strategies $\mathbb{N} \rightarrow \mathbb{N}$

Distinguish between values & computations
to make sense of $(\lambda x.M)V = M[V/x]$ in CBV

arbitrary
strategy

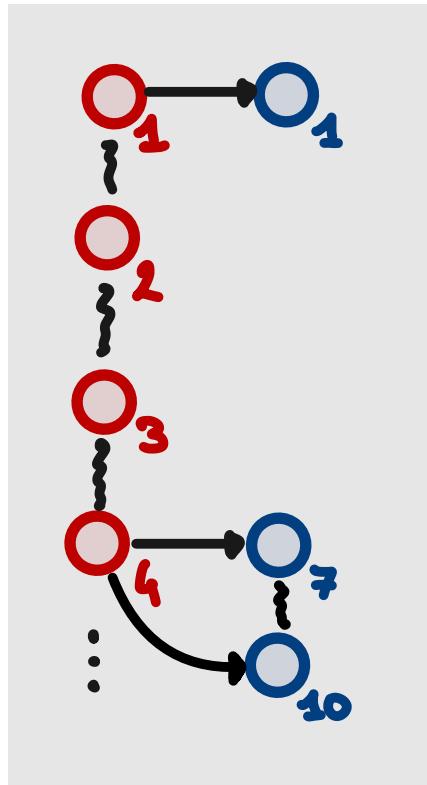


$\mathbb{N} \rightarrow \mathbb{N}$

Strategies $\mathbb{N} \rightarrow \mathbb{N}$

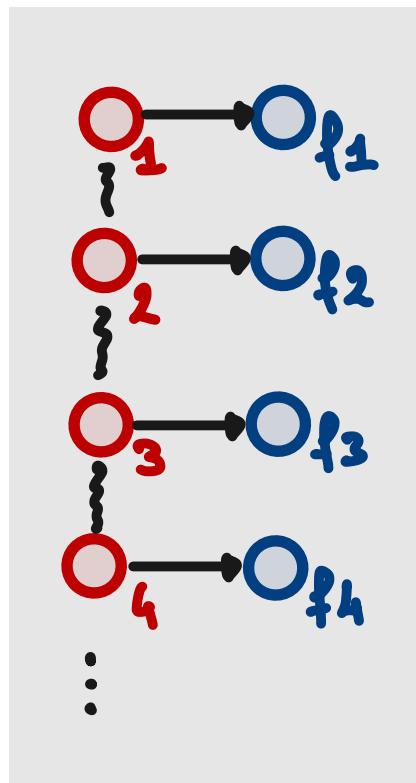
Distinguish between values & computations
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arbitrary
strategy



$\mathbb{N} \rightarrow \mathbb{N}$

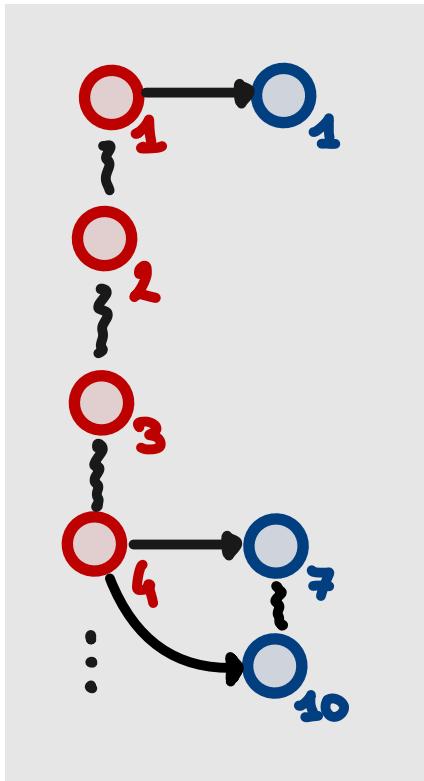
"value"
strategy



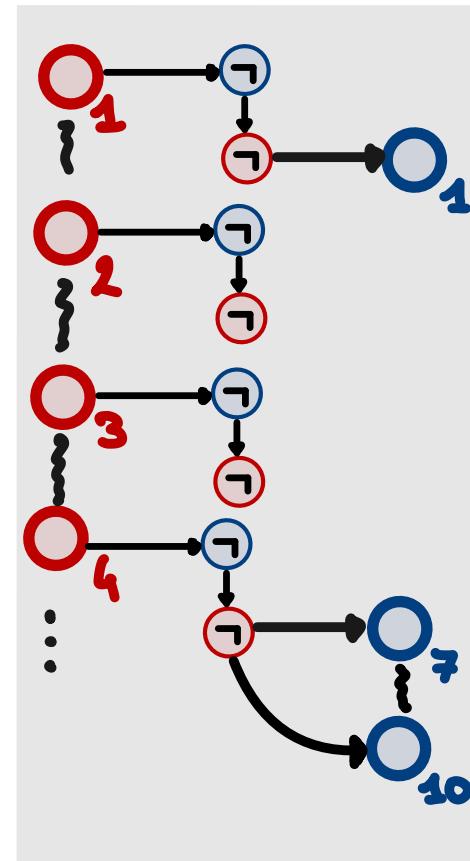
$\mathbb{N} \rightarrow \mathbb{N}$

(total,
deterministic)

Strategies $\mathbb{N} \rightarrow \mathbb{N}$

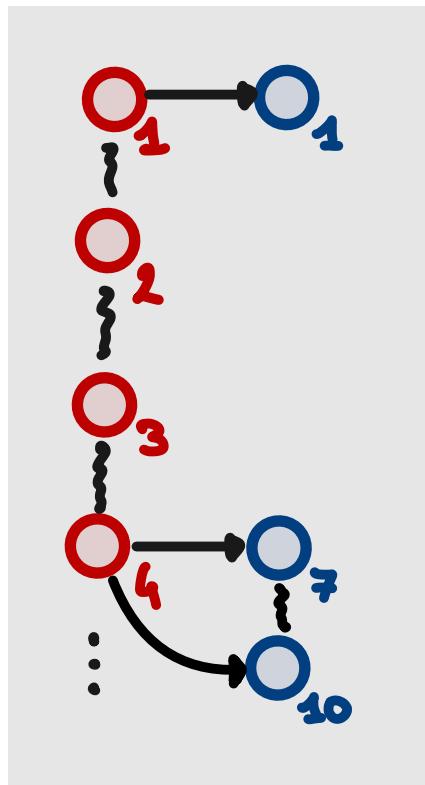


$\mathbb{N} \rightarrow \mathbb{N}$

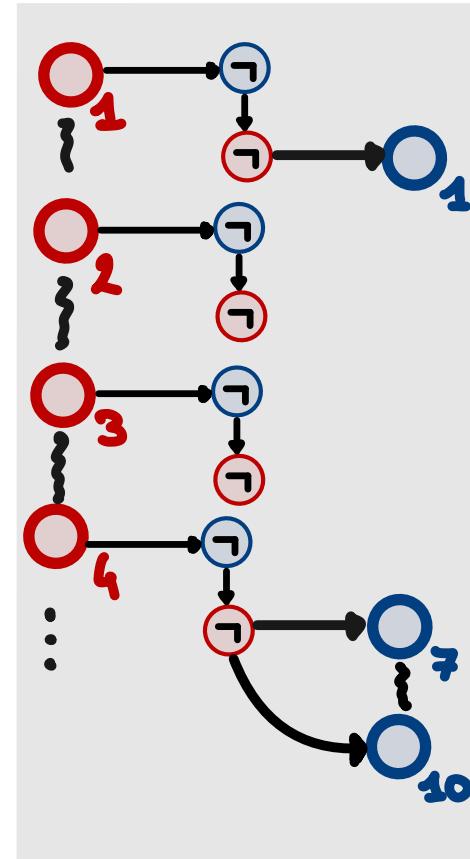


$\mathbb{N} \rightarrow \mathbb{N}$

Strategies $\mathbb{N} \rightarrow \mathbb{N}$



$\mathbb{N} \rightarrow \mathbb{N}$



$\mathbb{N} \rightarrow \vdash \mathbb{N}$

Prop:

$$\frac{\text{strategy } A \rightarrow B}{\text{value strategy } A \rightarrow \vdash B}$$

← monad ?

This talk

→ I. A bicategory of games

II. Strong pseudomonads

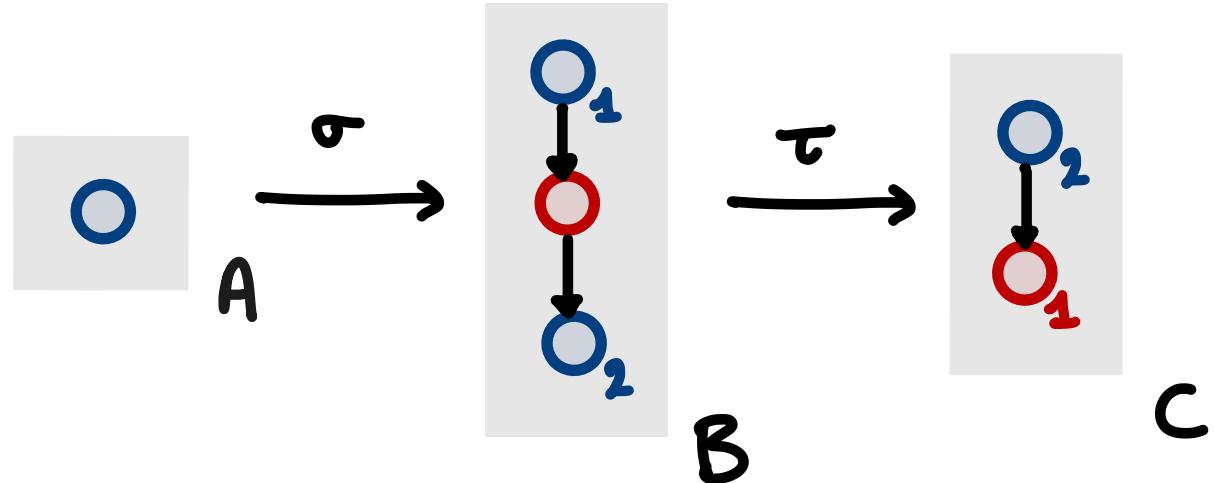
III. Resources and symmetries

Objects: games A, B, C, \dots

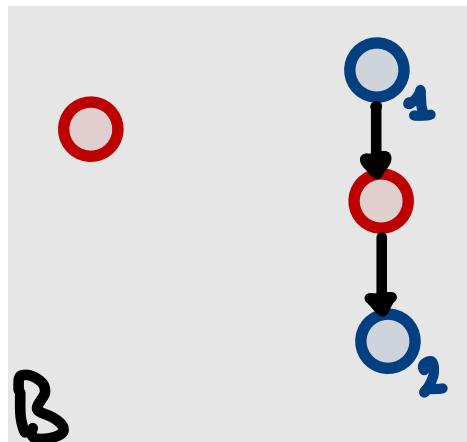
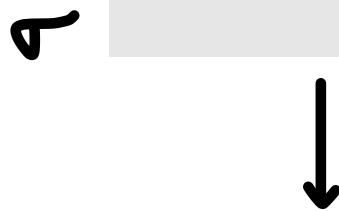
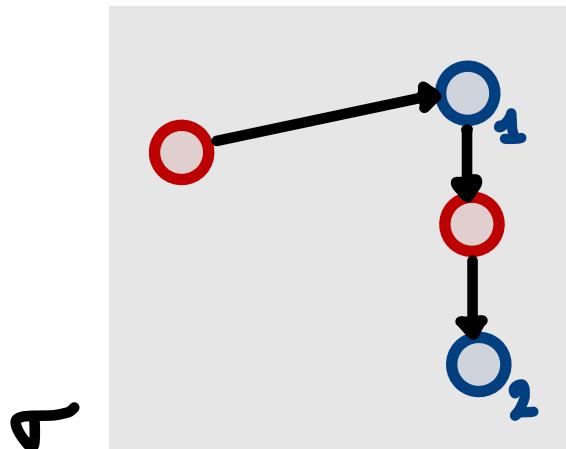
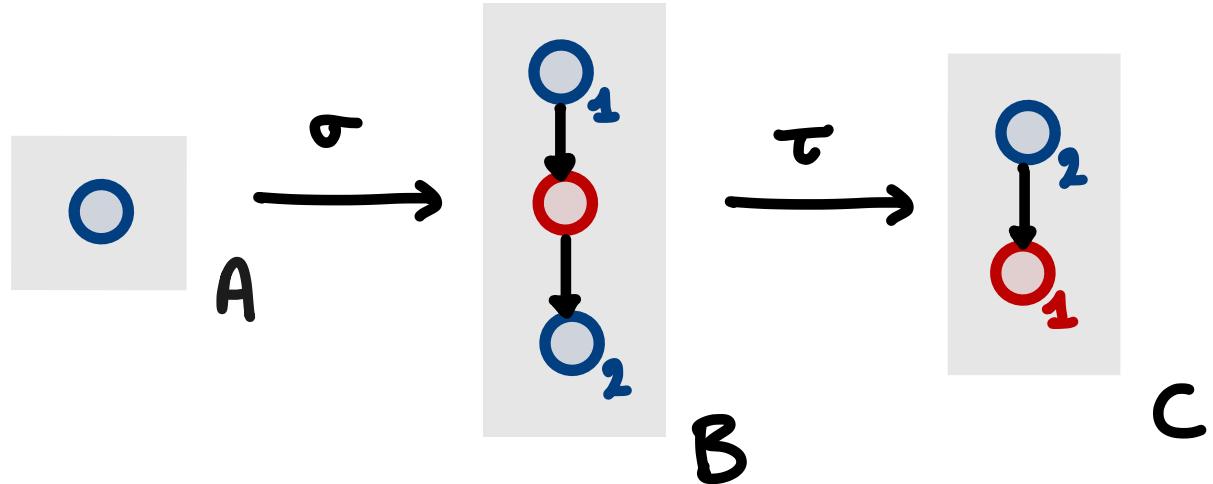
Morphisms $A \rightarrow B$. Strategies

$$\begin{array}{c} \sigma \\ \downarrow \\ A^\perp \otimes B \end{array}$$

Composition of strategies

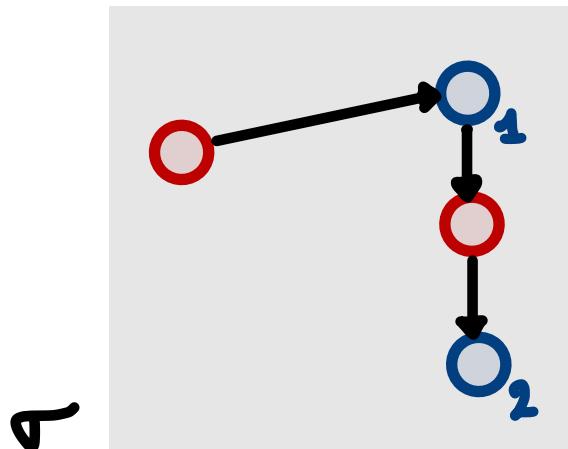
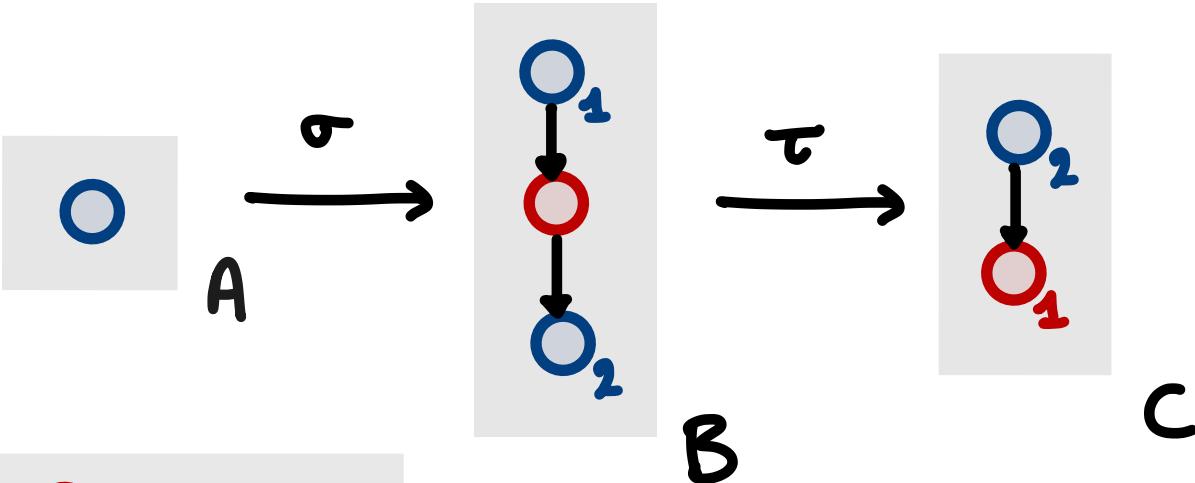


Composition of strategies

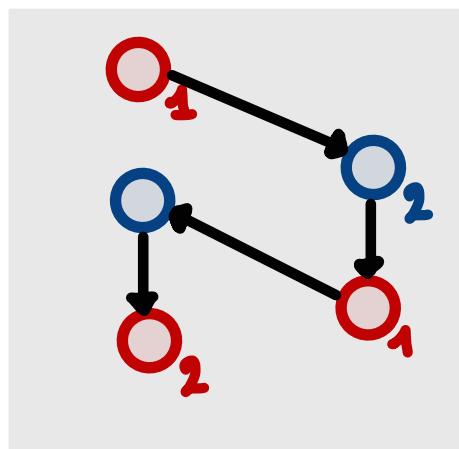


$A^L \otimes B$

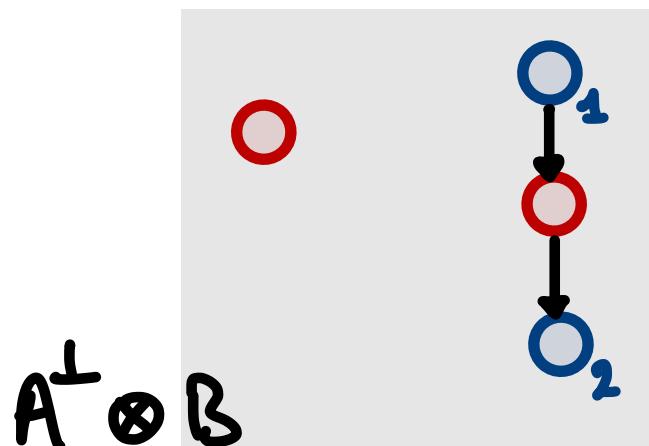
Composition of strategies



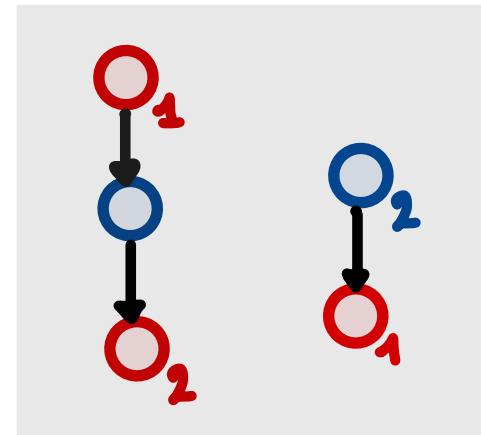
σ



τ

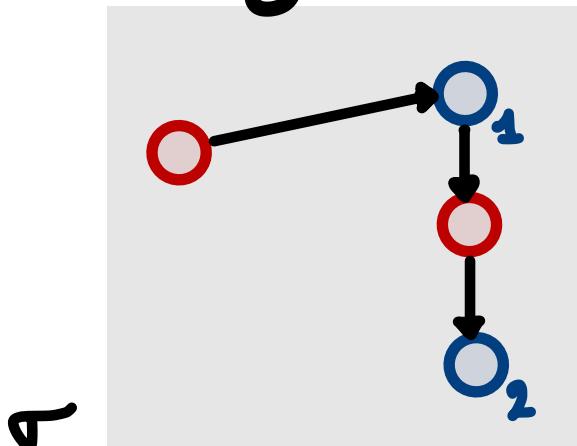


$A^\perp \otimes B$

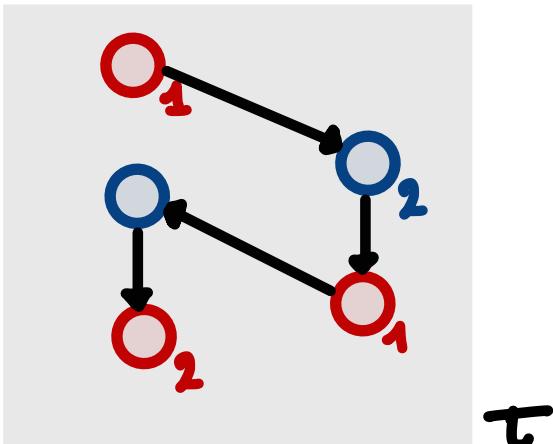


$B^\perp \otimes C$

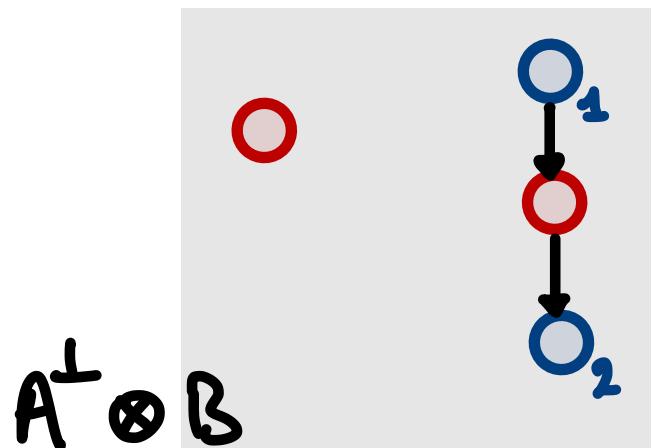
Composition of strategies



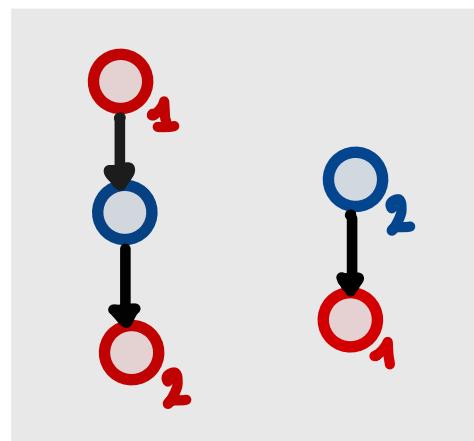
α



τ

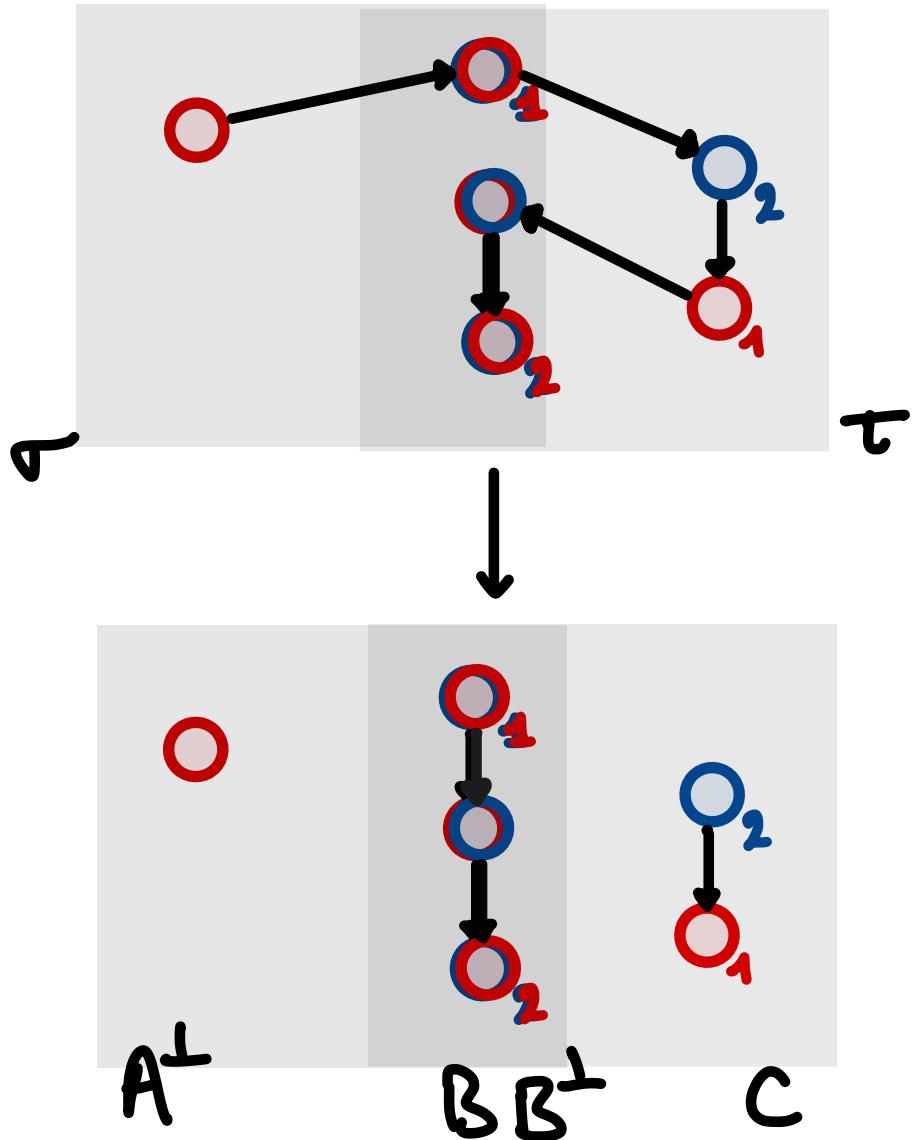


$\alpha^L \otimes \beta$

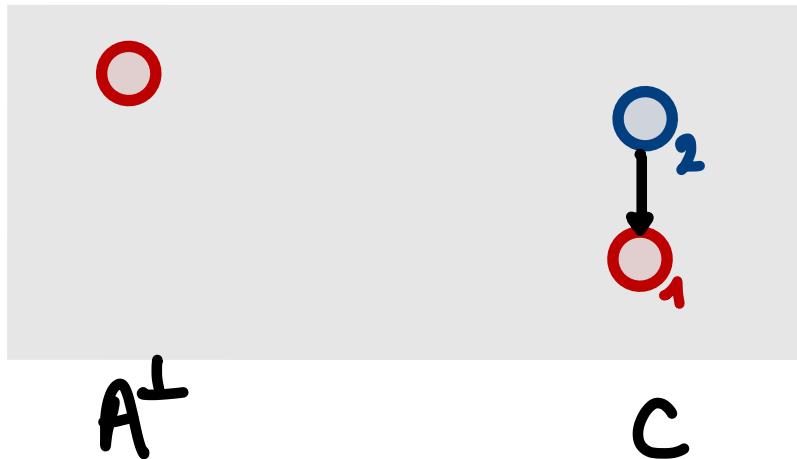


$\beta^L \otimes \gamma$

Composition of strategies



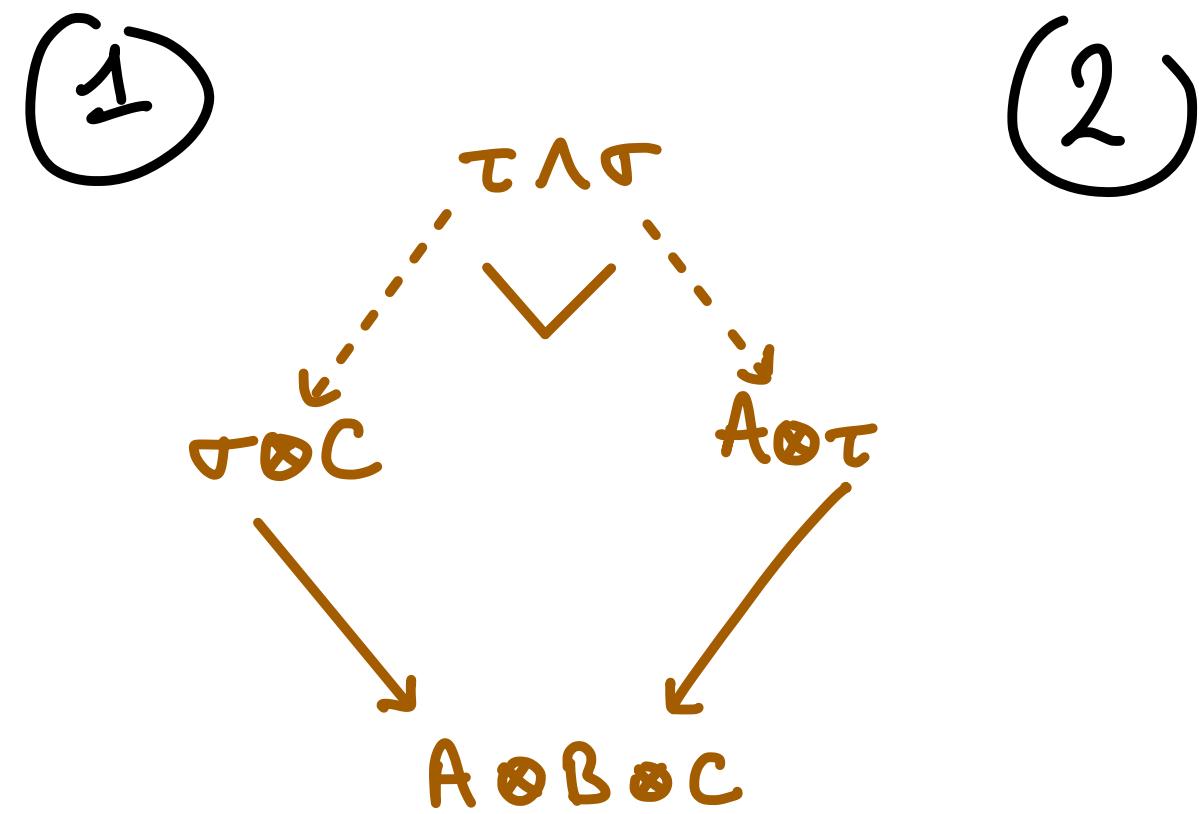
Composition of strategies



Composition of strategies

[Winskel, Clairambault,
Castellan, ...]

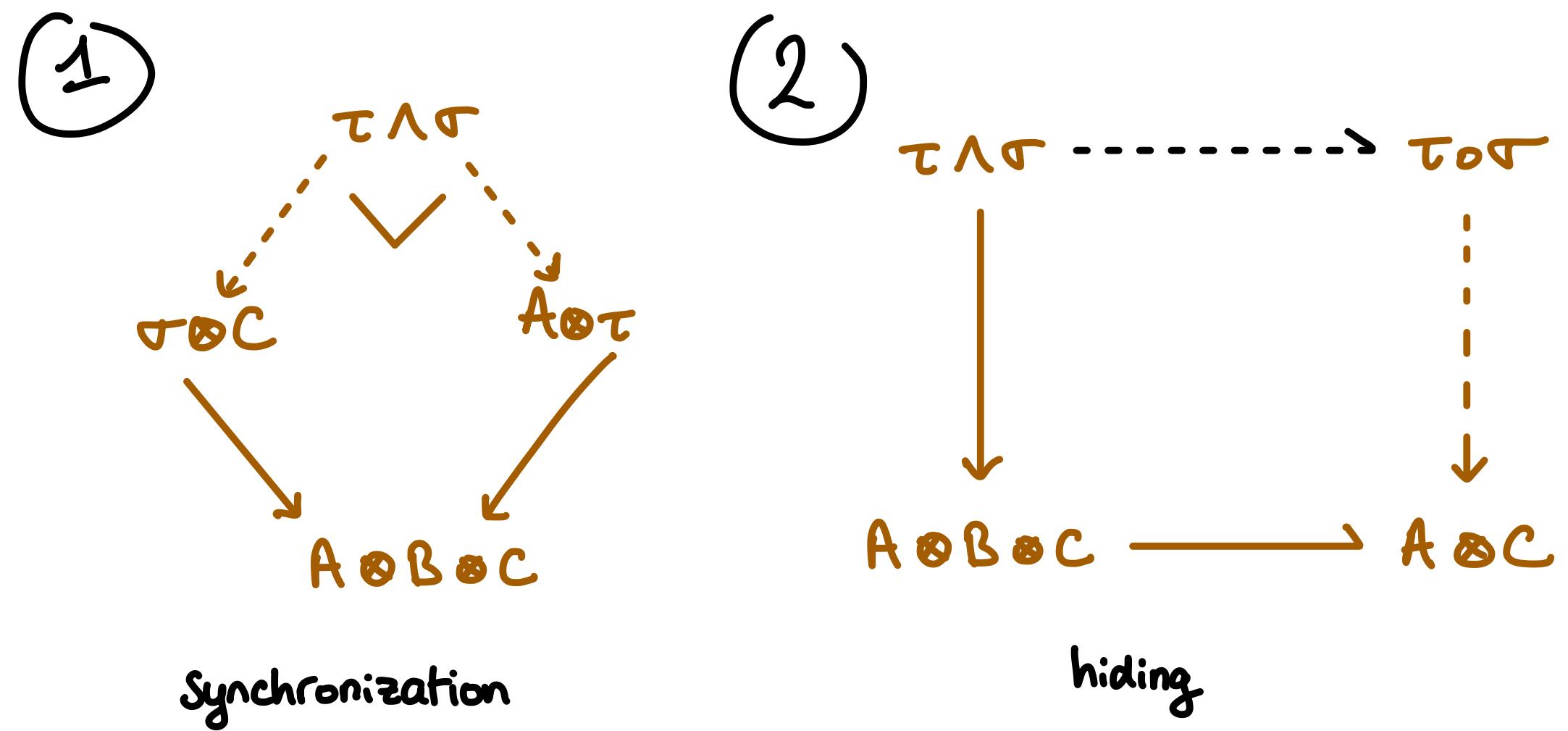
in the category of event structures:



Synchronization

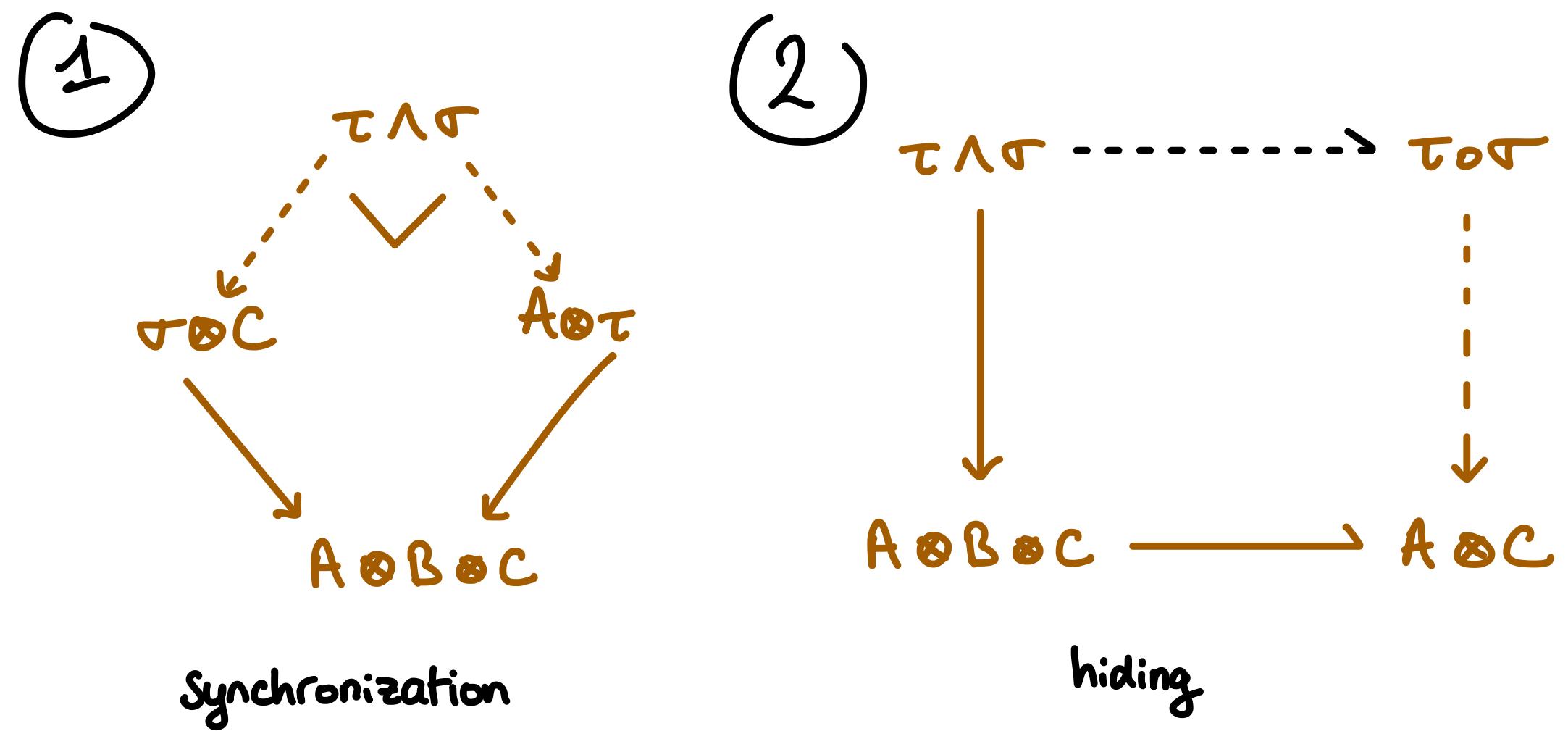
Composition of strategies

in the category of event structures:



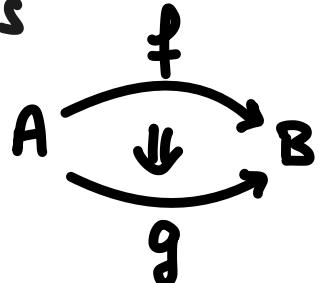
Composition of strategies (not strictly associative)

in the category of event structures:



Bicategories

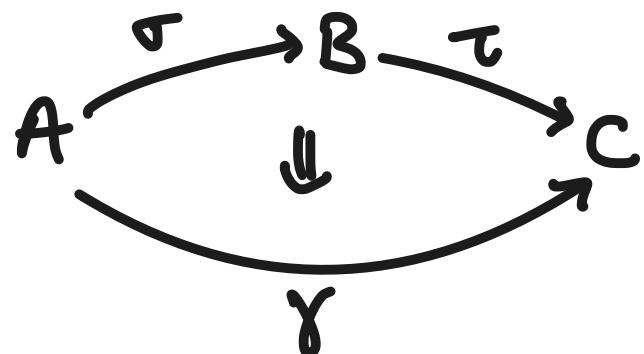
- objects A, B, \dots
- morphisms $f : A \rightarrow B, \dots$
- 2-cells
 - (with identity and composition)
- 2-cells



- associativity $\alpha_{f,g,h} : (h \circ g) \circ f \Rightarrow h \circ (g \circ f)$
- identity $r_f : f \circ \text{id} \Rightarrow f$
 $l_f : \text{id} \circ f \Rightarrow f$
- coherence axioms

Composition of strategies

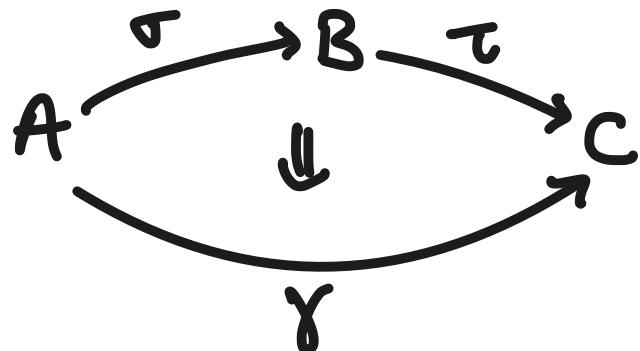
universal property of $\tau \circ \sigma$?



idea: send synchronized pair (x_σ, x_τ) to x_γ .

Composition of strategies

universal property of $\tau \circ \sigma$?



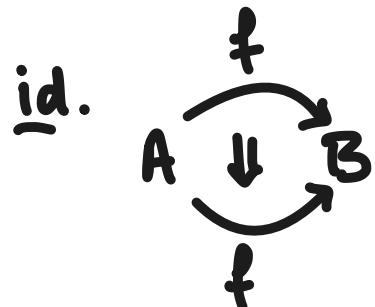
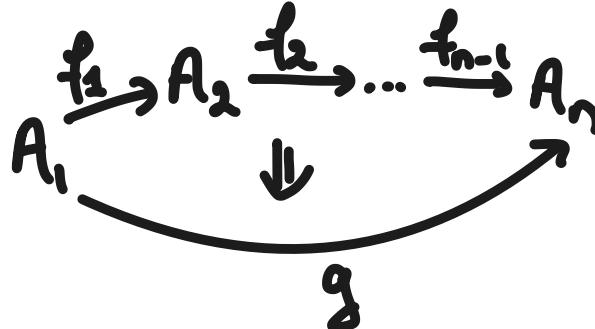
idea: send synchronized pair (x_σ, x_τ) to x_γ .

Prop: There is a universal multimap
 $\sigma, \tau \rightarrow \tau \circ \sigma$

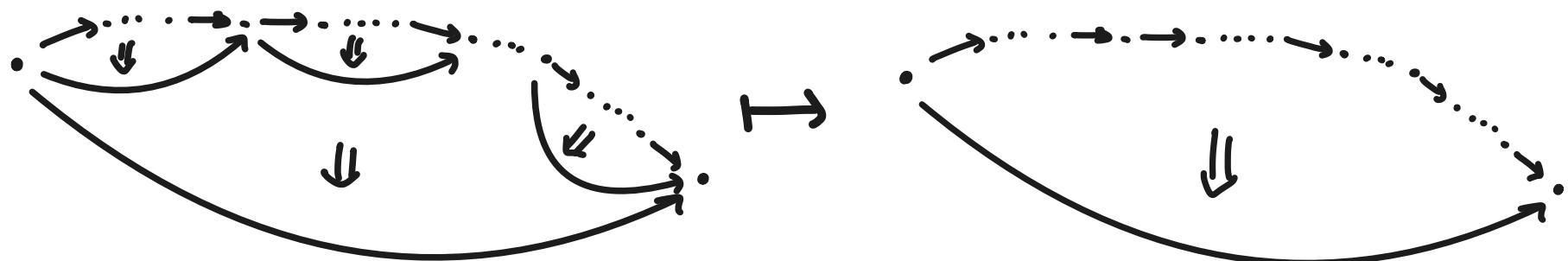
Cf. \otimes in Vect
& bilinear maps

Virtual 2-categories

- objects A, B, \dots
- morphisms $f : A \rightarrow B, \dots$ (no composition)
- multi-2-cells

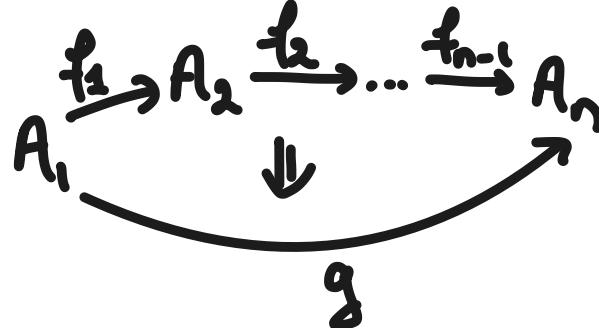


comp.

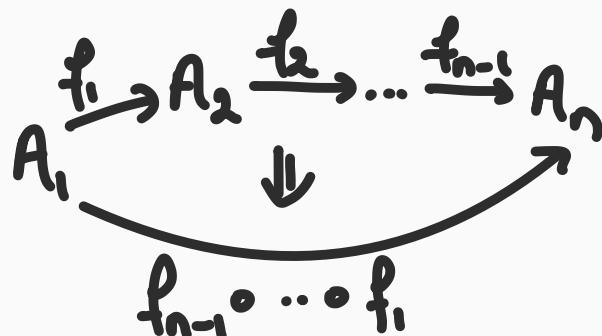


Virtual 2-categories

- objects A, B, \dots
- morphisms $f : A \rightarrow B, \dots$ (no composition)
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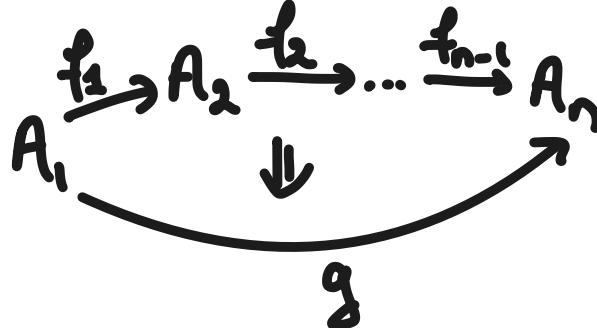


A virtual 2-cat. is **representable** if there is always a universal cell:

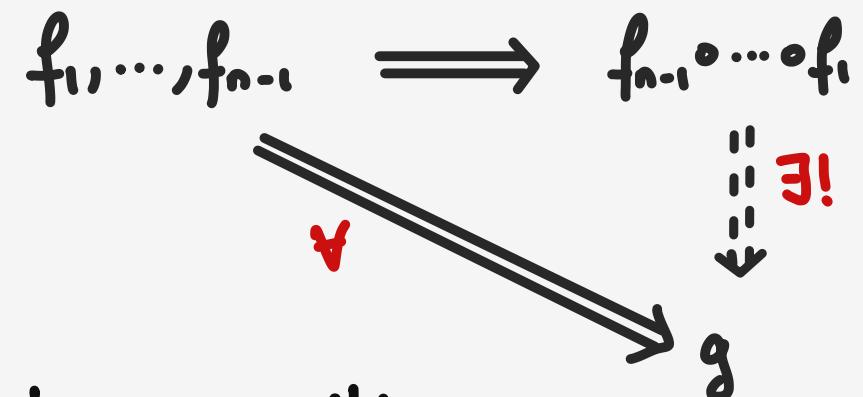
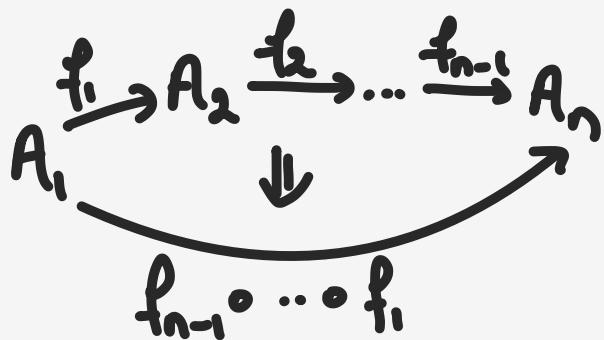


Virtual 2-categories

- objects A, B, \dots
- morphisms $f : A \rightarrow B, \dots$ (no composition)
- multi-2-cells



A virtual 2-cat. is **representable** if there is always a universal cell:



and the universal cells are closed under composition.

[Hermida 2000,
Leinster 2003]

representable
virtual
2-categories

\simeq

bicategories

↑

no structural 2-cells

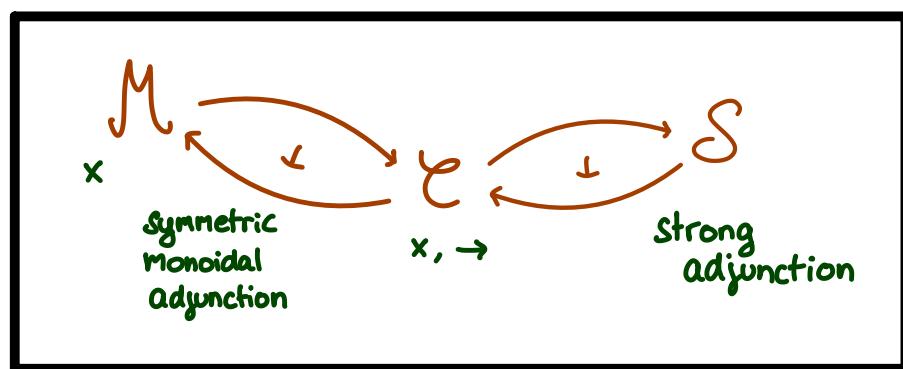
(Coherence is automatic)

Thm. The virtual 2-category of games, strategies, and multimaps is representable.

So the binary composition ToT gives a bicategory.

Thm. The virtual 2-category of games, strategies, and multimaps is representable.

So the binary composition Tot gives a bicategory.



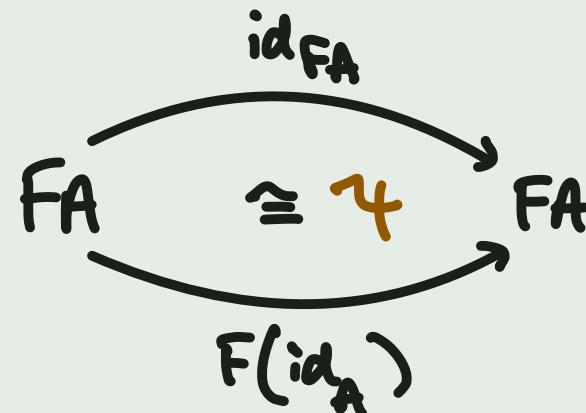
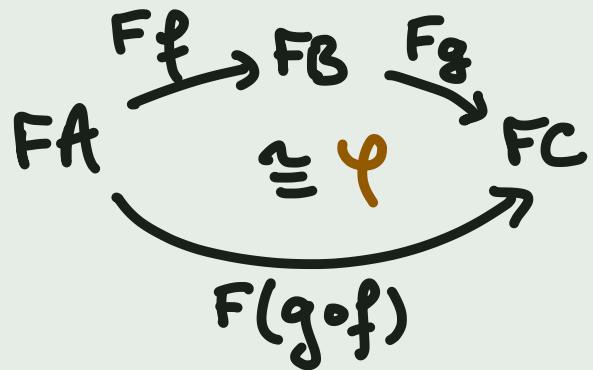
not well-understood
for bicategories.

Summary of bicategory theory

Everything holds up to coherent invertible 2-cells

pseudofunctor $F : \mathcal{B} \rightarrow \mathcal{C}$

- acts on objects, morphisms, 2-cells
- functor up to iso:



+ coherence axioms: φ, γ compatible with α, ρ, l .

Summary of bicategory theory

monoidal bicategory $(\mathcal{B}, \otimes, I)$:

$$\alpha: (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)$$

$$\lambda: I \otimes A \longrightarrow A$$

$$\rho: A \otimes I \longrightarrow A$$

}

pseudo-natural
equivalences

$$\begin{array}{ccc}
 ((AB)C)D & \xrightarrow{\alpha} & (AB)(CD) \\
 \alpha_D \downarrow & \cong & \downarrow \alpha \\
 (A(BC))D & & A(B(CD)) \\
 \alpha \downarrow & & \nearrow A\alpha \\
 & A((BC)D) &
 \end{array}$$

$$\begin{array}{ccc}
 (AI)B & \xrightarrow{\alpha} & A(I^B) \\
 \rho_B \searrow & \cong & \swarrow A\alpha \\
 & AB &
 \end{array}$$

...

- Similar notions of symmetric monoidal bicats, pseudomonads, etc.
- Main difficulty is to find the “right” axioms
- We give a definition of strong pseudomonads, axioms are justified by a correspondence theorem:

strengths \simeq actions on
the Kleisli bicat.

This talk

I. A bicategory of games

→ II. Strong pseudomonads

III. Resources and symmetries

Monads and Computation

Distinguish between : Values $A \rightarrow B$
Computations $A \rightarrow TB$

(So we can make sense of $(\lambda x.M)V = M[V/x]$ in C_{BV})

Monads and Computation

Distinguish between : Values $A \rightarrow B$
Computations $A \rightarrow TB$

(So we can make sense of $(\lambda x.M)V = M[V/x]$ in C_{BV})

Can compose computations: $\Gamma \vdash n : B$ $B \vdash N : C$

$$\Gamma \xrightarrow{M} TB \xrightarrow{T^N} T^2 C \not\xrightarrow{\mu} TC$$

Monads and Computation

Distinguish between : Values $A \rightarrow B$
Computations $A \rightarrow TB$

(So we can make sense of $(\lambda x.M)V = M[V/x]$ in C_{BV})

Can compose computations: $\Gamma \vdash n : B$ $B \vdash N : C$

$$\Gamma \xrightarrow{M} TB \xrightarrow{TN} T^2C \not\xrightarrow{\mu} TC$$

need **strength** in general: $\Gamma \vdash n : B$ $\Delta, B \vdash N : C$

$$\Delta \otimes \Gamma \xrightarrow{\Delta \otimes n} \Delta \otimes TB \xrightarrow{t} T(\Delta \otimes B) \xrightarrow{TN} T^2C \not\xrightarrow{\mu} TC$$

Definition: A strength for a pseudomonad $T: \mathcal{B} \rightarrow \mathcal{B}$ on a monoidal bicategory $(\mathcal{B}, \otimes, I)$ is a pseudo-natural transformation $t_{A,B}: A \otimes T B \rightarrow T(A \otimes B)$ equipped with 2-cells:

$$\begin{array}{ccc}
 & & (A \otimes B) \otimes T C \xrightarrow{\alpha} A \otimes (B \otimes T C) \\
 & & \downarrow t \quad \cong \quad \downarrow A t \\
 I \otimes T A \xrightarrow{t} T(I \otimes A) & & A \otimes T(B \otimes C) \\
 & \searrow \cong & \downarrow t \\
 & TA & T((A \otimes B) \otimes C) \xrightarrow{T\alpha} T(A \otimes (B \otimes C))
 \end{array}$$

$$\begin{array}{ccc}
 & & A \otimes T^2 B \xrightarrow{A\mu} A \otimes TB \\
 & & \uparrow t \\
 A \otimes B \xrightarrow{A\eta} A \otimes TB & \cong & \downarrow t \\
 & \searrow \eta & \downarrow T t \\
 & T(A \otimes B) & T^2(A \otimes B) \xrightarrow{\mu} T(A \otimes B)
 \end{array}$$

+ axioms

$$\begin{array}{ccc}
A(IT_B) & \xleftarrow{\alpha} & (AI)T_B \\
At \downarrow & A\lambda \searrow & \downarrow \rho T_B \\
AT_{IB} & \xrightarrow{AT_\lambda} & AT_B \\
t \downarrow & \cong & \downarrow t \\
T_{A(IB)} & \cong & T_{(AI)B} \\
& \searrow T_{A\lambda} & \downarrow T_{\rho B} \\
& T_{AB} &
\end{array}
=
\begin{array}{ccc}
A(IT_B) & \xleftarrow{\alpha} & (AI)T_B \\
At \downarrow & & \downarrow t \\
AT_{IB} & & y \\
t \downarrow & & \downarrow t \\
T_{A(IB)} & \xleftarrow{T_\alpha} & T_{(AI)B} \\
& \searrow T_{A\lambda} & \downarrow T_{\rho B} \\
& T_{AB} &
\end{array}$$

$$\begin{array}{ccc}
I(AT_B) & \xleftarrow{\alpha} & (IA)T_B \\
It \downarrow & \lambda \searrow & \downarrow \lambda T_B \\
IT_{AB} & \cong & AT_B \\
t \downarrow & \cong & \downarrow t \\
T_{I(AB)} & \xrightarrow{x} & T_{(IA)B} \\
& \searrow T_\lambda & \downarrow T_{\lambda B} \\
& T_{AB} &
\end{array}
=
\begin{array}{ccc}
I(AT_B) & \xleftarrow{\alpha} & (IA)T_B \\
It \downarrow & & \downarrow t \\
IT_{AB} & & y \\
t \downarrow & & \downarrow t \\
T_{I(AB)} & \xleftarrow{T_\alpha} & T_{(IA)B} \\
& \searrow T_\lambda & \downarrow T_\lambda B \\
& T_{AB} &
\end{array}$$

$$\begin{array}{c}
A((BC)T_D) \xleftarrow{\alpha} (A(BC))T_D \xleftarrow{\alpha T_D} ((AB)C)T_D \\
A(B(CT_D)) \xleftarrow{A\alpha} A(CT_D) \xleftarrow{\alpha} (AB)(CT_D) \xleftarrow{\alpha} ((AB)C)T_D \\
A(BT_{CD}) \xleftarrow{A(y)} AT_{CD} \xleftarrow{\alpha} (AB)T_{CD} \xleftarrow{(AB)t} ((AB)C)T_D \\
AT_{B(CD)} \xleftarrow{At} T_{B(CD)} \xleftarrow{y} T_{(AB)CD} \xleftarrow{T_\alpha} ((AB)C)T_D \\
T_{A(B(CD))} \xleftarrow{T_\alpha} T_{(AB)(CD)} \xleftarrow{T_\alpha} T_{((AB)C)D} \\
|| \\
\end{array}$$

$$\begin{array}{c}
A((BC)T_D) \xleftarrow{\alpha} (A(BC))T_D \xleftarrow{\alpha T_D} ((AB)C)T_D \\
A(B(CT_D)) \xleftarrow{A\alpha} A(CT_D) \xleftarrow{At} (AB)T_{CD} \xleftarrow{\alpha} ((AB)C)T_D \\
A(BT_{CD}) \xleftarrow{A(y)} AT_{CD} \xleftarrow{\alpha} (AB)T_{CD} \xleftarrow{(AB)t} ((AB)C)T_D \\
AT_{B(CD)} \xleftarrow{At} T_{B(CD)} \xleftarrow{\cong} T_{A((BC)D)} \xleftarrow{T_\alpha} ((AB)C)T_D \\
T_{A(B(CD))} \xleftarrow{T_\alpha} T_{(AB)(CD)} \xleftarrow{T_\alpha} T_{((AB)C)D} \\
|| \\
\end{array}$$

$$\begin{array}{ccc}
A(IT_B) \xleftarrow{\alpha} (AI)T_B & & A(IT_B) \xleftarrow{\alpha} (AI)T_B \\
\downarrow At \quad \downarrow A\lambda \quad \downarrow m & \cong & \downarrow At \quad \downarrow T_{AB} \\
AT_{IB} \xrightarrow{AT_\lambda} AT_B & & AT_{IB} \\
\downarrow t \quad \cong \quad \downarrow t & = & \downarrow t \\
T_{A(AB)} \cong & & T_{A(AB)} \xleftarrow{T_\alpha} T \\
\downarrow T_{A\lambda} \quad \downarrow T_{\rho B} & & \downarrow T_m \quad \downarrow T_{\rho l} \\
T_{AB} & & T_{AB}
\end{array}$$

$$\begin{array}{ccc}
I(AT_B) \xleftarrow{\alpha} (IA)T_B & & I(AT_B) \xleftarrow{\alpha} (I)T_B \\
\downarrow It \quad \downarrow \lambda \quad \downarrow \lambda T_B & & \downarrow It \quad \downarrow y \\
IT_{AB} \cong & & IT_{AB} \\
\downarrow t \quad \downarrow \lambda \quad \downarrow t & = & \downarrow t \\
T_{I(AB)} \xrightarrow{x} T_{AB} & & T_{I(AB)} \xleftarrow{T_\alpha} T \\
\downarrow T_\lambda \quad \downarrow T_{\lambda B} & & \downarrow T_l \quad \downarrow T_{\lambda l} \\
T_{AB} & & T_{AB}
\end{array}$$

$$\begin{array}{ccc}
A((BC)T_D) \xleftarrow{\alpha} (A(BC))T_D & & ((AB)C)T_D \xleftarrow{\alpha T_D} ((AB)C)T_D \\
\downarrow A\alpha \quad \downarrow p & & \downarrow \alpha \\
A(B(CT_D)) \xleftarrow{\alpha} A(BC)T_D & &
\end{array}$$

$$\begin{array}{ccc}
AT_B^2 \xleftarrow{A\eta} AT_B & & AT_B^2 \xleftarrow{A\eta} AT_B \\
\downarrow t \quad \downarrow A\mu \quad \parallel & & \downarrow t \quad \downarrow \eta \quad \parallel \\
T_{AT_B} \xrightarrow{w} AT_B & & T_{AT_B} \xrightarrow{z} AT_B \\
\downarrow T_t \quad \downarrow t & = & \downarrow T_t \quad \downarrow t \\
T_{AB}^2 \xrightarrow{\mu} T_{AB} & & T_{AB}^2 \xrightarrow{\mu} T_{AB} \\
\downarrow T_{AB}^2 \quad \downarrow \mu & & \downarrow T_{AB}^2 \quad \downarrow \mu \\
T_{I(AB)} \xrightarrow{x} T_{AB} & & T_{I(AB)} \xrightarrow{y} T_{AB} \\
\downarrow T_\lambda \quad \downarrow T_{\lambda B} & & \downarrow T_l \quad \downarrow T_{\lambda l} \\
T_{AB} & & T_{AB}
\end{array}$$

$$\begin{array}{ccc}
AT_B^3 \xrightarrow{A\mu} AT_B^2 & & AT_B^3 \xrightarrow{A\mu} AT_B^2 \\
\downarrow t \quad \downarrow A\mu \quad \parallel & & \downarrow t \quad \downarrow A\mu \quad \parallel \\
T_{AT_B^2} \xrightarrow{w} AT_B^2 & & T_{AT_B^2} \xrightarrow{w} AT_B^2 \\
\downarrow T_t \quad \downarrow t & = & \downarrow T_t \quad \downarrow t \\
T_{AB}^2 \xrightarrow{\mu} T_{AB} & & T_{AB}^2 \xrightarrow{\mu} T_{AB} \\
\downarrow T_{AB}^2 \quad \downarrow \mu & & \downarrow T_{AB}^2 \quad \downarrow \mu \\
T_{I(AB)} \xrightarrow{x} T_{AB} & & T_{I(AB)} \xrightarrow{y} T_{AB} \\
\downarrow T_\lambda \quad \downarrow T_{\lambda B} & & \downarrow T_l \quad \downarrow T_{\lambda l} \\
T_{AB} & & T_{AB}
\end{array}$$

$$\begin{array}{ccc}
IT_A \xleftarrow{I\eta} IA & & IT_A \xleftarrow{I\eta} IA \\
\downarrow t \quad \downarrow z \quad \downarrow \lambda & = & \downarrow t \quad \downarrow x \quad \downarrow \lambda \\
T_{IA} \xrightarrow{T\lambda} TA & & T_{IA} \xrightarrow{T\lambda} TA
\end{array}$$

$$\begin{array}{ccc}
T_{IT_A} \xleftarrow{t} IT_A^2 & & T_{IT_A} \xleftarrow{t} IT_A^2 \\
\downarrow T_t \quad \downarrow x \quad \downarrow \lambda & = & \downarrow T_t \quad \downarrow w \quad \downarrow I\mu \\
T_{IA}^2 \xrightarrow{\mu} T_A & & T_{IA}^2 \xrightarrow{\mu} T_A \\
\downarrow T_\lambda^2 \quad \downarrow \mu & & \downarrow T_\lambda^2 \quad \downarrow \mu \\
T_{IT_A} \xleftarrow{t} IT_A & & T_{IT_A} \xleftarrow{t} IT_A \\
\downarrow T_t \quad \downarrow x \quad \downarrow \lambda & & \downarrow T_t \quad \downarrow w \quad \downarrow I\mu \\
T_A & & T_A
\end{array}$$

$$\begin{array}{ccc}
(AB)T_C^2 \xrightarrow{(AB)\mu} (AB)T_C & & (AB)T_C^2 \xrightarrow{(AB)\mu} (AB)T_C \\
\downarrow \alpha \quad \downarrow t \quad \downarrow w & & \downarrow \alpha \quad \downarrow t \quad \downarrow w \\
A(BT_C^2) \xrightarrow{At} T_{(AB)T_C} \xrightarrow{Tt} T_{(AB)C}^2 \xrightarrow{\mu} T_{(AB)C} & & A(BT_C^2) \xrightarrow{At} T_{(AB)T_C} \xrightarrow{A(B\mu)} A(BT_C) \\
\downarrow At \quad \downarrow y \quad \downarrow T_\alpha & & \downarrow At \quad \downarrow Aw \quad \downarrow T_\alpha \\
AT_{BT_C} \xrightarrow{-t} T_{A(BT_C)} \xrightarrow{T_y} T_{AB}^2 \xrightarrow{\mu} T_{AB} & & AT_{BT_C} \xrightarrow{At} AT_{BC} \\
\downarrow AT_t \quad \cong \quad \downarrow T_{At} & & \downarrow AT_t \quad \downarrow A\mu \quad \downarrow T_\alpha \\
AT_{BC}^2 \xrightarrow{t} T_{AT_{BC}} \xrightarrow{T_t} T_{A(BC)}^2 \xrightarrow{\mu} T_{A(BC)} & & AT_{BC}^2 \xrightarrow{t} T_{AT_{BC}} \xrightarrow{T_t} T_{A(BC)}^2 \xrightarrow{\mu} T_{A(BC)}
\end{array}$$

$$\begin{array}{ccc}
(AB)T_C \xleftarrow{(AB)\eta} (AB)C & & (AB)T_C \xleftarrow{(AB)\eta} (AB)C \\
\downarrow \alpha \quad \cong \quad \downarrow \alpha & & \downarrow \alpha \quad \cong \quad \downarrow \alpha \\
A(BT_C) \xleftarrow{A(B\eta)} A(BC) & & A(BT_C) \xleftarrow{A(B\eta)} A(BC) \\
\downarrow At \quad \downarrow Az \quad \downarrow A\eta & = & \downarrow At \quad \downarrow y \quad \downarrow T_\alpha \\
AT_{BC} \xrightarrow{t} T_{A(BC)} & & AT_{BC} \xrightarrow{t} T_{A(BC)}
\end{array}$$

Another view : strengths as actions of (\mathcal{B}, \otimes) on \mathcal{B}_T

Another view : strengths as actions of (\mathcal{B}, \otimes) on \mathcal{B}_T

$$\begin{array}{ccc} \mathcal{B} \times \mathcal{B}_T & \longrightarrow & \mathcal{B}_T \\ A, B & \longmapsto & A \otimes B \end{array}$$

Another view : strengths as actions of (\mathfrak{F}, \otimes) on \mathfrak{F}_T

$$\begin{array}{ccc} \mathfrak{F} \times \mathfrak{F}_T & \longrightarrow & \mathfrak{F}_T \\ A, B & \longmapsto & A \otimes B \\ \downarrow v & & \downarrow v \otimes M \\ A' & & T B' \\ & & \downarrow t \\ & & T(A' \otimes B') \end{array}$$

Another view : strengths as actions of (\mathcal{B}, \otimes) on \mathcal{B}_T

$$\begin{array}{ccc}
 \mathcal{B} \times \mathcal{B}_T & \longrightarrow & \mathcal{B}_T \\
 A, B & \longmapsto & A \otimes B \\
 \downarrow v & & \downarrow v \otimes M \\
 A' & & A' \otimes T B' \\
 & \downarrow t & \\
 & & T(A' \otimes B')
 \end{array}$$

Theorem:

strengths
for T

\approx

actions
 $*: \mathcal{B} \times \mathcal{B}_T \rightarrow \mathcal{B}_T$

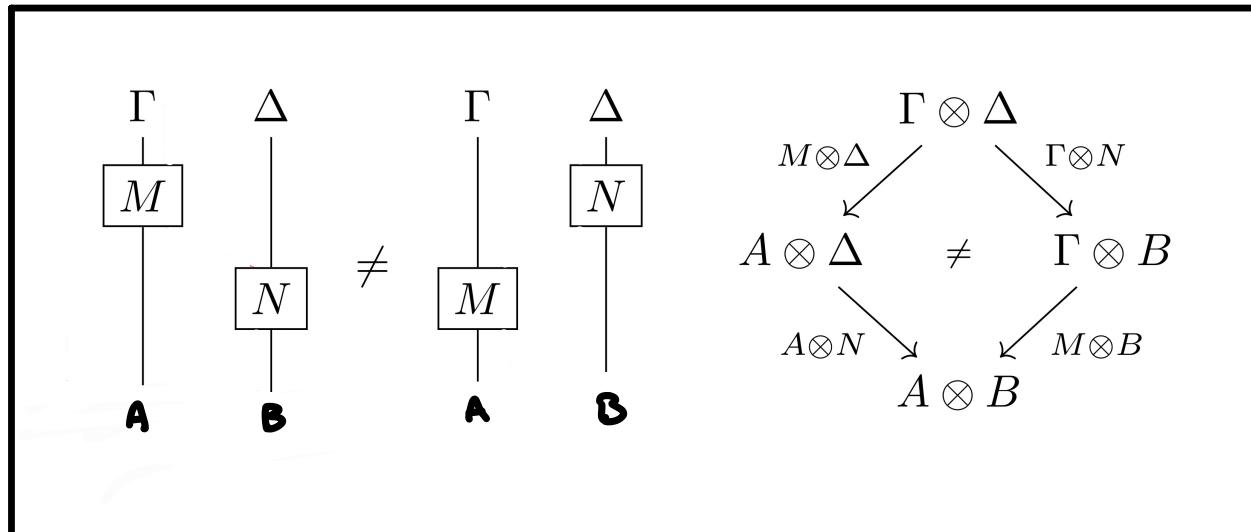
s.t.

$$\begin{array}{ccc}
 \mathcal{B} \times \mathcal{B}_T & \xrightarrow{*} & \mathcal{B}_T \\
 \uparrow \mathcal{B} \times \mathcal{B} & \cong & \uparrow \mathcal{B} \\
 & \otimes &
 \end{array}$$

Premonoidal bicategories

premonoidal cats.
[Power & Robinson]
97

\mathcal{B}_+ has a tensor product, but no interchange law:

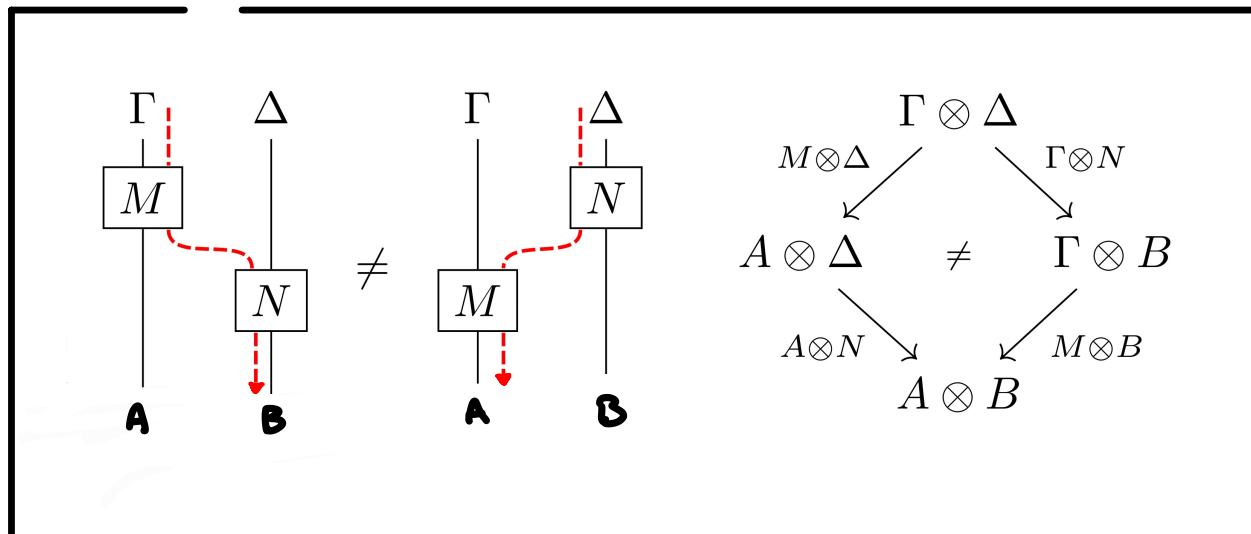


$\Gamma \vdash M : A$
 $\Delta \vdash N : B$

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97

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Premonoidal bicategories

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A premonoidal bicategory K has

- $\otimes A$ and $A \otimes -$ for all $A \in K$
- + the coherence data for a monoidal bicategory.

Premonoidal bicategories

premonoidal cats.
[Power & Robinson]
97

A premonoidal bicategory K has

- $\otimes A$ and $A \otimes -$ for all $A \in K$
- + the coherence data for a monoidal bicategory.

Prop: (\mathcal{B}, \otimes) symmetric monoidal bicategory

T strong pseudomonad

$\Rightarrow \mathcal{B}_T$ is premonoidal.

$$\mathcal{B} \longrightarrow \mathcal{B}_T$$

values

computations

Dialogue bicategories

(tensor & negation)

Dialogue categories
[Mellies & Tabareau]

A dialogue bicategory is

a symmetric monoidal \mathcal{B}

an object \perp

such that $\mathcal{B}(A \otimes B, \perp) \simeq \mathcal{B}(A, \neg B)$ for some $\neg B$.

Dialogue bicategories

(tensor & negation)

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[Mellies & Tabareau]

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such that $\mathcal{B}(A \otimes B, \perp) \simeq \mathcal{B}(A, \neg B)$ for some $\neg B$.

Sym. monoidal
closed, \perp
 $\neg A = A \multimap \perp$

\subseteq

Dialogue

Dialogue bicategories

(tensor & negation)

Dialogue categories
[Mellies & Tabareau]

A dialogue bicategory is

a symmetric monoidal \mathcal{B}

an object \perp

such that $\mathcal{B}(A \otimes B, \perp) \simeq \mathcal{B}(A, \neg B)$ for some $\neg B$.

*-autonomous

$$(A \multimap \perp) \multimap \perp \cong A$$

\subseteq

Sym. monoidal
closed, \perp
 $\neg A = A \multimap \perp$

\subseteq

Dialogue

Dialogue bicategories

(tensor & negation)

Dialogue categories
[Mellies & Tabareau]

A dialogue bicategory is

a symmetric monoidal \mathcal{B}

an object \perp

such that $\mathcal{B}(A \otimes B, \perp) \simeq \mathcal{B}(A, \neg B)$ for some $\neg B$.

Key properties:

- \neg is a pseudofunctor $\mathcal{B} \rightarrow \mathcal{B}^{\text{op}}$
- $\neg\neg$ is a strong pseudomonad on \mathcal{B}
- $\mathcal{B}_{\neg\neg}$ is premonoidal

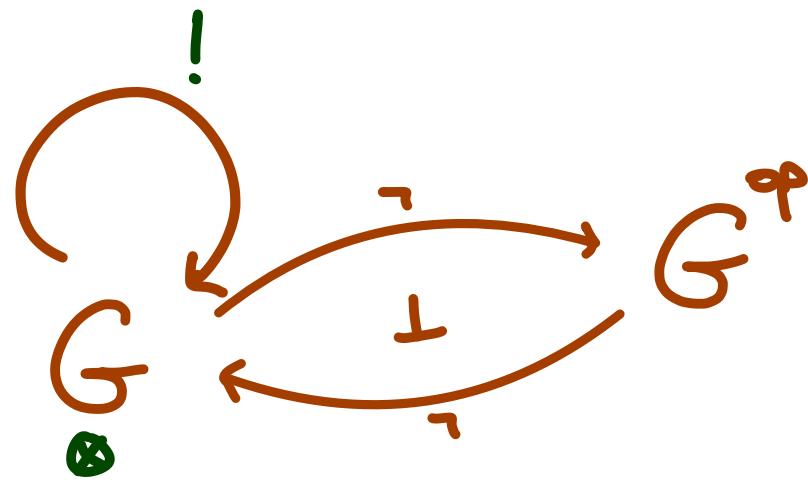
This talk

I. A bicategory of games

II. Strong pseudomonads

→ III. Resources and symmetries

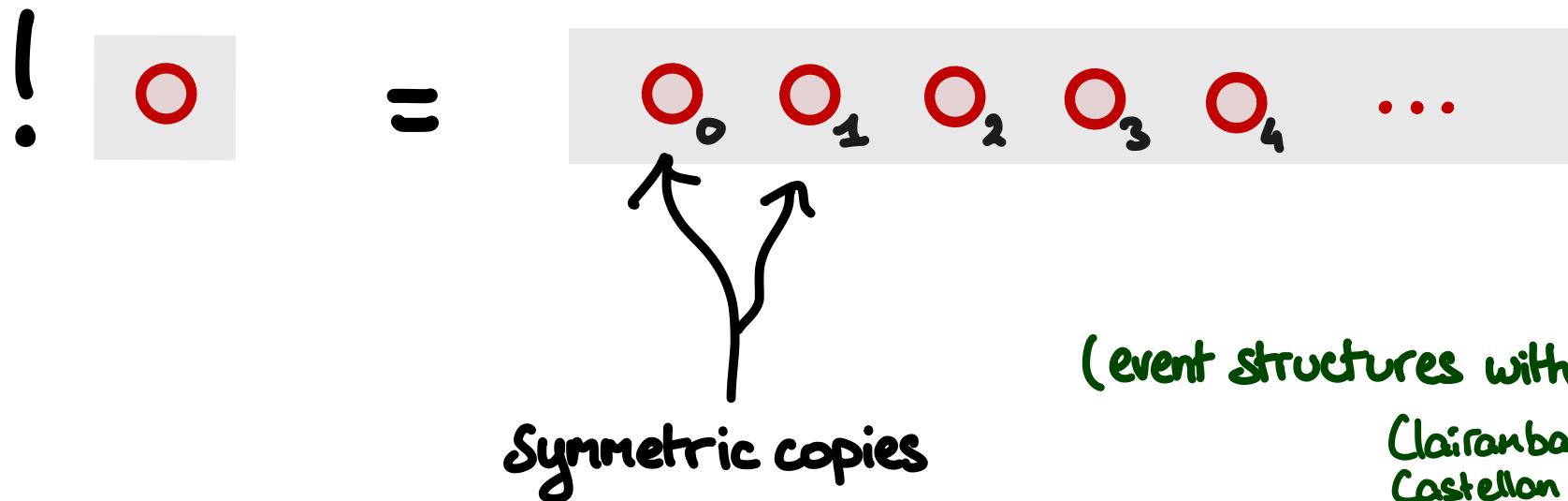
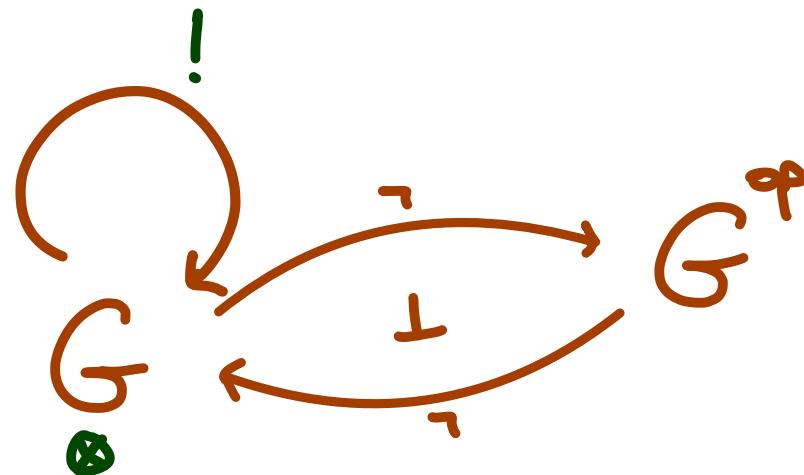
Adding an exponential modality :



$$! \circ = \begin{array}{ccccccc} \circ_0 & \circ_1 & \circ_2 & \circ_3 & \circ_4 & \dots \end{array}$$

The symbol \circ is enclosed in a gray square. The equals sign $=$ is positioned between the \circ and the sequence of circles. Below the sequence, a black wavy line points upwards towards the first circle, with the text "Symmetric copies" written below it.

Adding an exponential modality :

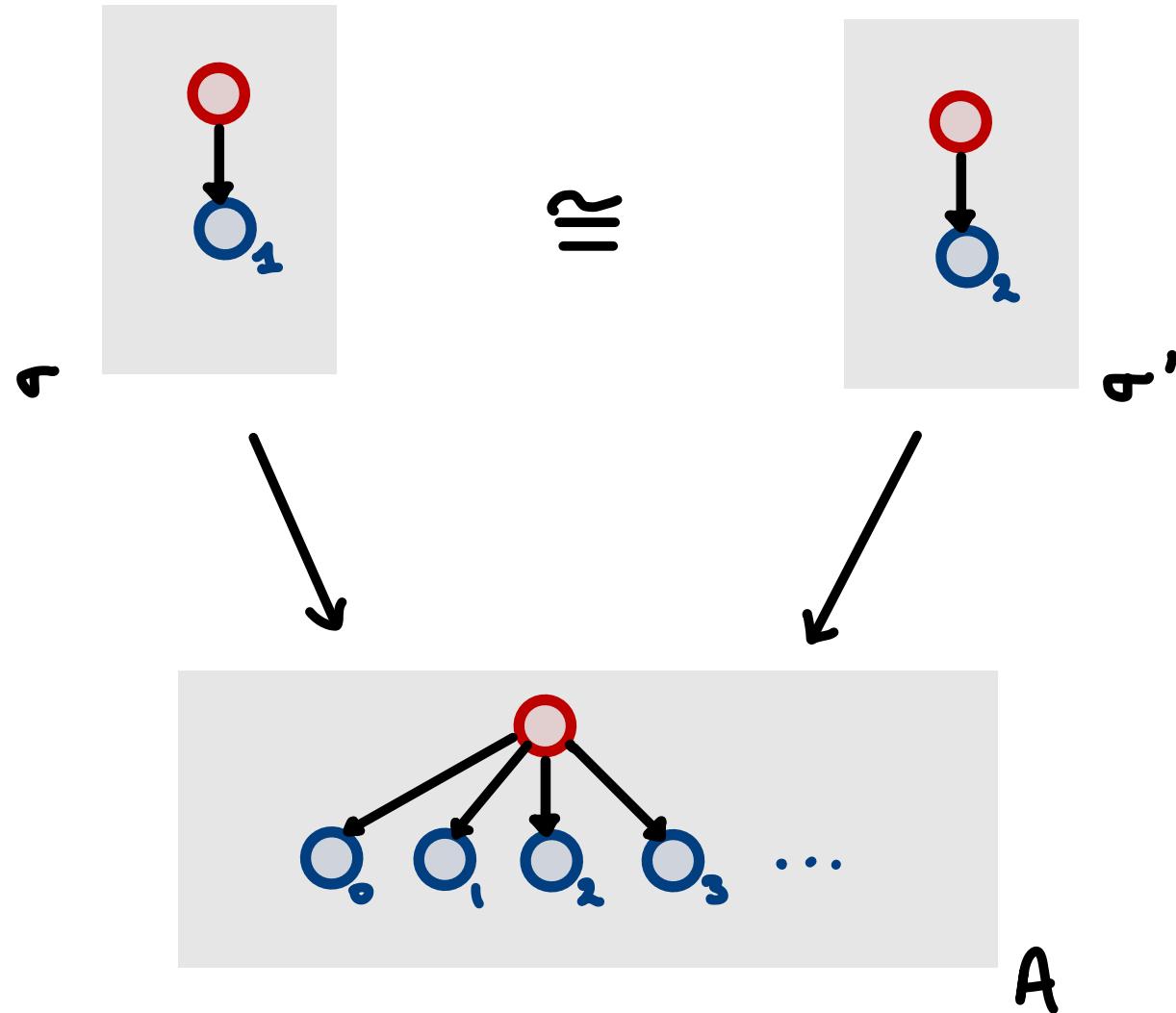


Symmetric copies

(event structures with symmetry)

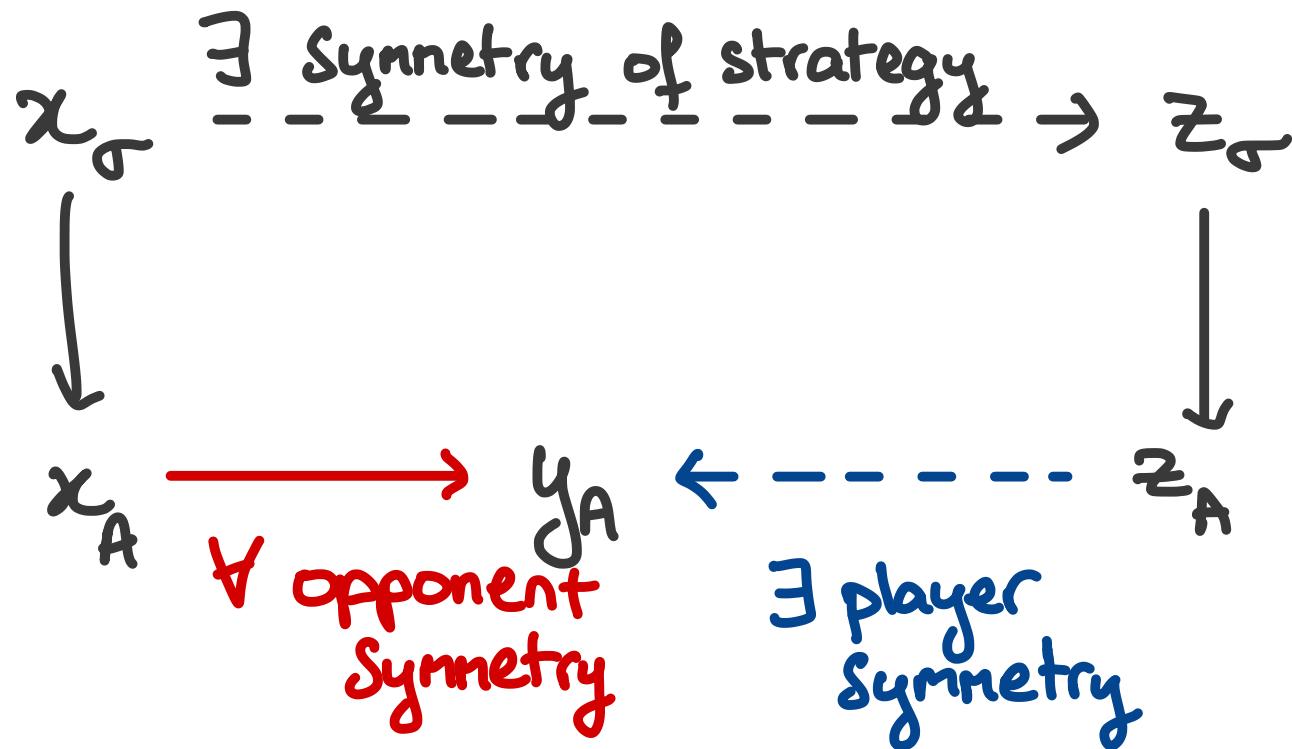
Clairambault,
Castellon,
Winskel

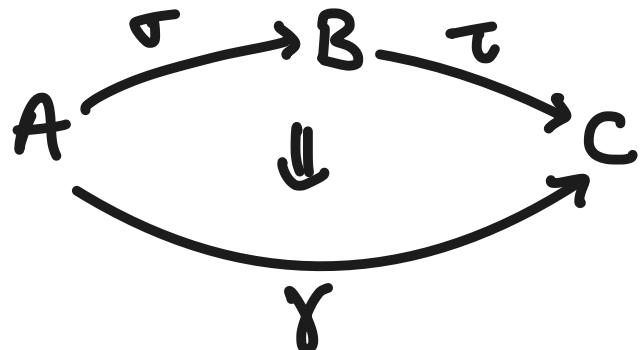
Strategies playing Symmetrically :



Strategies need to be bi-invariant:

Strategies need to be bi-invariant:





idea: send synchronized pair (x_σ, x_τ) up to symmetry to x_γ .

Thm. The virtual 2-category of games
with symmetry, strategies,
 and multimaps is representable.

Summary

- Generalize the foundations of game semantics from categories to bicategories
- 2-dimensional setting:
“proof-relevant” all the time
- Other applications of premonoidal bicategories:
graded monads, “PARA” construction

