Call-by-value in Bicategories of Games

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joint work with Philip Saville
Resources and effects

Models of Linear Logic

! - \otimes, \oplus

exponential Comonad
Resources and effects

Models of Linear Logic

\[ ! \]

Exponential Comonad

\[ \otimes, \circ \]

Linear/non-linear adjunction

\[ \otimes, \circ \]
Resources and effects

Models of effectful languages

Strong monad

\( T \)

\( X \Rightarrow \)
Resources and effects

Models of effectful languages

Strong monad

Strong adjunction
Resources and effects

Models of effectful languages

\[(CBV)\] (strong \text{monad})

\[(CBPV)\] (strong \text{adjunction})
Resources and effects

- Can combine the two viewpoints ("Linear CBPV")

\[
M \uparrow \quad S \downarrow
\]

linear/non-linear adjunction

Strong adjunction

- Several canonical examples.
Resources and effects

- Can combine the two viewpoints ("Linear CBPV")
- e.g.

\[ M \xrightarrow{\times} C \xleftarrow{\otimes} C^\text{op} \]

linear/non-linear adjunction
Resources and effects

- Can combine the two viewpoints ("Linear CBPV")
- e.g.

\[ \text{linear/non-linear adjunction} \]
Resources and effects

Game Semantics:

\[
\begin{align*}
  & M \\
  & \downarrow \\
  & \times \\
  & \downarrow \\
  & e \\
  & \downarrow \\
  & \circ^0
\end{align*}
\]
Resources and effects

Game Semantics:

\[ M \rightarrow E \rightarrow M \]

\[ \times \rightarrow \times \rightarrow \]
Game Semantics

Two players

- **player**
- **opponent**

A simple game:
Game Semantics

Two players

- player
- opponent

A simple game:

Strategies over the game:
Game Semantics

Two players

- Player
- Opponent

A simple game:

Strategies over the game:
Game Semantics

Two players

- ○ player
- ○ opponent

A simple game:

A

Strategies over the game:

\begin{align*}
&\bigcirc_1 & \bigcirc_1 \\
&\bigcirc_2 & \bigcirc_2 \\
\end{align*}

\begin{align*}
&\bigcirc_4 \\
&\bigcirc_2 \\
\end{align*}

Use event structures [Winskel, Clairambault, Castellan, ...]
Strategies $\mathbb{N} \rightarrow \mathbb{N}$

Distinguish between values & computations to make sense of $(\forall x. M) V = M[V/x]$ in CBV
Strategies $\mathbb{N} \rightarrow \mathbb{N}$

Distinguish between values & computations to make sense of $(\forall x.M)V = M[V/x]$ in CBV
Strategies $\mathbb{N} \rightarrow \mathbb{N}$

Distinguish between values & computations to make sense of $(\forall x. M)\, V = M[\forall x]$ in CBV

*arbitrary* strategy

```
N ----> N
```

*“value”* strategy

```
N ----> N
```

(total, deterministic)
Strategies \( N \rightarrow N \)
Strategies \( N \rightarrow N \)

\[
\begin{array}{c}
\text{Prop:} \\
\text{strategy } A \rightarrow B \\
\text{value strategy } A \rightarrow \omega \omega B
\end{array}
\]
This talk

I. A bicategory of games

II. Strong pseudomonads

III. Resources and symmetries
Objects: games $A, B, C, \ldots$

Morphisms $A \rightarrow B$: Strategies

$\sigma$

$A \otimes B$
Composition of strategies

\[ A \xrightarrow{\sigma} \xrightarrow{\tau} \]

\[ \text{Composition of strategies} \]
Composition of strategies

\( A \rightarrow \sigma \rightarrow B \rightarrow \tau \rightarrow C \)

\( A \mathbin{\otimes} B \)
Composition of strategies

At

$A \circ \sigma$ $\rightarrow$ $A$

$B \circ \tau$ $\rightarrow$ $B$

$B \circ C$

$A^\perp \circ B$

$\sigma$

$\tau$
Composition of strategies
Composition of strategies
Composition of strategies
Composition of strategies

in the category of event structures:

1. \( \tau \land \sigma \)

2. \( \sigma \circ C \)

3. \( A \circ \tau \)

4. \( A \circ B \circ C \)

Synchronization
Composition of strategies

in the category of event structures:

1. Synchronization

2. Hiding
Composition of strategies (not strictly associative)
in the category of event structures:

(1) $\tau \land \sigma$

\[ \sigma \circ C \quad \tau \circ \theta \tau \quad \vartheta \circ \sigma \circ C \]

synchronization

(2) $\tau \land \sigma$

\[ \tau \circ \sigma \quad \vartheta \circ \sigma \circ C \]

hiding
**Bicategories**

- objects $A, B, ...$
- morphisms $f: A \rightarrow B, ...$ (with identity and composition)
- 2-cells

```
\[ A \xrightarrow{f} \downarrow \xleftarrow{g} B \]
```

- associativity
  \[ a_{f,g,h}: (h \circ g) \circ f \Rightarrow h \circ (g \circ f) \]

- identity
  \[ r_f: f \circ \text{id} \Rightarrow f \]
  \[ l_f: \text{id} \circ f \Rightarrow f \]

- coherence axioms
Composition of strategies

universal property of $\tau \circ \sigma$?

\[ \begin{array}{ccc}
A & \xrightarrow{\sigma} & B \\
\downarrow & & \downarrow \tau \\
\gamma & \rightarrow & C \\
\end{array} \]

idea: send synchronized pair $(x_\sigma, x_C)$ to $x_\gamma$. 

Composition of strategies

universal property of $\tau \circ \sigma$?

![Diagram]

idea: send synchronized pair $(x_\sigma, x_\tau)$ to $x_\gamma$.

Prop: There is a universal multimap

$$\sigma, \tau \rightarrow \tau \circ \sigma$$

$c$ \& bilinear maps
Virtual 2-categories

- objects $A, B, ...$
- morphisms $f: A \rightarrow B, ...$ (no composition)
- multi-2-cells $f_1:A_1 \rightarrow A_2 \rightarrow \ldots \rightarrow A_n$
Virtual 2-categories

- objects $A, B, \ldots$
- morphisms $f: A \to B, \ldots$ (no composition)
- multi-2-cells

$$\begin{array}{c}
f_1 \Rightarrow A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_n \\
A_1 \downarrow \quad \Downarrow g \quad \downarrow \\
A_1 \end{array}$$

A virtual 2-cat. is representable if there is always a universal cell:

$$\begin{array}{c}
f_1 \Rightarrow A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_n \\
A_1 \downarrow \quad \Downarrow f_1 \\
A_1 \end{array}$$
Virtual 2-categories

- objects $A, B, ...$
- morphisms $f: A \rightarrow B, ...$ (no composition)
- multi-2-cells $f_1: A_1 \rightarrow A_2 \rightarrow ... \rightarrow A_n$

A virtual 2-cat. is **representable** if there is always a universal cell:

and the universal cells are closed under composition.
representable virtual 2-categories $\simeq$ bicategories

↑

no structural 2-cells

(Coherence is automatic)
Thm. The virtual 2-category of games, strategies, and multimaps is representable.

So the binary composition \( \tau \circ \tau \) gives a bicategory.
Thm. The virtual 2-category of games, strategies, and multimaps is representable.

So the binary composition $\circ \circ$ gives a bicategory.

---

Not well-understood for bicategories.
Summary of bicategory theory

Everything holds up to coherent invertible 2-cells

**pseudofunctor** $F : B \to C$

- acts on objects, morphisms, 2-cells
- functor up to iso:
  
  $F(f) \Rightarrow F(g)$
  $F(g \circ f) \Rightarrow F(g) \circ F(f)$
  $F(id_a) \Rightarrow id_{F(a)}$

+ coherence axioms: $\Psi, \Psi$ compatible with $\alpha, \phi, \rho, \lambda$. 
Summary of bicategory theory

monoidal bicategory $(B, \otimes, I)$:

\[ \alpha: (A \otimes B) \otimes C \rightarrow A \otimes (B \otimes C) \]
\[ \rho: A \otimes I \rightarrow A \]
\[ \tau: I \otimes A \rightarrow A \]

\[ ((AB)C)D \rightarrow (AB)(CD) \]
\[ (A(BC))D \rightarrow A(B(CD)) \]

\[ (A\otimes B) \rightarrow A(\otimes B) \]

\[ \alpha \]

\[ \rho \]

\[ \tau \]

\[ \Rightarrow \]

\[ \Rightarrow \]

\[ \Rightarrow \]

\[ \Rightarrow \]

\[ \Rightarrow \]

\[ \Rightarrow \]
• Similar notions of symmetric monoidal bicats, pseudomonads, etc.

• Main difficulty is to find the “right” axioms

• We give a definition of strong pseudomonads, axioms are justified by a correspondence theorem:

  strengths $\cong$ actions on the Kleisli bicat.
This talk

I. A bicategory of games

II. Strong pseudomonads

III. Resources and symmetries
Monads and computation

Distinguish between:

Values: $A \rightarrow B$

Computations: $A \rightarrow TB$

(If we can make sense of $(\lambda x. M) V = M[\nu x] \text{ in CBV}$)
Monads and computation

Distinguish between:
- Values: $A \rightarrow B$
- Computations: $A \rightarrow TB$

(So we can make sense of $(\lambda x. M) V = M[V/x]$ in CBV)

Can compose computations:

\[
\Gamma \vdash M : B \quad B \vdash N : C
\]

\[
\Gamma \vdash M \rightarrow TB \quad B \vdash N \rightarrow TC
\]

\[
\Gamma \vdash TB \xrightarrow{TN} TC \xrightarrow{\alpha} TC
\]
Monads and computation

Distinguish between:
- Values $A \to B$
- Computations $A \to TB$

(So we can make sense of $(\forall x. M) V = M[V/x]$ in CBV)

Can compose computations:
\[
\Gamma \vdash M : B \quad B \vdash N : C
\]
\[
\begin{array}{c}
\Gamma \\ M \\
\xrightarrow{T B} \\
\xrightarrow{T N} \\
\xrightarrow{T^2 C} \\
\xrightarrow{\mu} \\
\xrightarrow{T C}
\end{array}
\]

need strength in general:
\[
\Delta \otimes \Gamma \ construed \Rightarrow \Delta \otimes TB \\
\xrightarrow{t} T(\Delta \otimes B) \\
\xrightarrow{T N} T^2 C \\
\xrightarrow{\mu} T C
\]
**Definition:** A strength for a pseudomonad $T : B \to B$ on a monoidal bicategory $(B, \otimes, I)$ is a pseudo-natural transformation $t_{A,B} : A \otimes TB \longrightarrow T(A \otimes B)$ equipped with 2-cells:

\[
\begin{align*}
I \otimes TA & \xrightarrow{t} T(I \otimes A) \\
& \cong (A \otimes TA) \\
& \xrightarrow{\lambda} TA \\
A \otimes B & \xrightarrow{\eta} A \otimes TB \\
& \cong T(A \otimes B)
\end{align*}
\]

\[
\begin{align*}
(A \otimes B) \otimes TC & \xrightarrow{\alpha} A \otimes (B \otimes TC) \\
& \cong T((A \otimes B) \otimes C) \\
& \xrightarrow{T(\alpha)} T(A \otimes (B \otimes C)) \\
T((A \otimes B) \otimes C) & \cong T(A \otimes (B \otimes C)) \\
& \xrightarrow{Tt} T(A \otimes (B \otimes C)) \\
T(A \otimes (B \otimes C)) & \cong T(A \otimes (B \otimes C)) \\
& \xrightarrow{T^2(B \otimes C)} T(A \otimes (B \otimes C))
\end{align*}
\]

\[+ \text{ axioms}\]
Another view: strengths as actions of $(\mathfrak{g}, \mathfrak{h})$ on $\mathfrak{h}$. 
Another view: strengths as actions of $(\mathcal{B}, \otimes)$ on $\mathcal{B}_T$

\[\mathcal{B} \times \mathcal{B}_T \rightarrow \mathcal{B}_T\]

\[A, B \rightarrow A \otimes B\]
Another view: strengths as actions of $(B, \emptyset)$ on $B_T$.
Another view: strengths as actions of \((B, \otimes)\) on \(B_T\)

\[
\begin{align*}
B \times B_T & \longrightarrow B_T \\
A, B \quad & \mapsto A \otimes B \\
A' \quad & \mapsto A' \otimes B'
\end{align*}
\]

Theorem:

\[
\text{strengths for } T \quad \Rightarrow \quad \text{actions \(*:* B \times B_T \longrightarrow B_T\)}
\]

s.t.

\[
\begin{align*}
B \times B_T & \rightarrow B_T \\
\uparrow \equiv \uparrow & \\
B \times B & \rightarrow B
\end{align*}
\]
Premonoidal bicategories

$B_T$ has a tensor product, but no interchange law:

$\Gamma \oplus \Delta : A$  
$\Delta \oplus \Gamma : B$
Premonoidal bicategories

$\mathcal{B}_T$ has a tensor product, but no interchange law:

$\Gamma + M \colon A$

$\Delta + N \colon B$
Premonoidal bicategories

A premonoidal bicategory $K$ has $\otimes A$ and $A \otimes -$ for all $A \in K$ + the coherence data for a monoidal bicategory.
Premonoidal bicategories

A premonoidal bicategory $K$ has

1. $A \otimes A$ and $A \otimes -$ for all $A \in K$
2. the coherence data for a monoidal bicategory.

Prop: $(B, \otimes)$ symmetric monoidal bicategory

$T$ strong pseudomonad

$\Rightarrow B_\tau$ is premonoidal.

$B \rightarrow B_\tau$

values Computations
A dialogue bicategory is a symmetric monoidal 2-

an object 1

such that \( B(A \otimes B, 1) = B(A, \neg B) \) for some \( \neg B \).
Dialogue bicategories
(tensor & negation)

A dialogue bicategory is

a symmetric monoidal $\mathcal{B}$
on an object $\bot$

such that $\mathcal{B}(A \otimes B, \bot) = \mathcal{B}(A, \neg B)$ for some $\neg B$. 

$\Rightarrow$ Synm. monoidal closed, $\bot$

$\neg A = A \rightarrow \bot$

$\subseteq$ Dialogue
Dialogue bicategories
(tensor & negation)

A dialogue bicategory is
a symmetric monoidal \( \mathcal{B} \)
an object \( \bot \)
such that \( \mathcal{B}(A \otimes B, \bot) \cong \mathcal{B}(A, \neg B) \) for some \( \neg B \).

\[ \begin{array}{ccc}
\ast\text{-autonomous} & \cong & \text{Symm. monoidal closed, } \bot \\
(A \to \bot) \to \bot \cong A & \cong & \neg A = A \to \bot \\
\end{array} \]

[Dialogue categories]
[Mellies & Tabareau]
Dialogue bicategories

(tensor & negation)

A dialogue bicategory is

a symmetric monoidal \( B \)

an object \( \bot \)

such that \( B(A \otimes B, \bot) = B(A, \neg B) \) for some \( \neg B \).

Key properties:

- \( \neg \) is a pseudofunctor \( B \rightarrow B^\text{op} \)
- \( \neg \neg \) is a strong pseudomonad on \( B \)
- \( B_{\neg \neg} \) is premonoidal
This talk

I. A bicategory of games

II. Strong pseudomonads

→ III. Resources and symmetries
Adding an exponential modality:

\[ G \xrightarrow{!} \quad \xrightarrow{\perp} \quad G^\otimes \]

\[ ! \circ = \circ_0 \circ_1 \circ_2 \circ_3 \circ_4 \ldots \]

Symmetric copies
Adding an exponential modality:

\[
\begin{align*}
\text{!} \quad 0 &= \{ 0_0, 0_1, 0_2, 0_3, 0_4, \ldots \} \\
\text{Symmetric copies} &\quad \text{(event structures with symmetry)}
\end{align*}
\]

Clairambault, Castellan, Winskel
Strategies playing symmetrically:

\[ O_1 \sim O_2 \]

\[ q \quad \Rightarrow \quad q' \]

\[ A \]
Strategies need to be bi-invariant:
Strategies need to be bi-invariant:

\[ x_\sigma \xrightarrow{\text{Symmetry of strategy}} z_{\sigma} \]

\[ x_A \xrightarrow{\text{Opponent Symmetry}} y_A \xleftarrow{\text{Player Symmetry}} z_A \]
idea: send synchronized pair \((x_\sigma, x_\tau)\) up to symmetry to \(x_y\).

Thm. The virtual 2-category of games with symmetry, strategies, and multimaps is representable.
Summary

- Generalize the foundations of game semantics from categories to bicategories
- 2-dimensional setting: "proof-relevant" all the time
- Other applications of premonoidal bicategories: graded monads, "PARA" construction