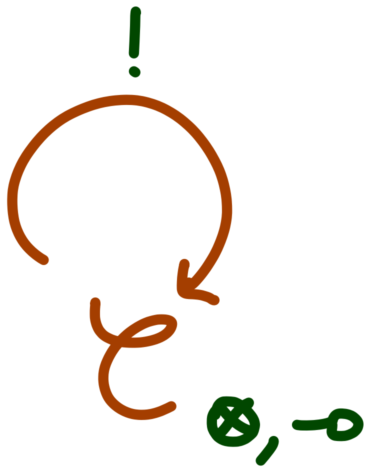
Call-by-value in Bicategories of Games

Hugo Paquet
LIPN

joint work with Philip Saville

Resources and effects

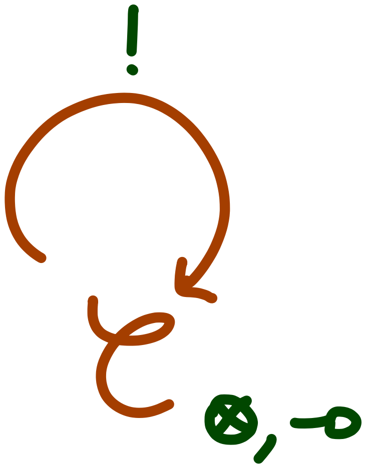
Models of Linear Logic



exponential comonad

Resources and effects

Models of Linear Logic



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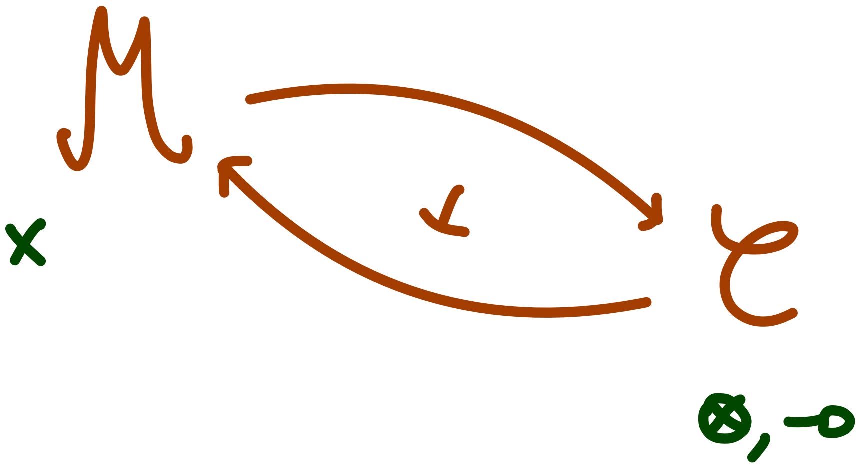


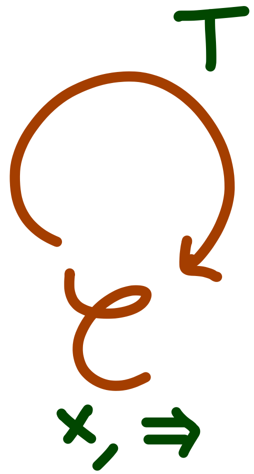
linear/non-linear adjunction

Resources and effects

Moggi, Levy, ...

Models of effectful languages

Strong monad



Resources and effects

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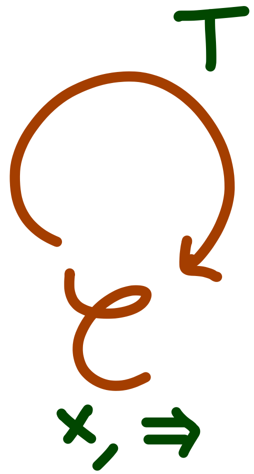
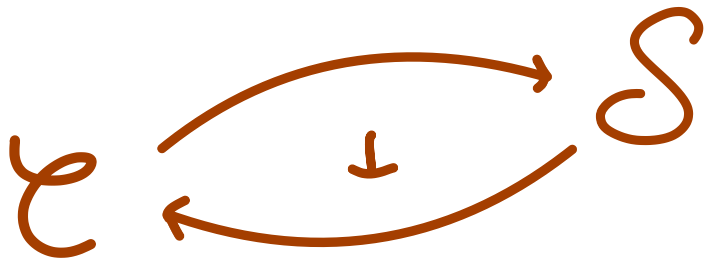


strong adjunction

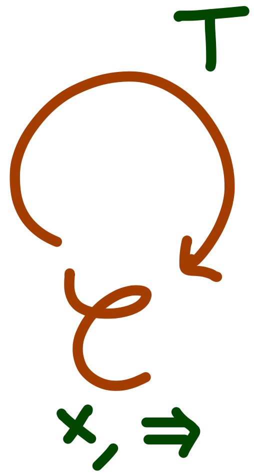


Resources and effects

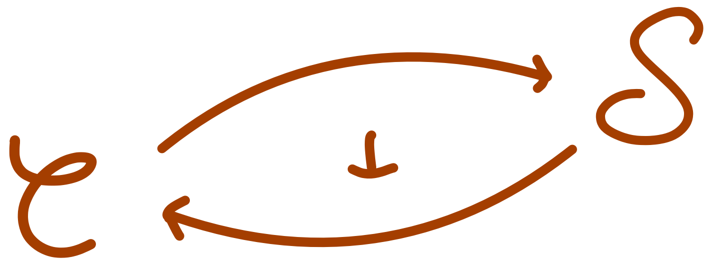
Moggi, Levy, ...

Models of effectful languages

(CBV) strong monad



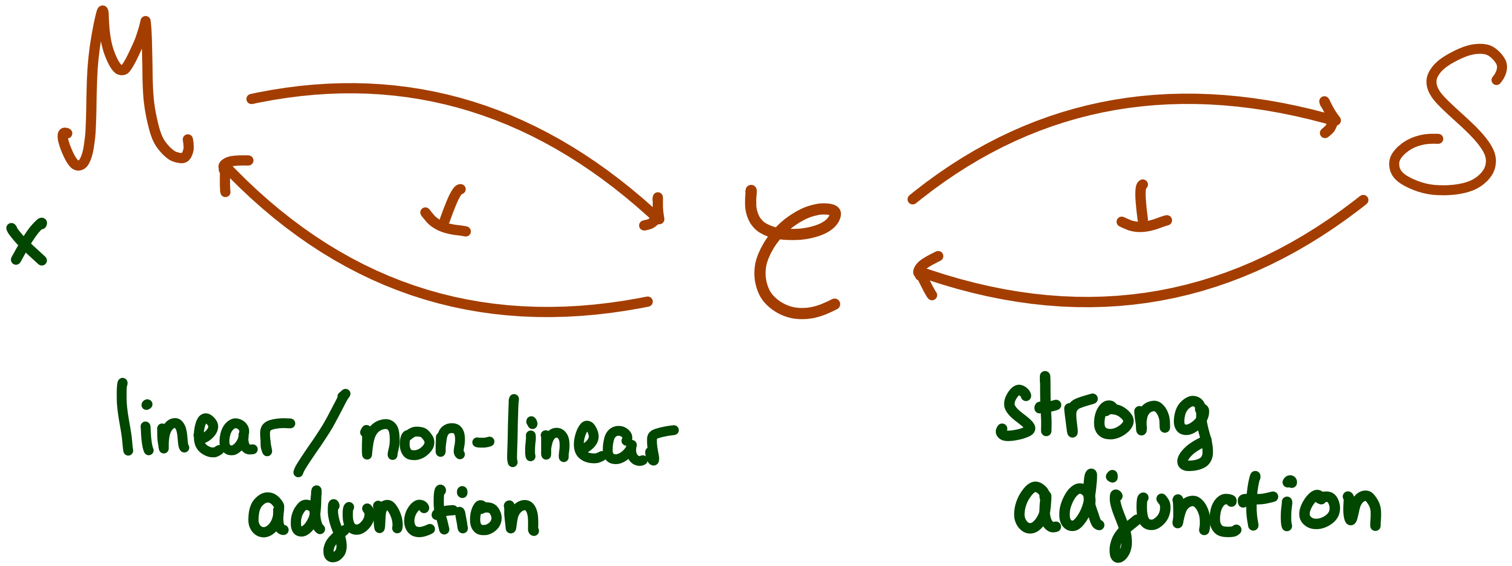
(CBPV) strong adjunction



Resources and effects

Munch-Maccagnoni, Fiore

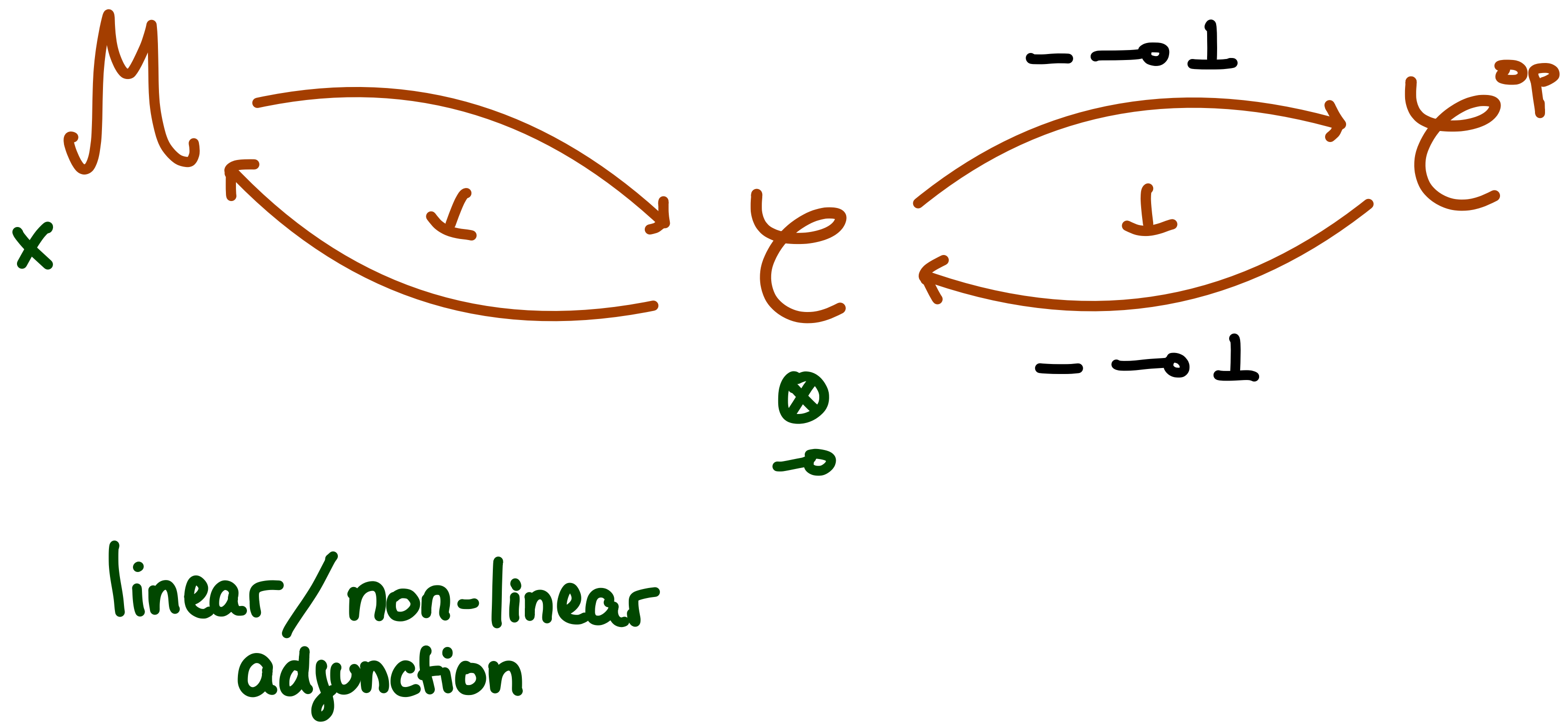
- Can combine the two viewpoints ("Linear CBPV")



- Several canonical examples.

Resources and effects

- Can combine the two viewpoints ("Linear CBPV")
- e.g.



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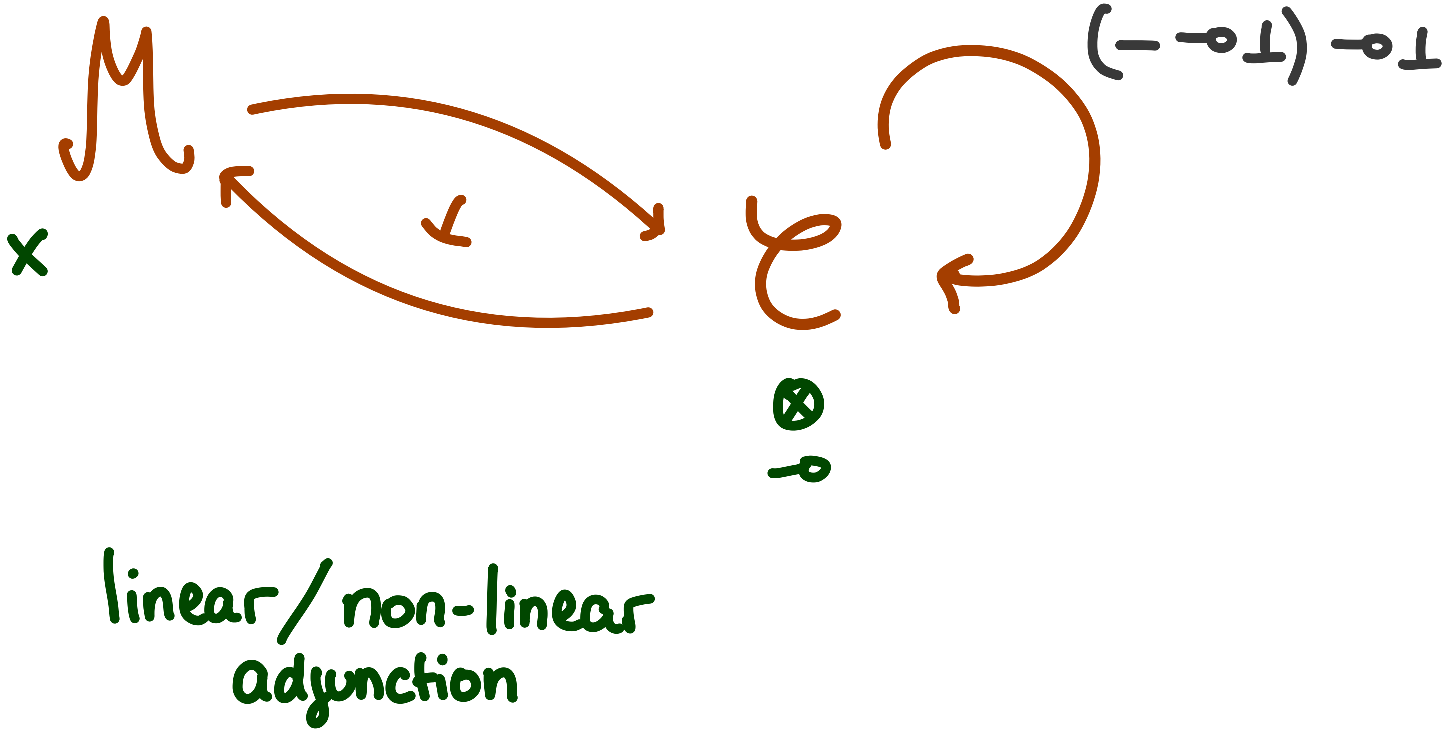
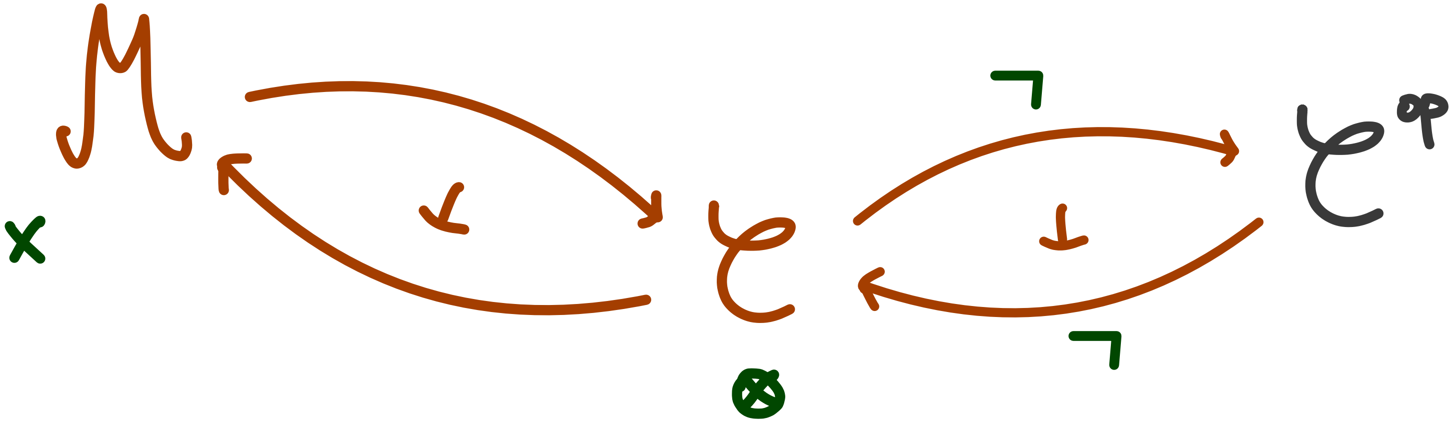


Resources and effects

Game Semantics:



Resources and effects

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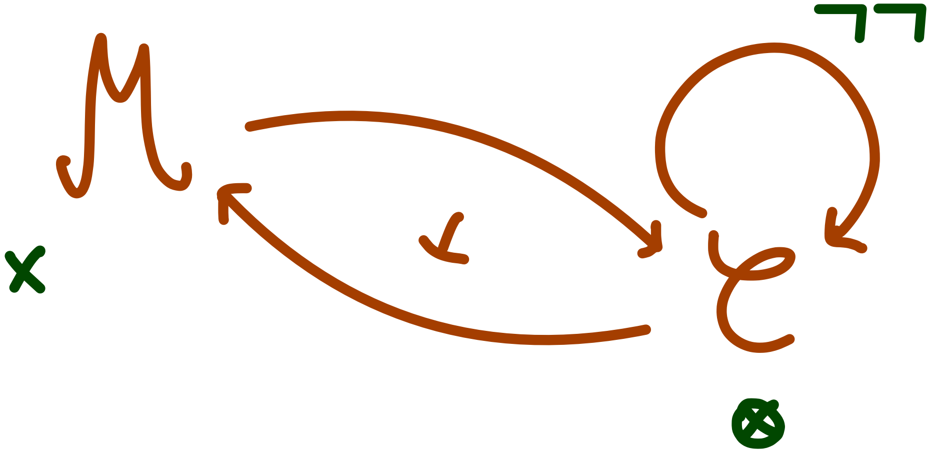
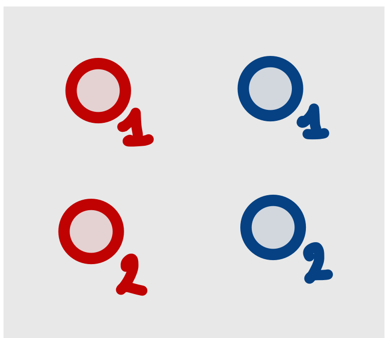


Game Semantics

Two players

- player
- opponent

A simple game:



A

Game Semantics

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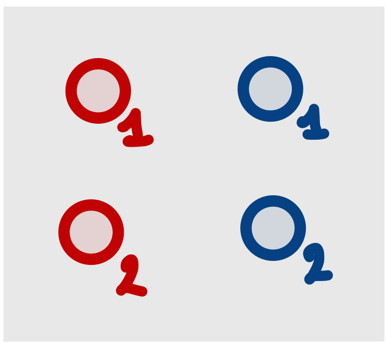
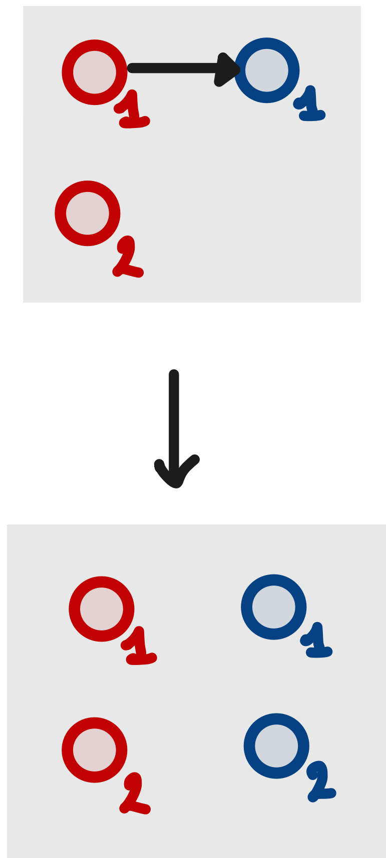


A

Strategies over the game:



σ

A

Game Semantics

Two players

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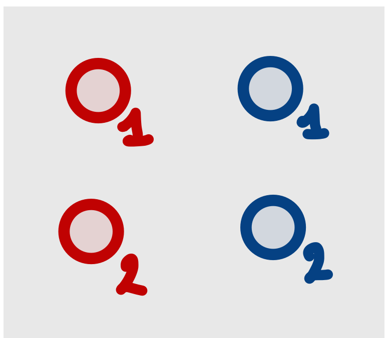

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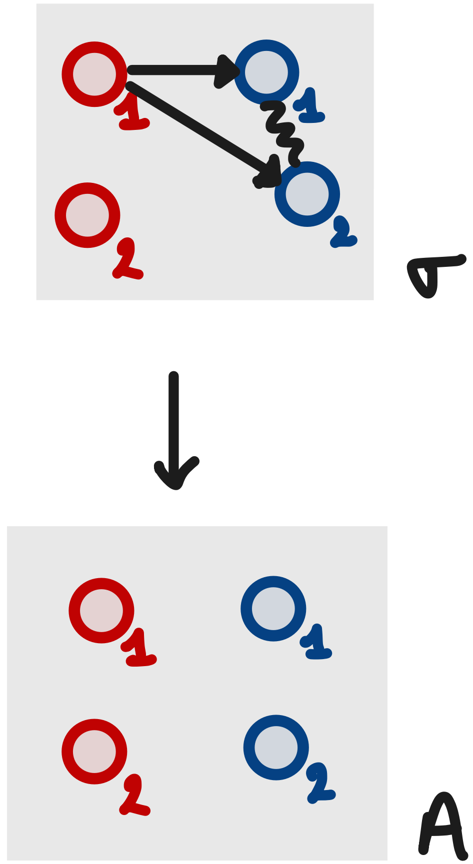
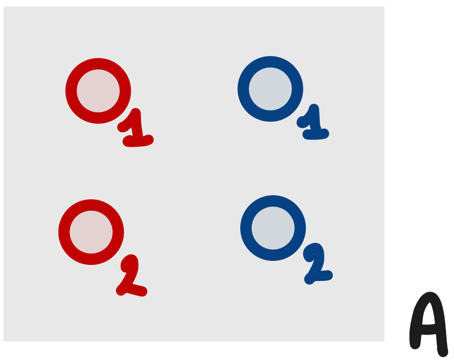


Game Semantics

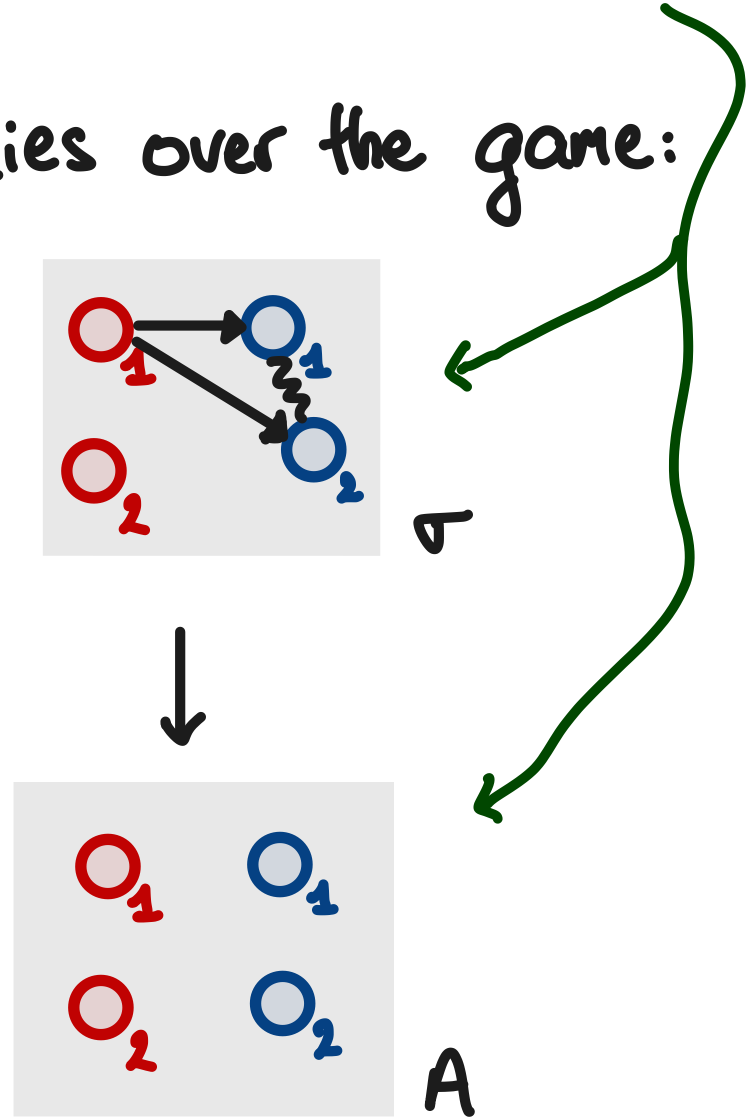
use event structures [Winskel, Clairambault, Castellan, ...]

Two players

A simple game:



Strategies over the game:



Strategies $\mathbb{N} \longrightarrow \mathbb{N}$

Distinguish between values & computations to make sense of $(\lambda x.M)\,V = M[V/x]$ in CBV

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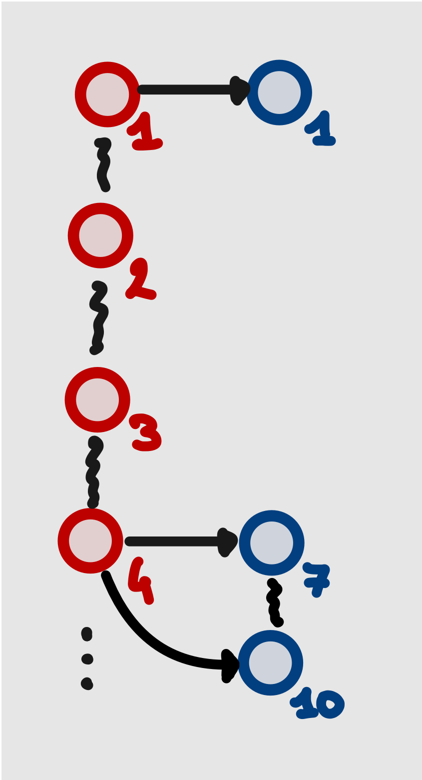
arbitrary strategy



$\mathbb{N} \longrightarrow \mathbb{N}$

Strategies $\mathbb{N} \to \mathbb{N}$

Distinguish between values & computations
to make sense of $(\lambda x.M)\,V = M[V/x]$ in CBV

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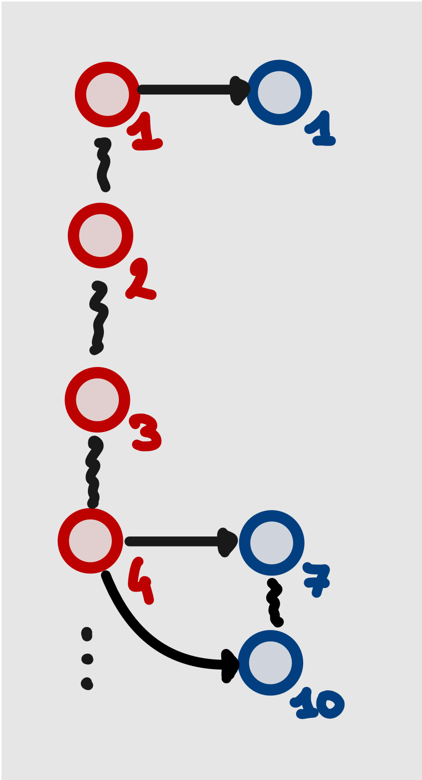
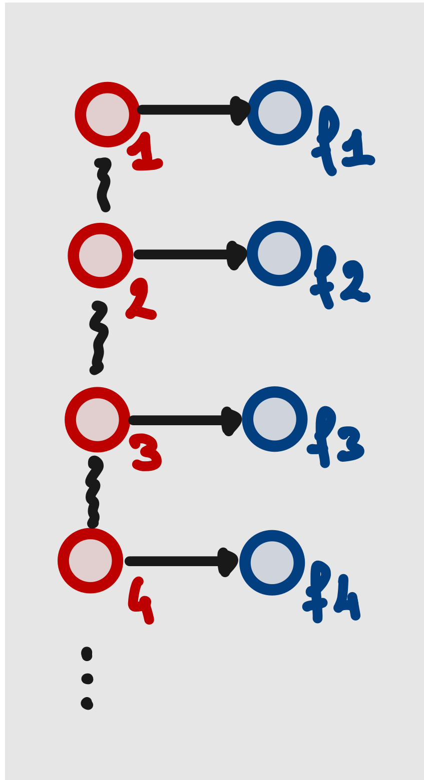


$\mathbb{N} \to \mathbb{N}$

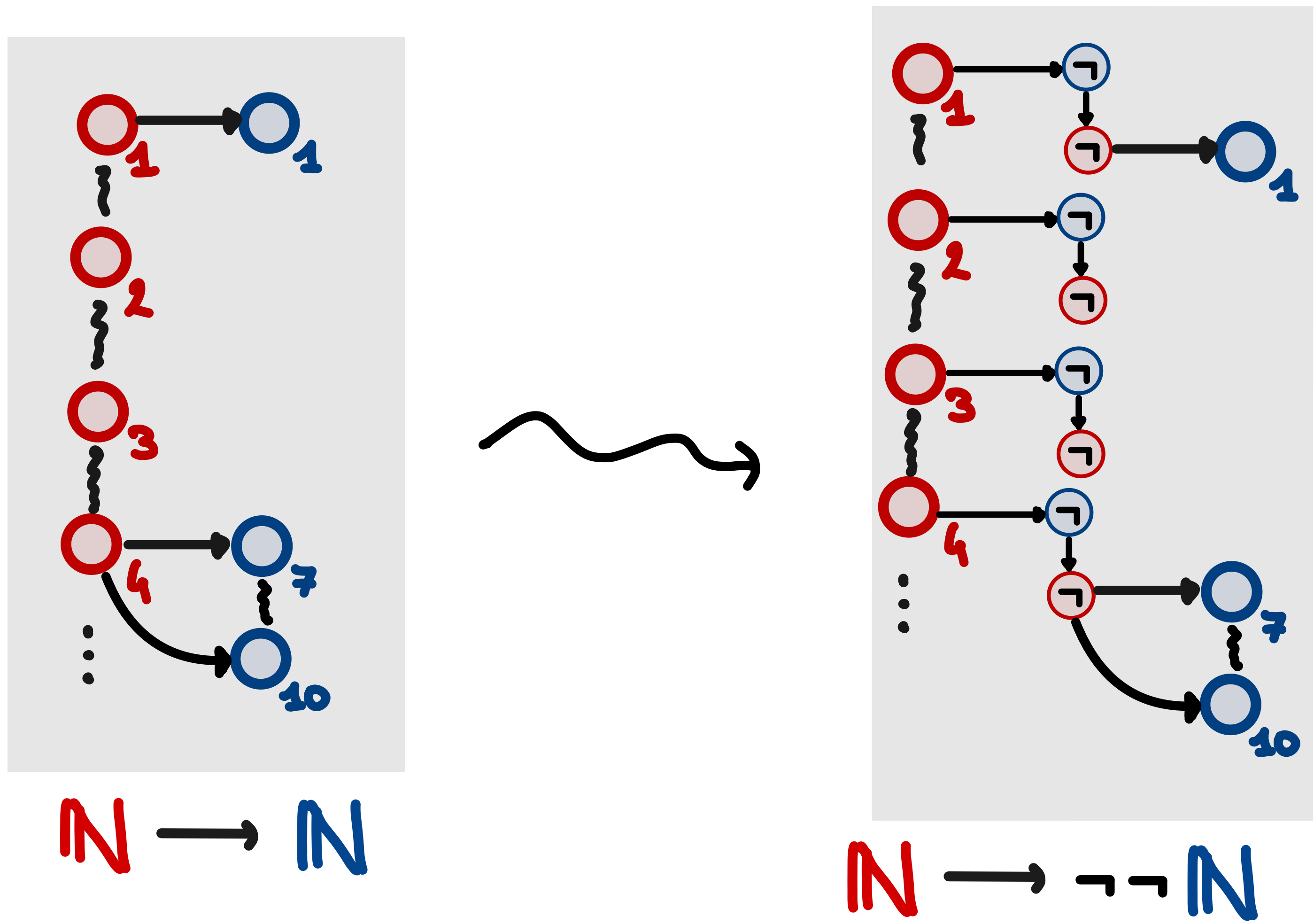
"value" strategy



(total, deterministic)

$\mathbb{N} \to \mathbb{N}$

Strategies $\mathbb{N} \longrightarrow \mathbb{N}$



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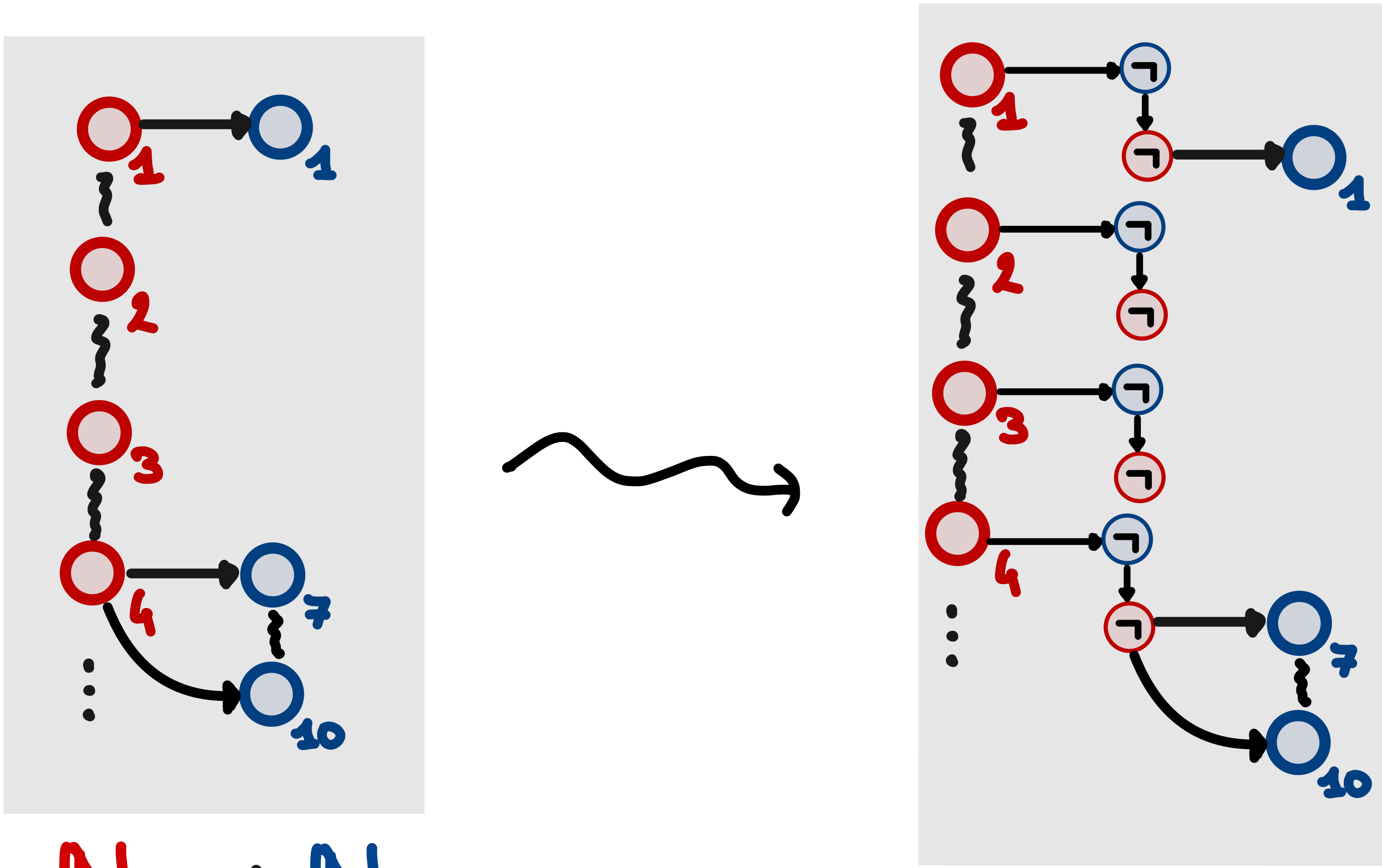


$\mathbb{N} \to \mathbb{N}$ $\qquad\qquad$ $\mathbb{N} \to \neg\neg\mathbb{N}$

Prop:

$$\frac{\text{strategy } A \to B}{\text{value strategy } A \to \neg\neg B}$$

← monad ?

This talk

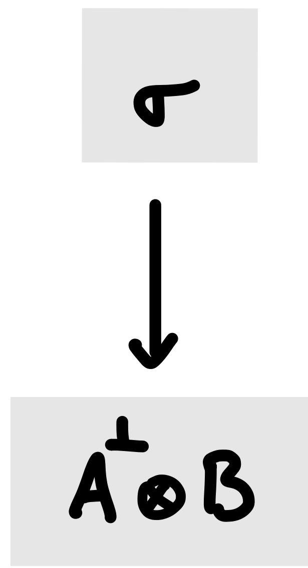
→ I. A bicategory of games

II. Strong pseudomonads

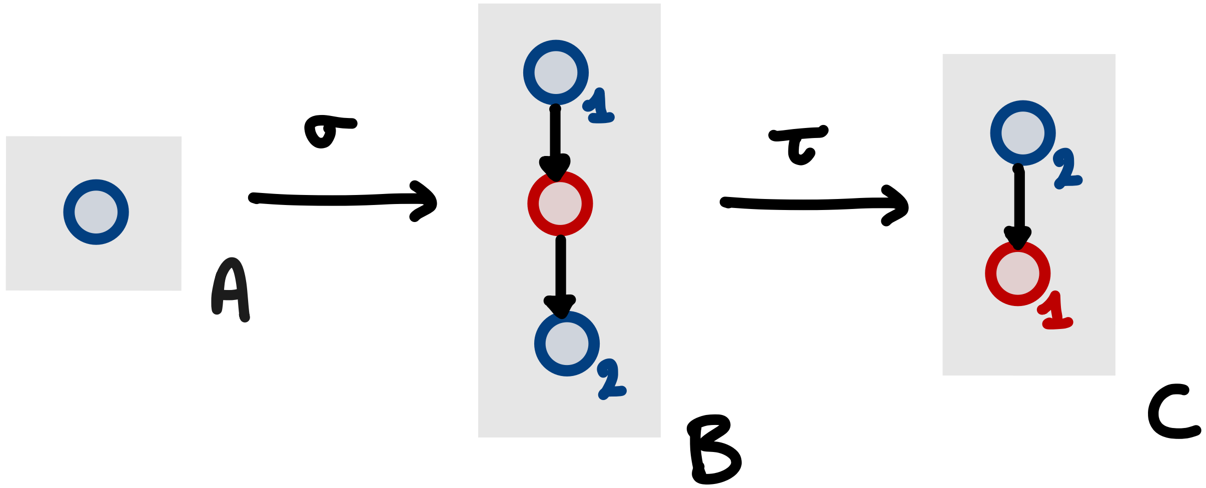
III. Resources and symmetries

Objects: games $A, B, C, \ldots$

Morphisms $A \longrightarrow B$: strategies



Composition of strategies



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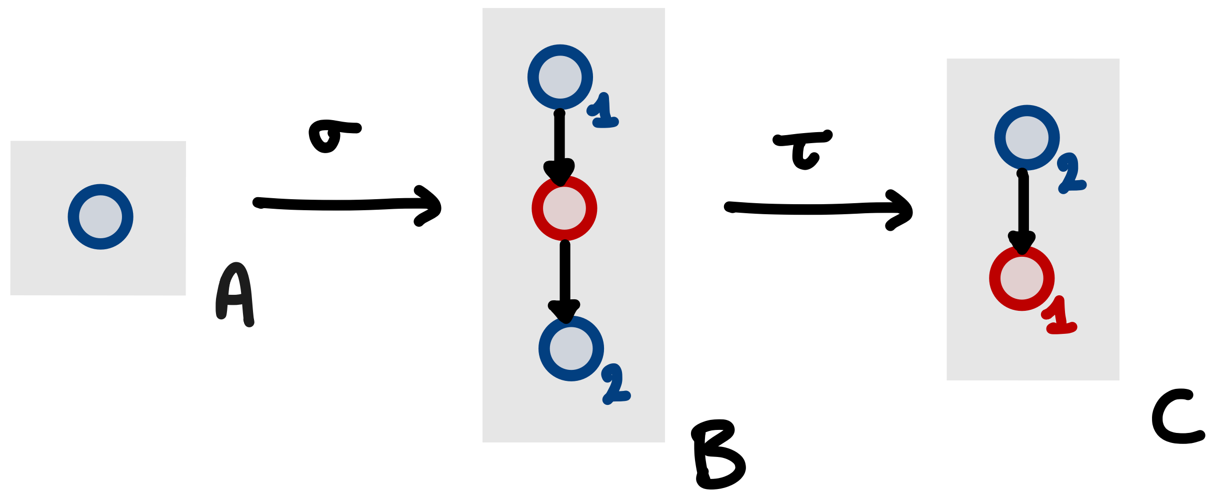

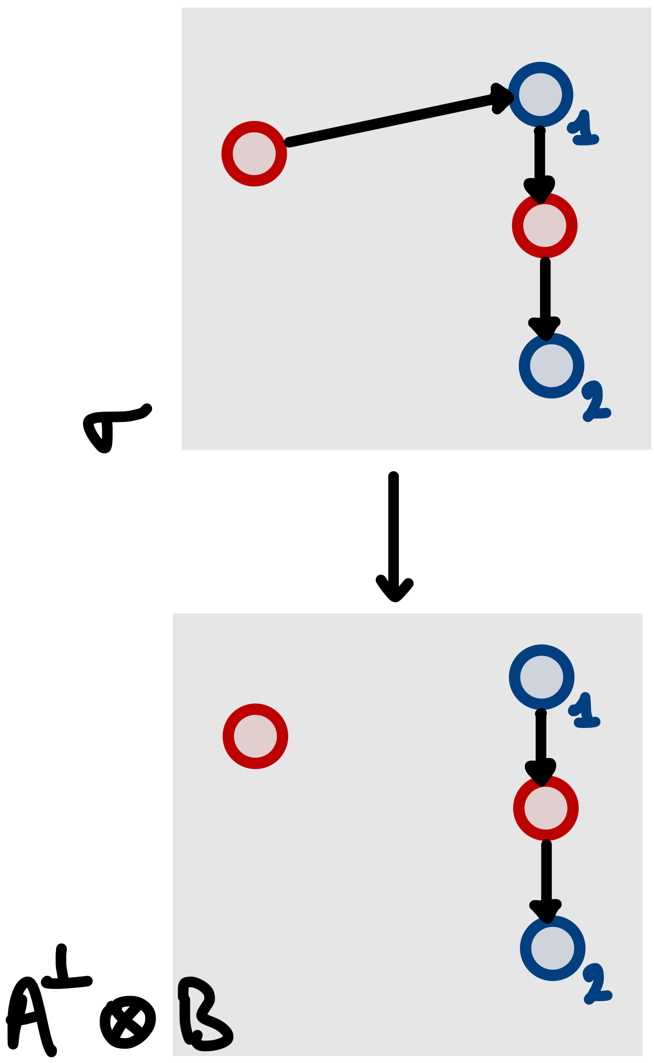

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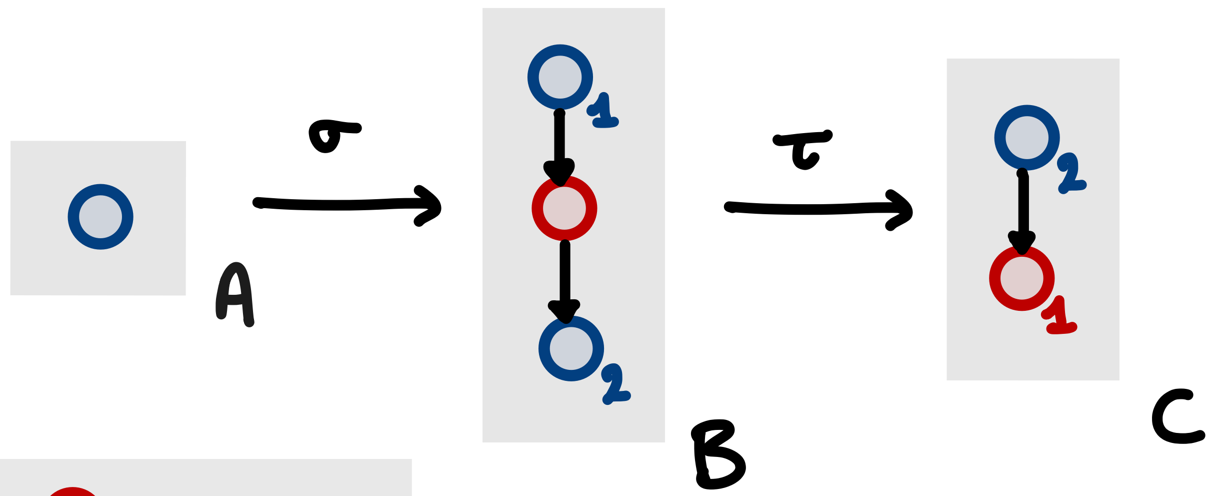

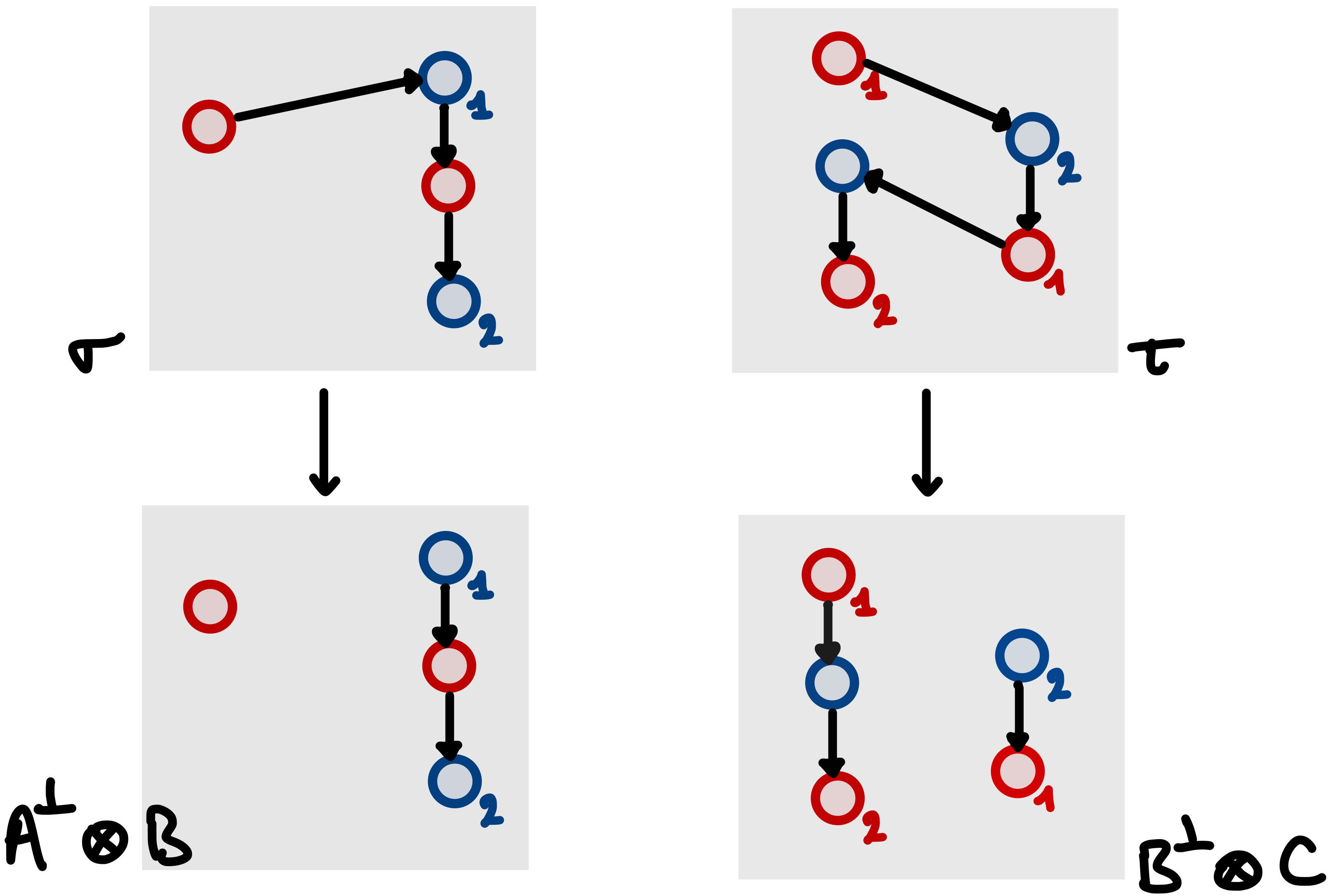

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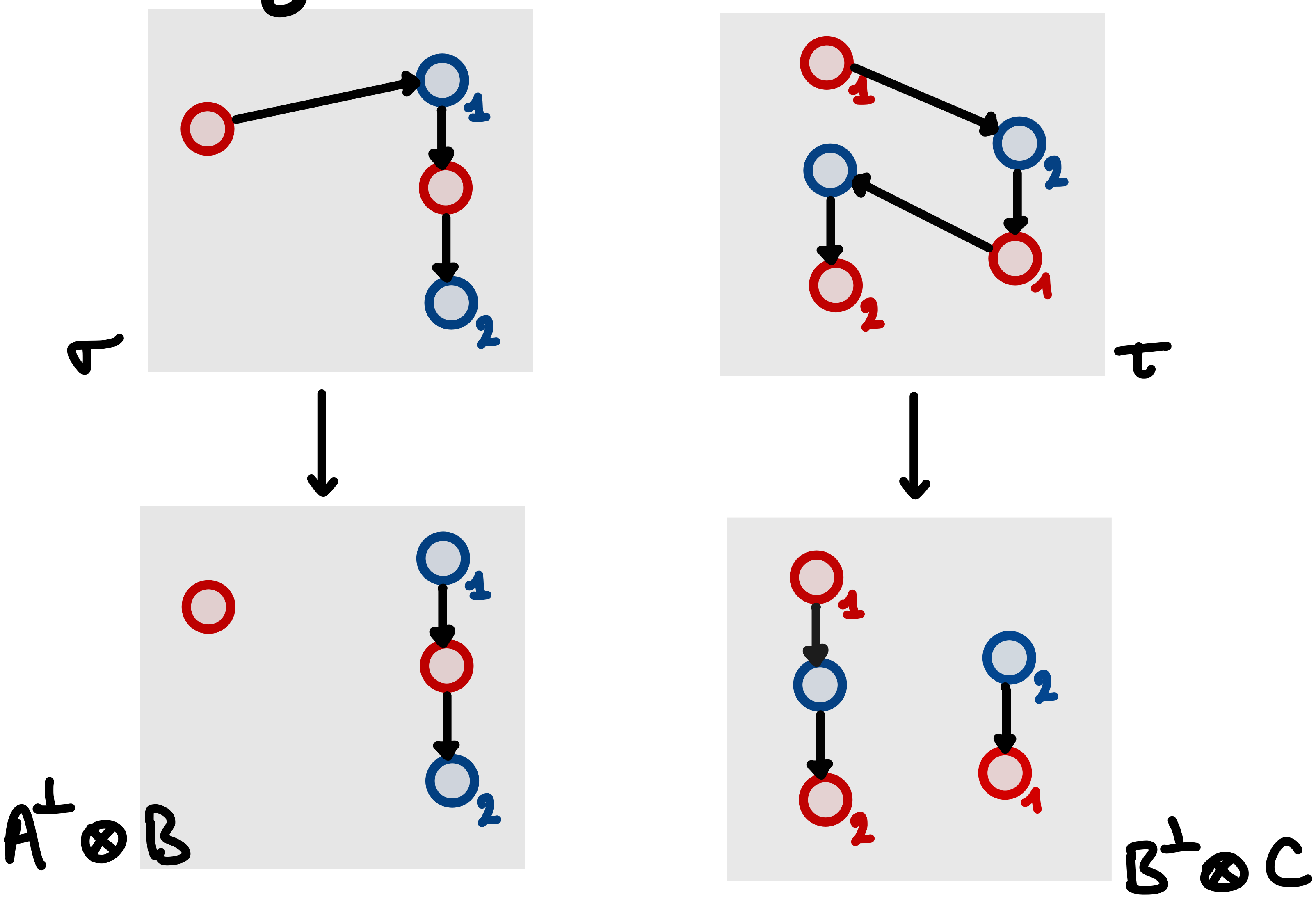

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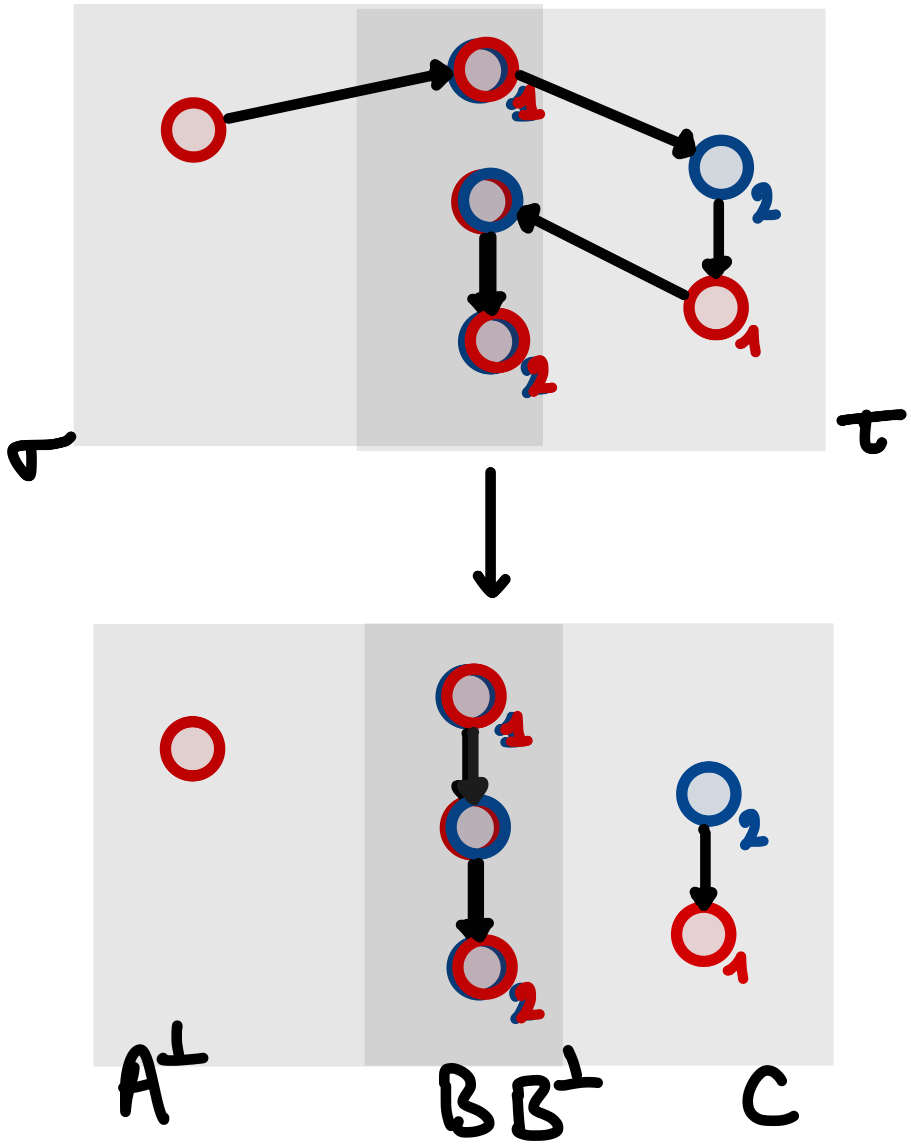

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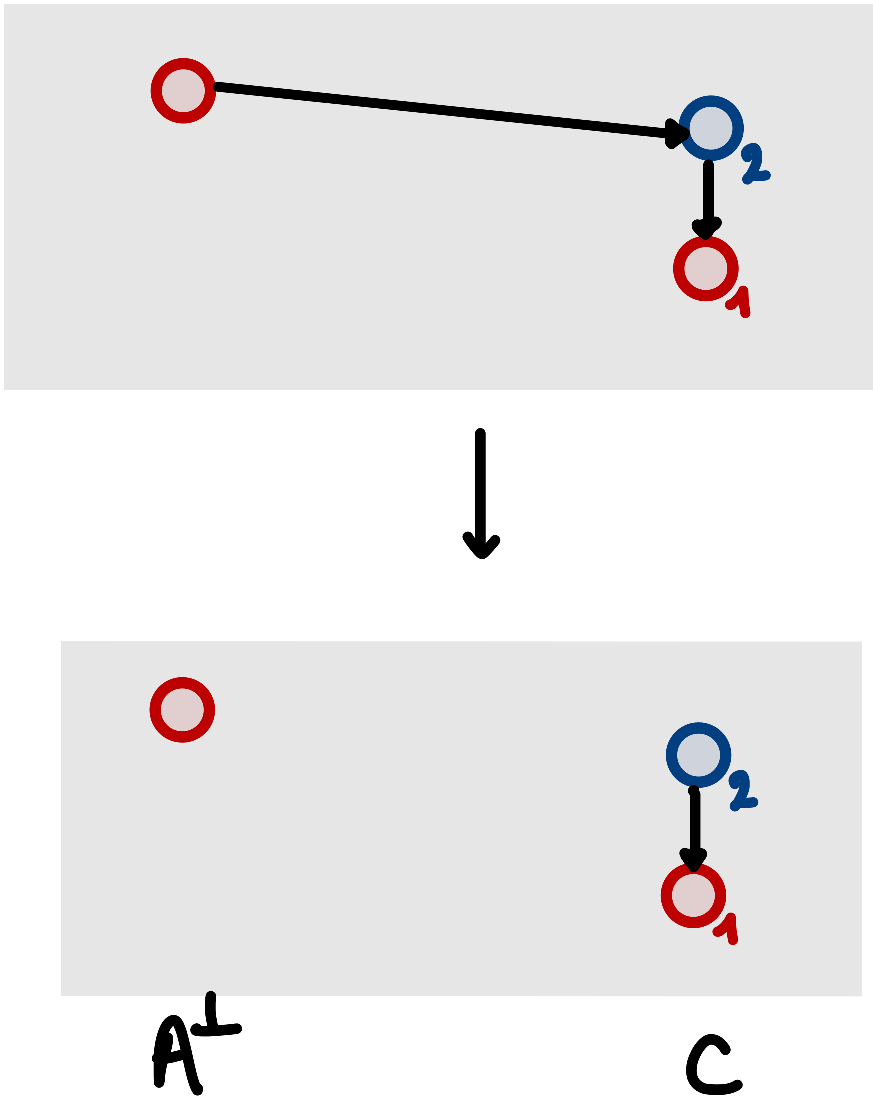
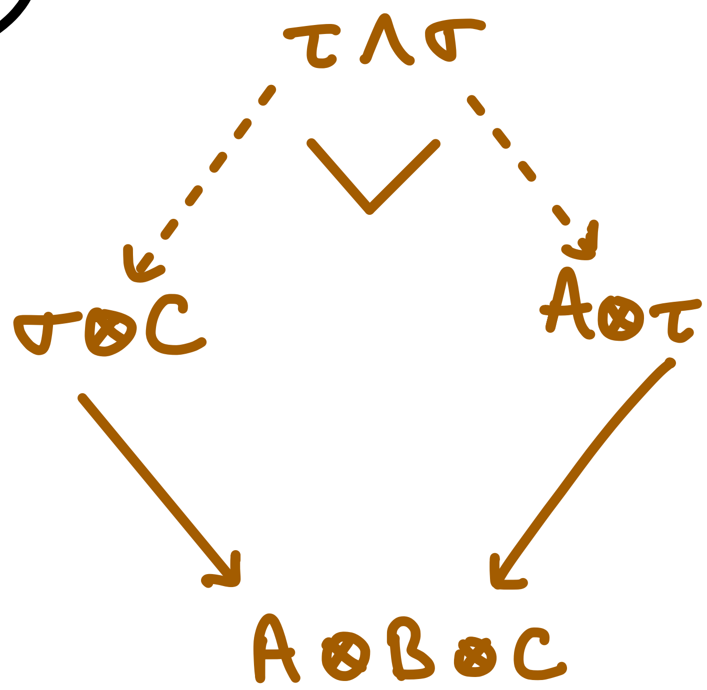


Composition of strategies

[Winskel, Clairambault, Castellan, ...]

in the category of event structures:

(1)

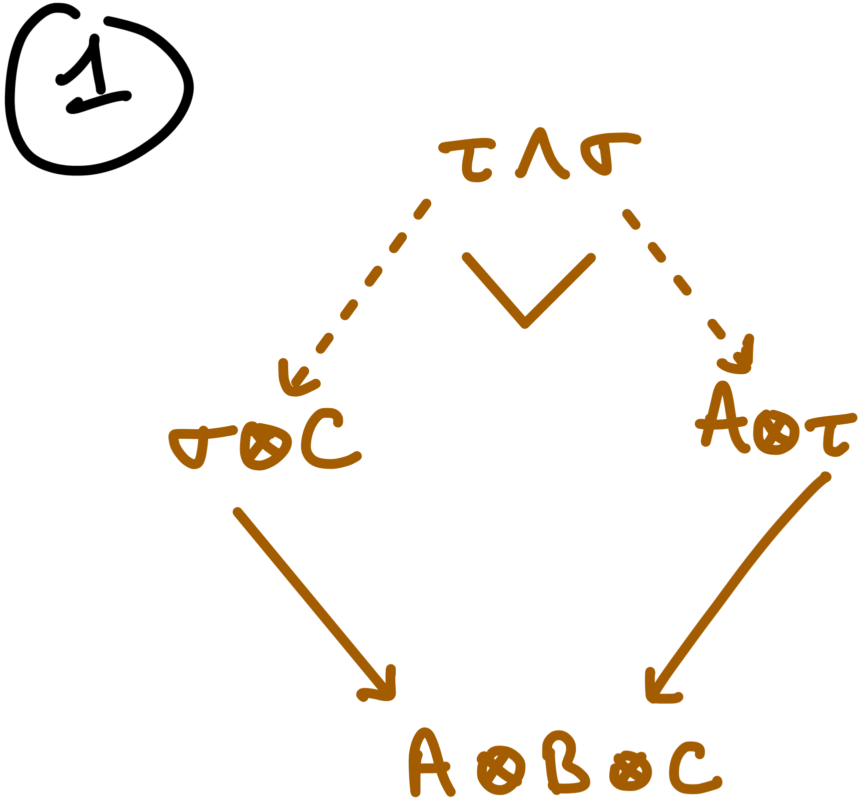


Synchronization

(2)

Composition of strategies

in the category of event structures:



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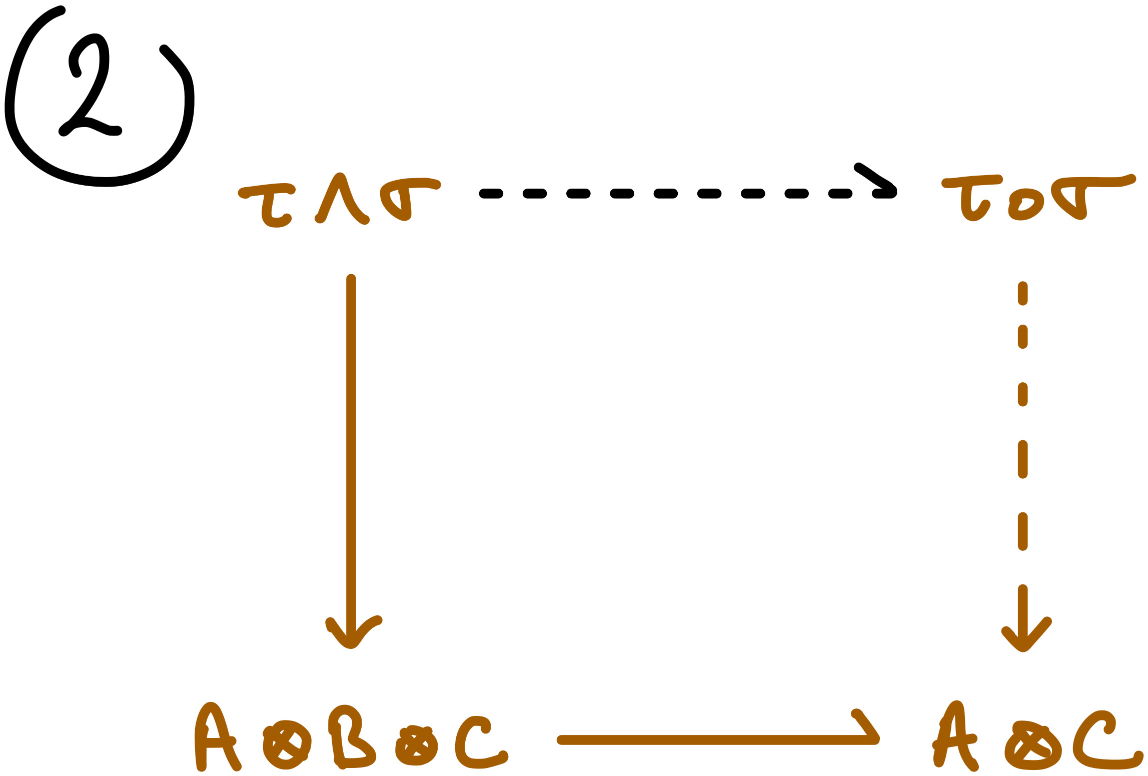

hiding

Composition of strategies (not strictly associative)

in the category of event structures:

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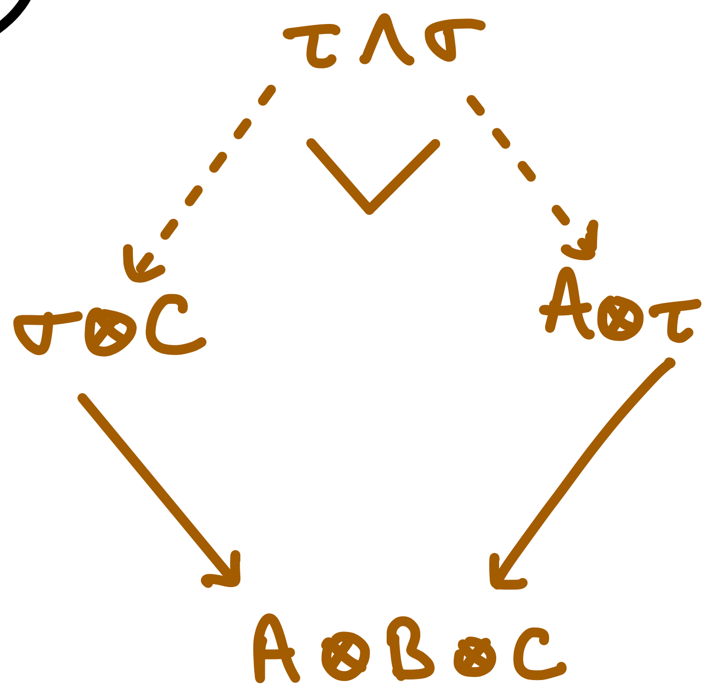
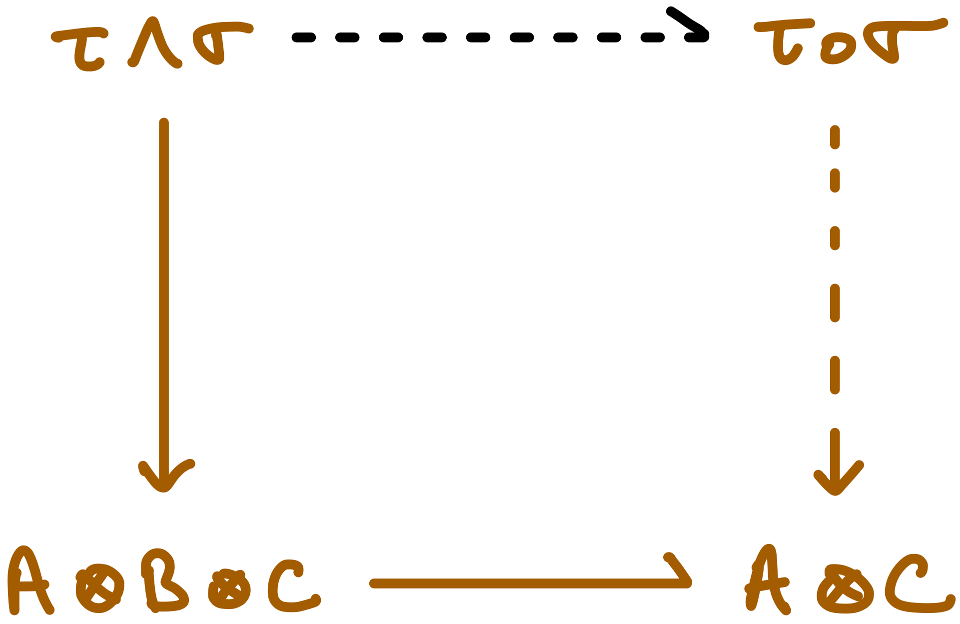


Synchronization

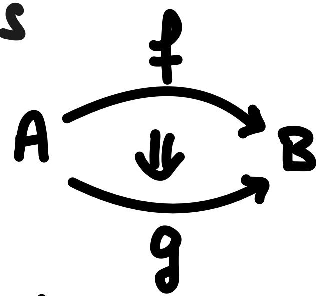
(2)



hiding

Bicategories

- objects $A, B, \ldots$
- morphisms $f : A \longrightarrow B, \ldots$ (with identity and composition)
- 2-cells

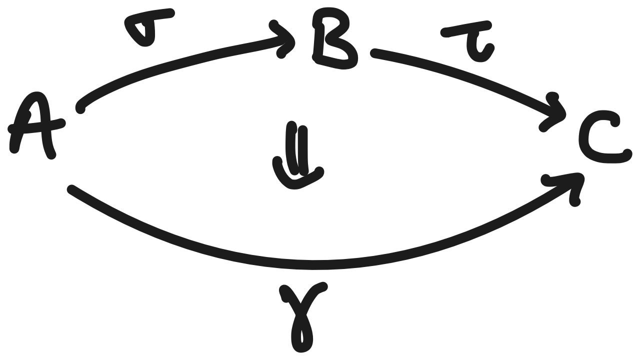


- associativity $a_{f,g,h} : (h \circ g) \circ f \Longrightarrow h \circ (g \circ f)$
- identity $r_f : f \circ id \Longrightarrow f$

  $l_f : id \circ f \Longrightarrow f$
- coherence axioms

Composition of strategies

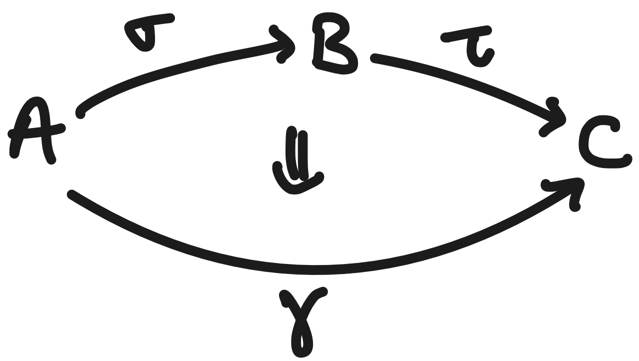
universal property of $\tau \circ \sigma$?



idea: send synchronized pair (x_σ, x_τ) to x_γ.

Composition of strategies

universal property of $\tau \circ \sigma$?



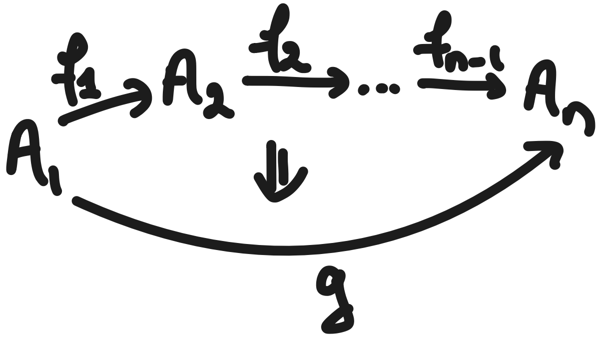
idea: send synchronized pair (x_σ, x_τ) to x_γ.

Prop: There is a universal multimap
$$\sigma, \tau \longrightarrow \tau \circ \sigma$$

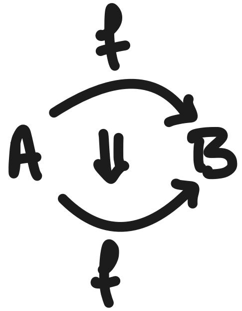
cf. ⊗ in Vect & bilinear maps

Virtual 2-categories

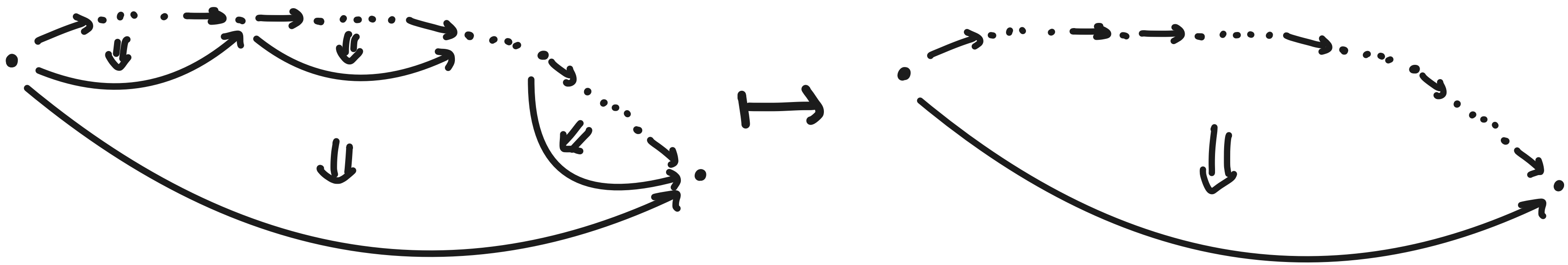
- objects $A, B, \ldots$
- morphisms $f : A \longrightarrow B, \ldots$ (no composition)
- multi-2-cells



id.



comp.



Virtual 2-categories

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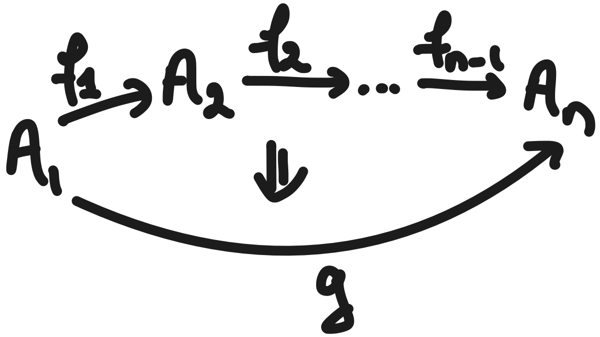
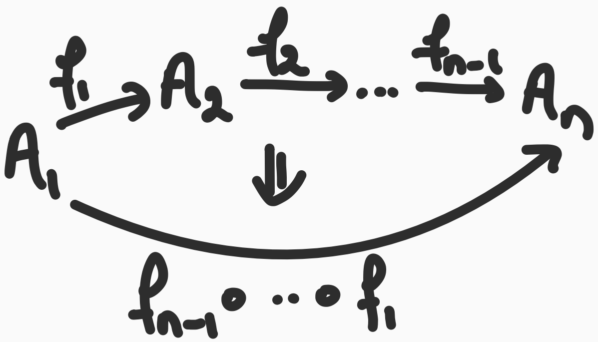


A virtual 2-cat. is representable if there is always a universal cell:



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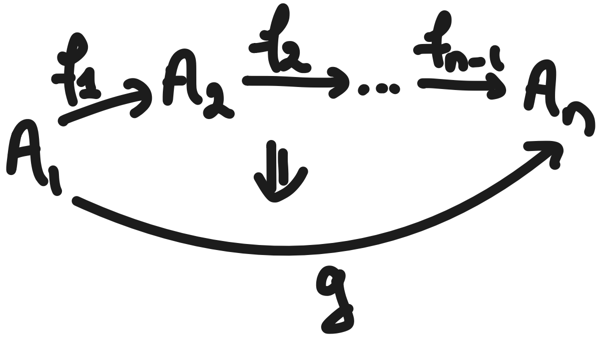

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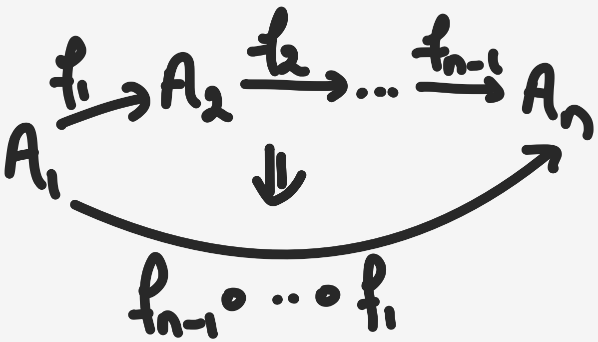

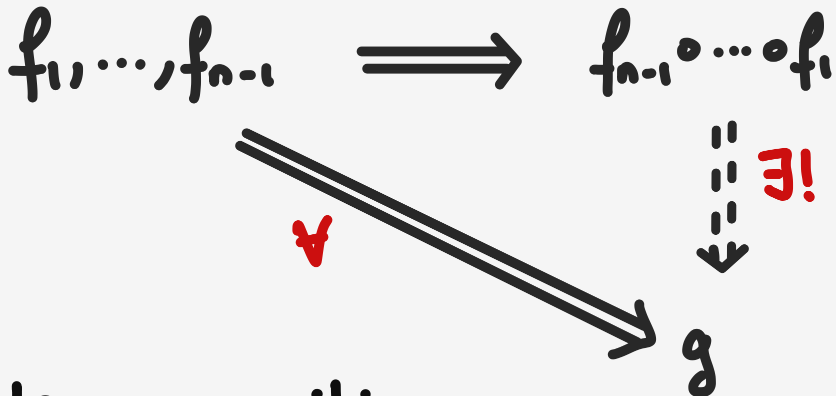

and the universal cells are closed under composition.

[Hermida 2000, Leinster 2003]

representable virtual 2-categories $\simeq$ bicategories

↑

no structural 2-cells

(Coherence is automatic)

Thm. The virtual 2-category of games, strategies, and multimaps is representable.

So the binary composition $\tau \circ \sigma$ gives a **bicategory**.

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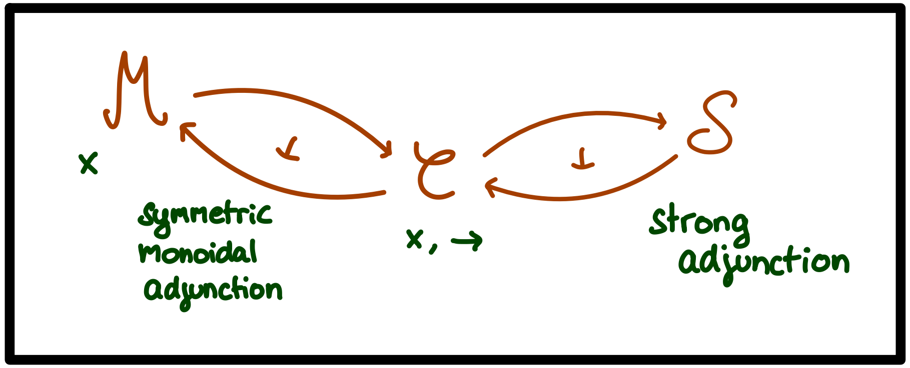
So the binary composition $\tau \circ \sigma$ gives a <u>bicategory</u>.



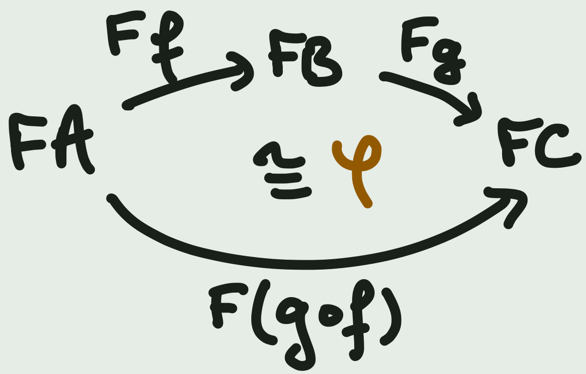
not well-understood for bicategories.

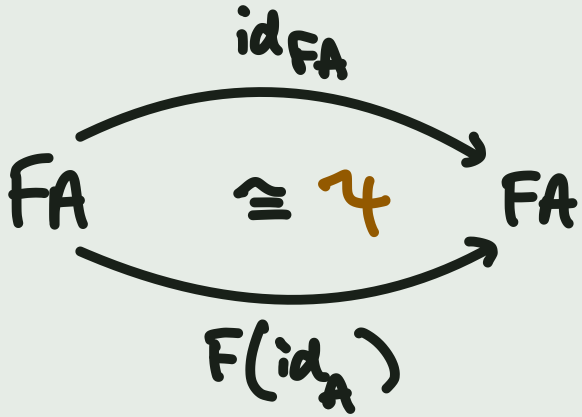
Summary of bicategory theory

Everything holds up to coherent invertible 2-cells

pseudofunctor $F : \mathcal{B} \longrightarrow \mathcal{C}$

- acts on objects, morphisms, 2-cells,
- functor up to iso:





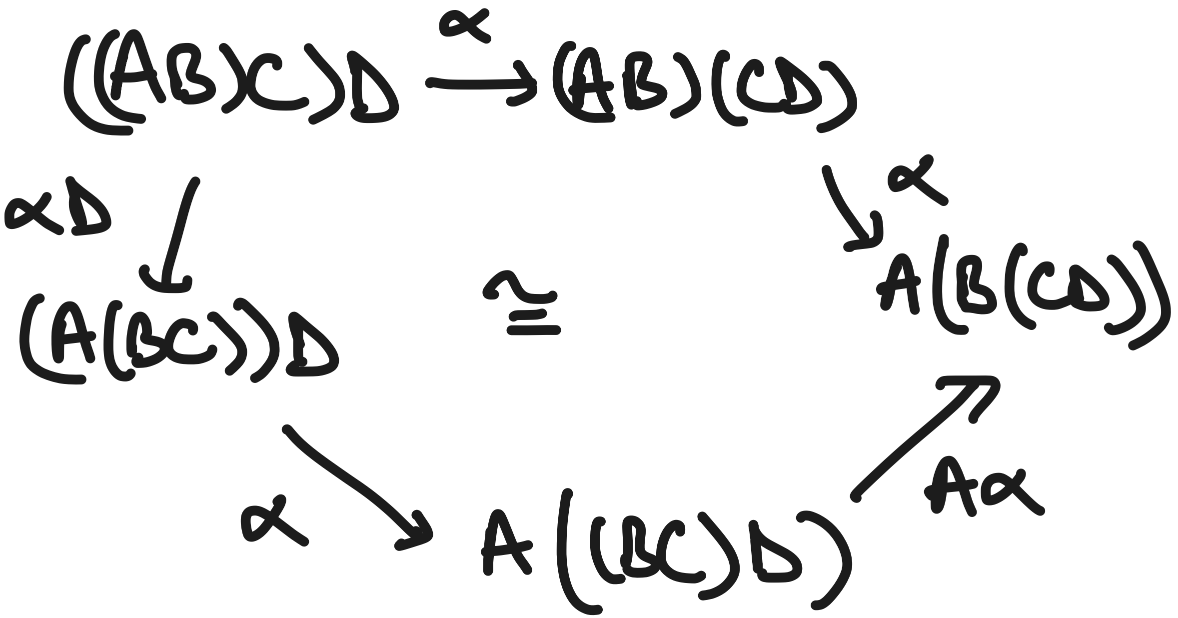
+ coherence axioms : φ, ψ compatible with a, r, ℓ.

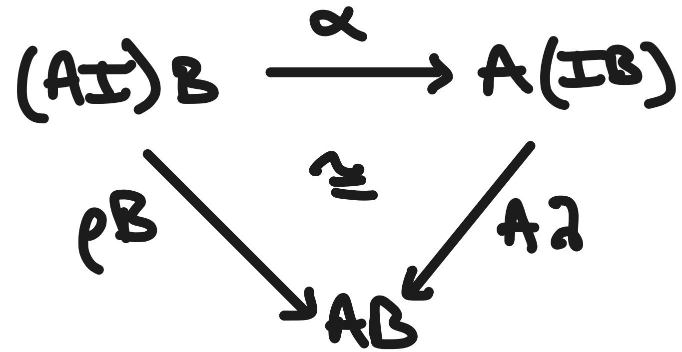
Summary of bicategory theory

monoidal bicategory $(\mathcal{B}, \otimes, I)$:

$$\alpha : (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)$$
$$\lambda : I \otimes A \longrightarrow A$$
$$\rho : A \otimes I \longrightarrow A$$

} pseudo-natural equivalences





…

- Similar notions of symmetric monoidal bicats, pseudomonads, etc.
- Main difficulty is to find the "right" axioms
- We give a definition of strong pseudomonads, axioms are justified by a correspondence theorem:

strengths $\simeq$ actions on the Kleisli bicat.

This talk

I. A bicategory of games

→ II. Strong pseudomonads

III. Resources and symmetries

Monads and computation

Distinguish between: Values $A \longrightarrow B$
Computations $A \longrightarrow TB$

(So we can make sense of $(\lambda x.M)\,V = M[V/x]$ in CBV)

Monads and computation

Distinguish between: Values $A \to B$
Computations $A \to TB$

(So we can make sense of $(\lambda x.M)V = M[V/x]$ in CBV)

Can compose computations: $\Gamma \vdash M : B$ $\quad B \vdash N : C$

$$\Gamma \xrightarrow{M} TB \xrightarrow{TN} T^2C \xrightarrow{\mu} TC$$

Monads and computation

Distinguish between: Values $A \to B$
Computations $A \to TB$

(So we can make sense of $(\lambda x.M)\,V = M[V/x]$ in CBV)

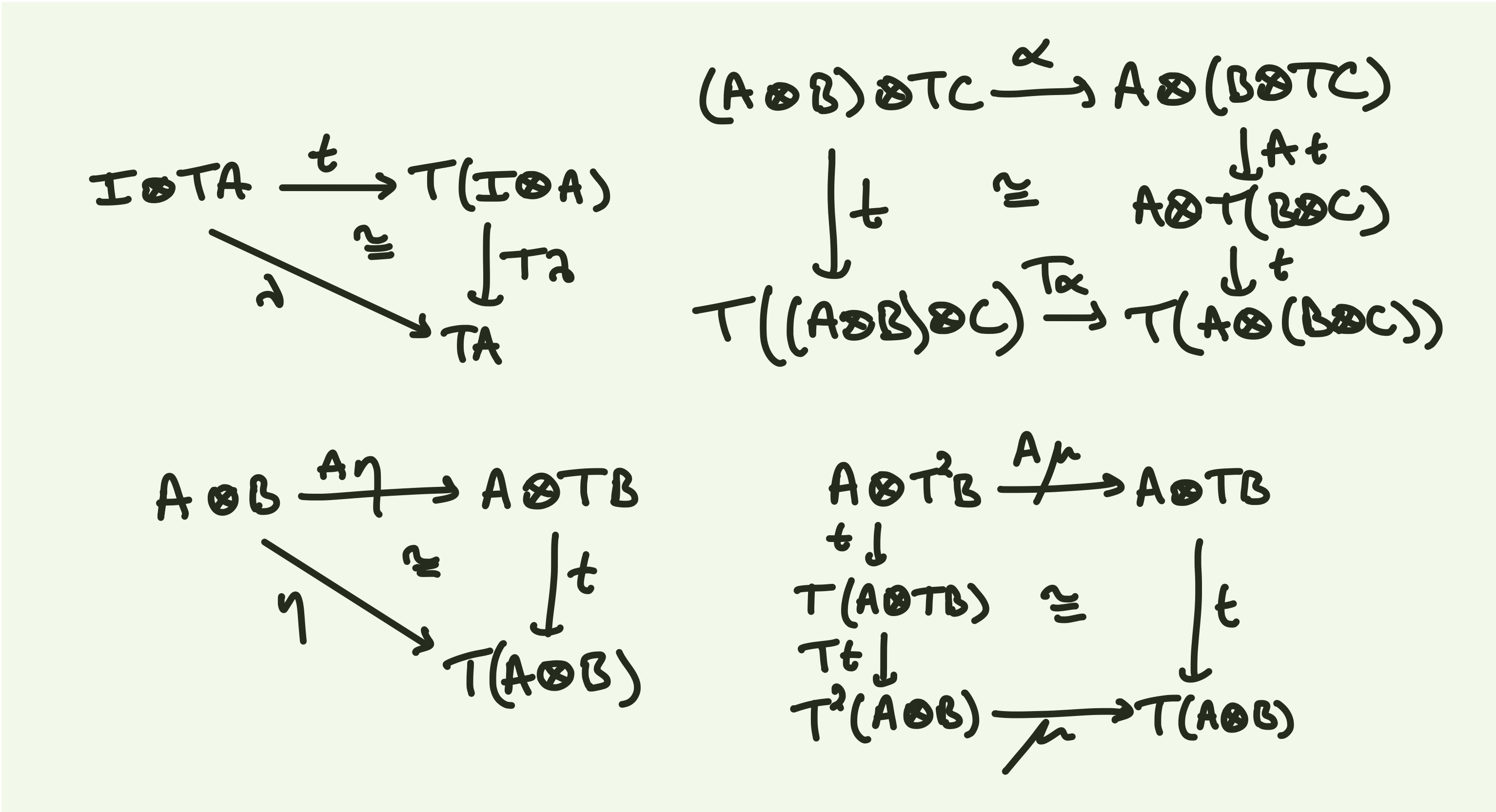
Can compose computations: $\Gamma \vdash M : B \qquad B \vdash N : C$

$$\Gamma \xrightarrow{M} TB \xrightarrow{TN} T^2C \xrightarrow{\mu} TC$$

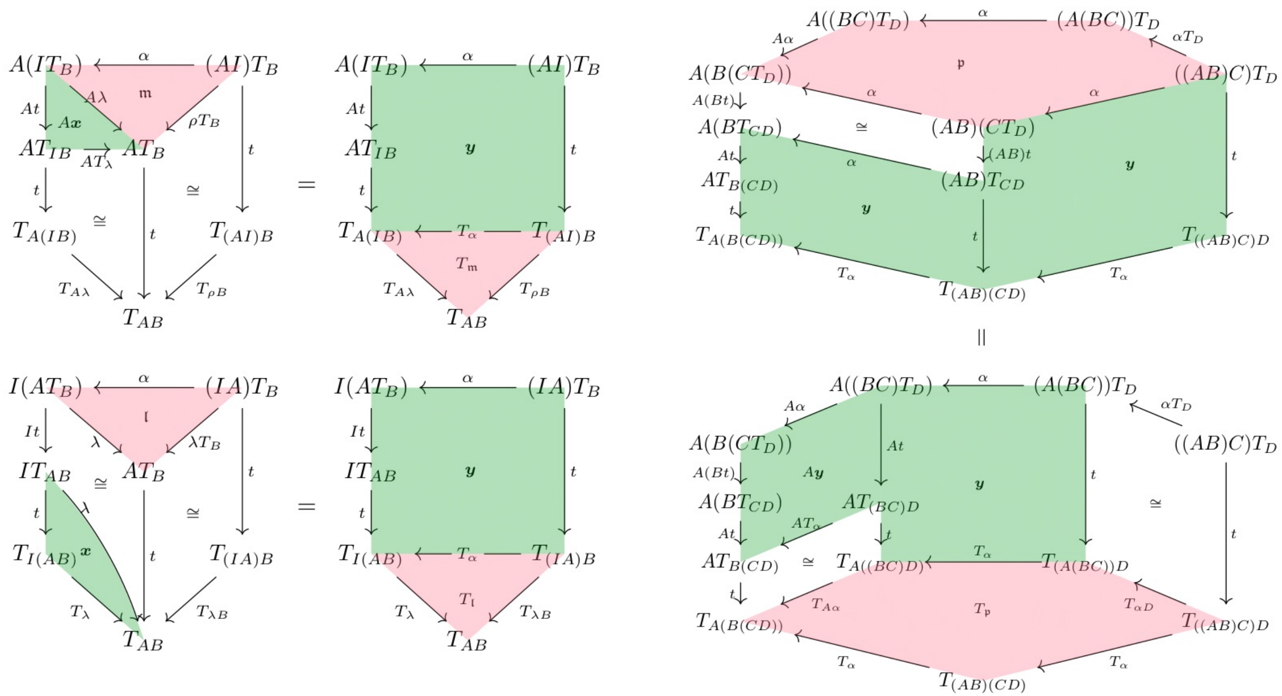
need strength in general: $\Gamma \vdash M : B \qquad \Delta, B \vdash N : C$

$$\Delta \otimes \Gamma \xrightarrow{\Delta \otimes M} \Delta \otimes TB \xrightarrow{t} T(\Delta \otimes B) \xrightarrow{TN} T^2C \xrightarrow{\mu} TC$$

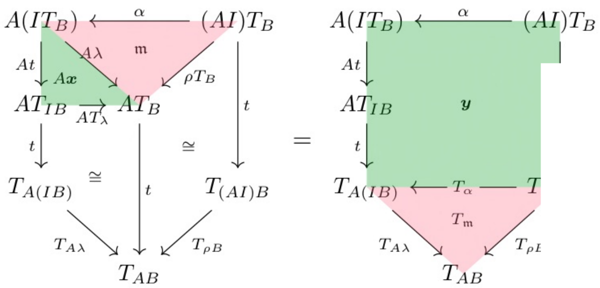
Definition: A strength for a pseudomonad $T : \mathcal{B} \to \mathcal{B}$ on a monoidal bicategory $(\mathcal{B}, \otimes, I)$ is a pseudo-natural transformation $t_{A,B} : A \otimes TB \longrightarrow T(A \otimes B)$ equipped with 2-cells:

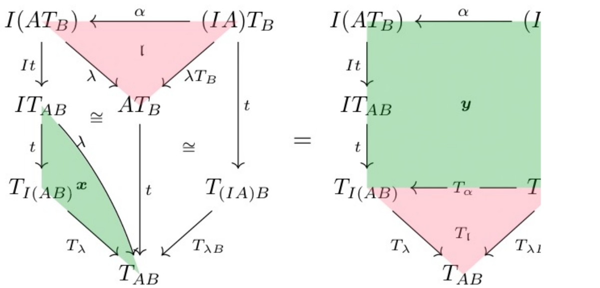


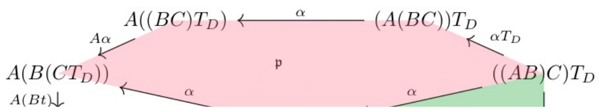
+ axioms

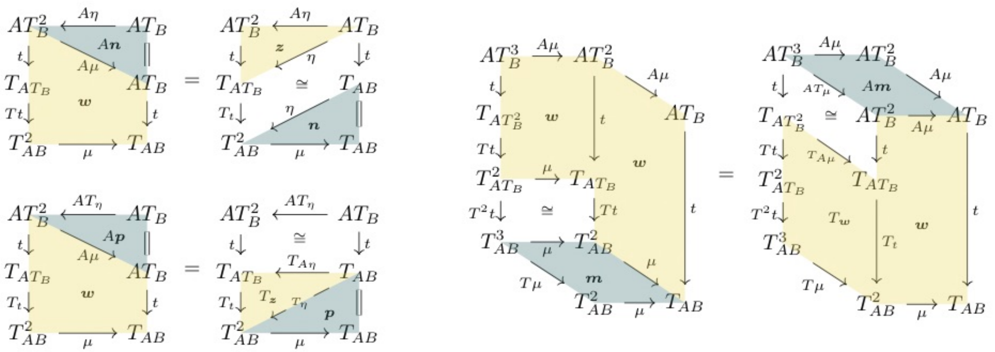


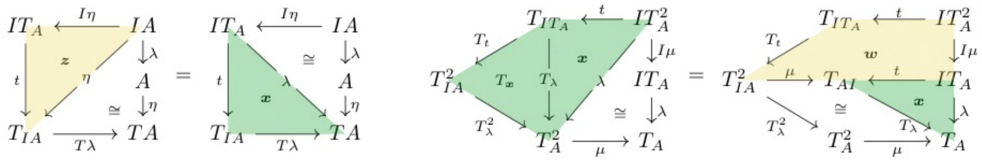
$A(IT_B)$ α $(AI)T_B$
$A\lambda$ $\mathfrak{m}$ ρT_B
At $A\boldsymbol{x}$ t
AT_{IB} AT_λ AT_B
t $\cong$ $\cong$
$T_{A(IB)}$ t $T_{(AI)B}$
$T_{A\lambda}$ $T_{\rho B}$
T_{AB}
$=$
$A(IT_B)$ α $(AI)T_B$
At AT_{IB} $\boldsymbol{y}$ t
t $T_{A(IB)}$ T_α $T_{(AI)B}$
$T_{\mathfrak{m}}$ $T_{A\lambda}$ $T_{\rho B}$
T_{AB}
$I(AT_B)$ α $(IA)T_B$
It $\mathfrak{l}$ λ λT_B
IT_{AB} $\cong$ AT_B t
t λ $\cong$
$T_{I(AB)}$ $\boldsymbol{x}$ t $T_{(IA)B}$
T_λ $T_{\lambda B}$
T_{AB}
$=$
$I(AT_B)$ α $(IA)T_B$
It IT_{AB} $\boldsymbol{y}$ t
t $T_{I(AB)}$ T_α $T_{(IA)B}$
$T_{\mathfrak{l}}$ T_λ $T_{\lambda B}$
T_{AB}
$A((BC)T_D)$ α $(A(BC))T_D$
$A\alpha$ $\mathfrak{p}$ αT_D
$A(B(CT_D))$ α α $((AB)C)T_D$
$A(Bt)$ $A(BT_{CD})$ $\cong$ $(AB)(CT_D)$
At $(AB)t$ α $\boldsymbol{y}$ t
$AT_{B(CD)}$ $(AB)T_{CD}$
t $\boldsymbol{y}$ t
$T_{A(B(CD))}$ $T_{((AB)C)D}$
T_α $T_{(AB)(CD)}$ T_α
$\|$
$A((BC)T_D)$ α $(A(BC))T_D$
$A\alpha$ αT_D
$A(B(CT_D))$ At $((AB)C)T_D$
$A(Bt)$ $A\boldsymbol{y}$ $\boldsymbol{y}$ t
$A(BT_{CD})$ $AT_{(BC)D}$ $\cong$
At AT_α t t
$AT_{B(CD)}$ $\cong$ $T_{A((BC)D)}$ T_α $T_{(A(BC))D}$
t $T_{A\alpha}$ $T_{\mathfrak{p}}$ $T_{\alpha D}$
$T_{A(B(CD))}$ $T_{((AB)C)D}$
T_α $T_{(AB)(CD)}$ T_α

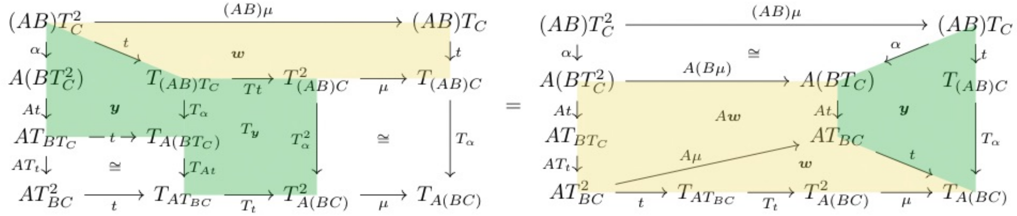


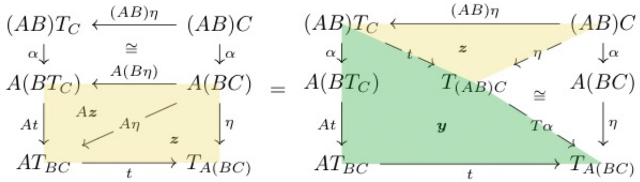












Another view : strengths as actions of $(\mathcal{B}, \otimes)$ on $\mathcal{B}_T$

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$$A \,,\; B \longmapsto A \otimes B$$

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Theorem:

strengths for T $\simeq$ actions $*: \mathcal{B}\times\mathcal{B}_T \longrightarrow \mathcal{B}_T$

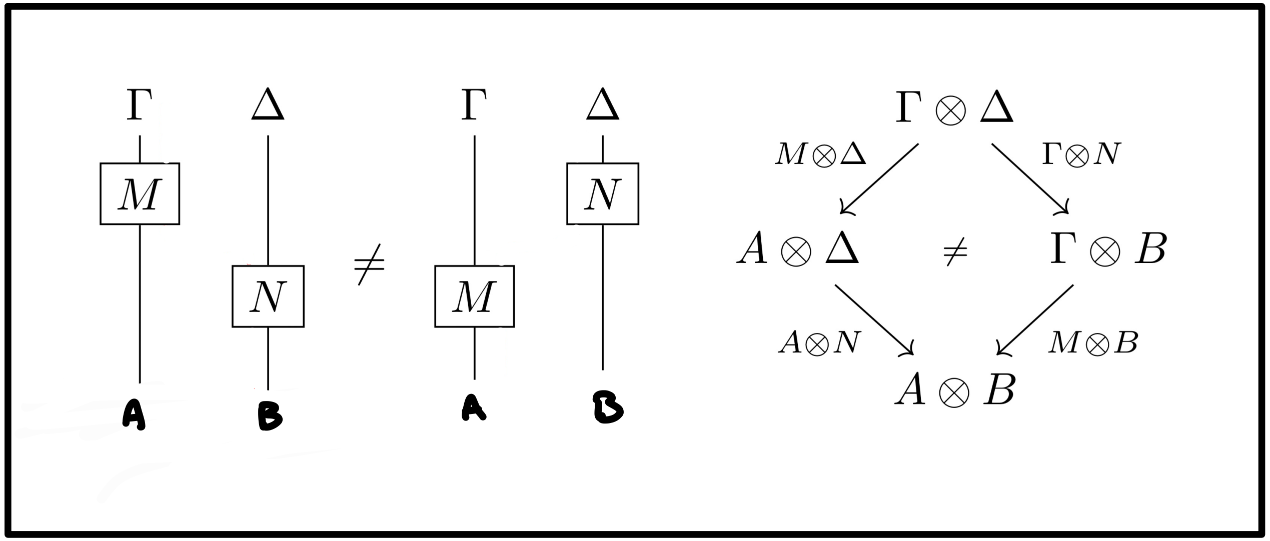
s.t.

$$\begin{array}{ccc} \mathcal{B}\times\mathcal{B}_T & \xrightarrow{*} & \mathcal{B}_T \\ \uparrow & \cong & \uparrow \\ \mathcal{B}\times\mathcal{B} & \xrightarrow[\otimes]{} & \mathcal{B} \end{array}$$

Premonoidal bicategories

premonoidal cats.
[Power & Robinson 97]

$\mathcal{B}_T$ has a tensor product, but no interchange law:



$\Gamma \vdash M : A$

$\Delta \vdash N : B$

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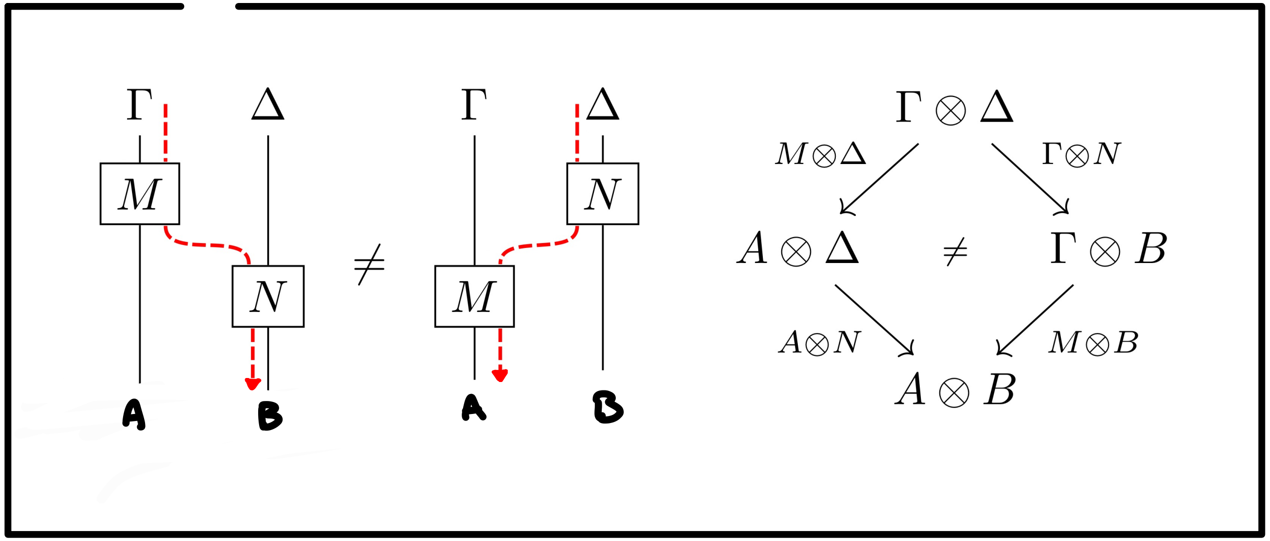

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A **premonoidal bicategory** K has

$-\otimes A$ and $A\otimes -$ for all $A \in K$

+ the coherence data for a monoidal bicategory.

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+ the coherence data for a monoidal bicategory.

Prop: $(\mathcal{B}, \otimes)$ symmetric monoidal bicategory

T strong pseudomonad

$\Rightarrow \mathcal{B}_T$ is premonoidal.

$$\mathcal{B} \longrightarrow \mathcal{B}_T$$

values — computations

Dialogue bicategories

Dialogue categories [Melliès & Tabareau]

(tensor & negation)

A dialogue bicategory is

a symmetric monoidal $\mathcal{B}$

an object $\perp$

such that $\mathcal{B}(A \otimes B, \perp) \simeq \mathcal{B}(A, \neg B)$ for some $\neg B$.

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Symm. monoidal closed, $\bot$, $\neg A = A \multimap \bot$ $\subseteq$ Dialogue

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A dialogue bicategory is
- a symmetric monoidal $\mathcal{B}$
- an object $\perp$

such that $\mathcal{B}(A \otimes B, \perp) \simeq \mathcal{B}(A, \neg B)$ for some $\neg B$.

*-autonomous: $(A \multimap \perp) \multimap \perp \cong A$ $\subseteq$ Symm. monoidal closed, $\perp$; $\neg A = A \multimap \perp$ $\subseteq$ Dialogue

Dialogue bicategories

Dialogue categories [Melliès & Tabareau]

(tensor & negation)

A dialogue bicategory is

a symmetric monoidal $\mathcal{B}$

an object $\bot$

such that $\mathcal{B}(A \otimes B, \bot) \simeq \mathcal{B}(A, \neg B)$ for some $\neg B$.

Key properties:

- $\neg$ is a pseudofunctor $\mathcal{B} \longrightarrow \mathcal{B}^{op}$
- $\neg\neg$ is a strong pseudomonad on $\mathcal{B}$
- $\mathcal{B}_{\neg\neg}$ is premonoidal

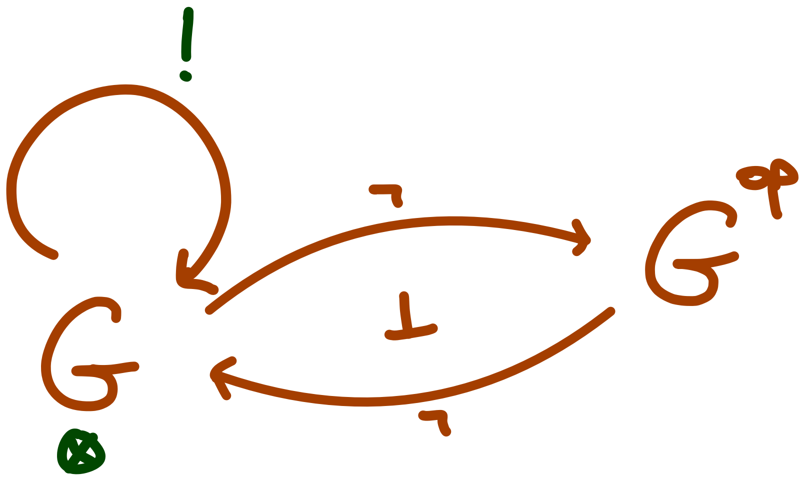
This talk

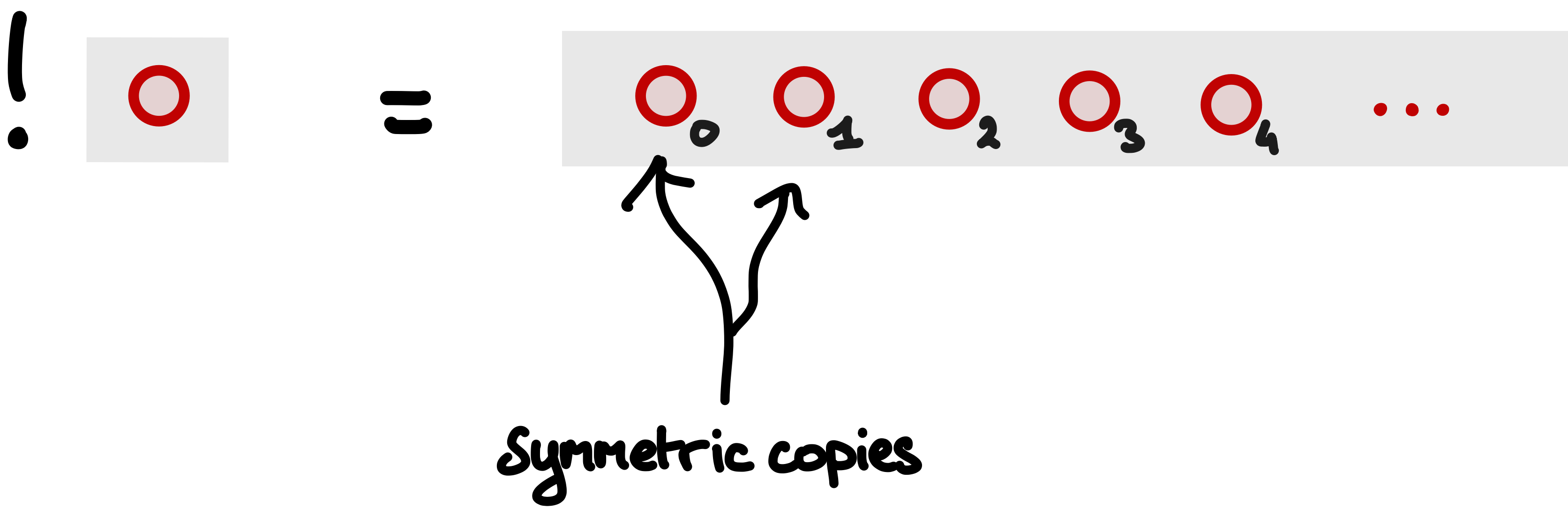
I. A bicategory of games

II. Strong pseudomonads

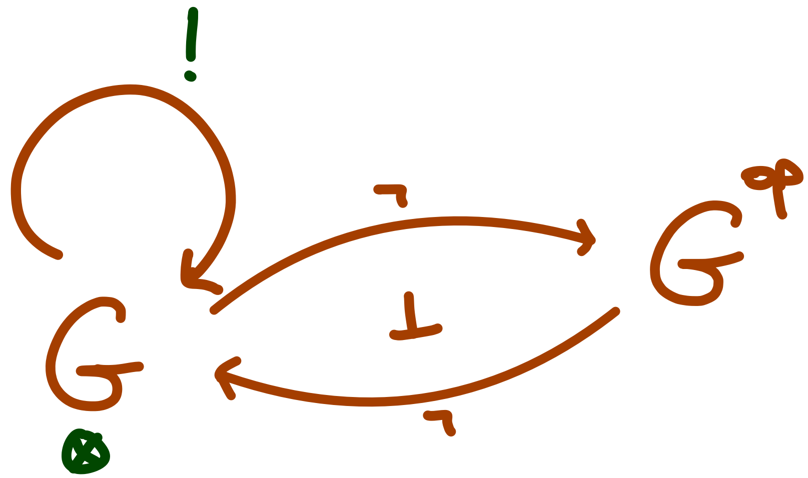
→ III. Resources and symmetries

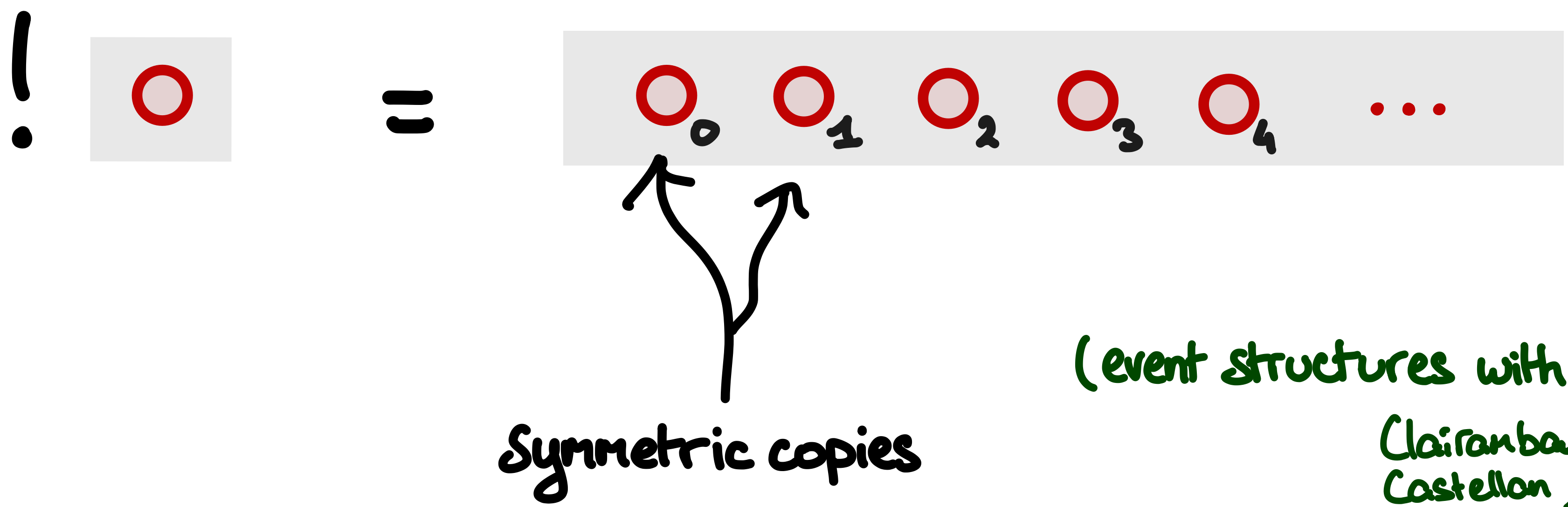
Adding an exponential modality:





Adding an exponential modality:

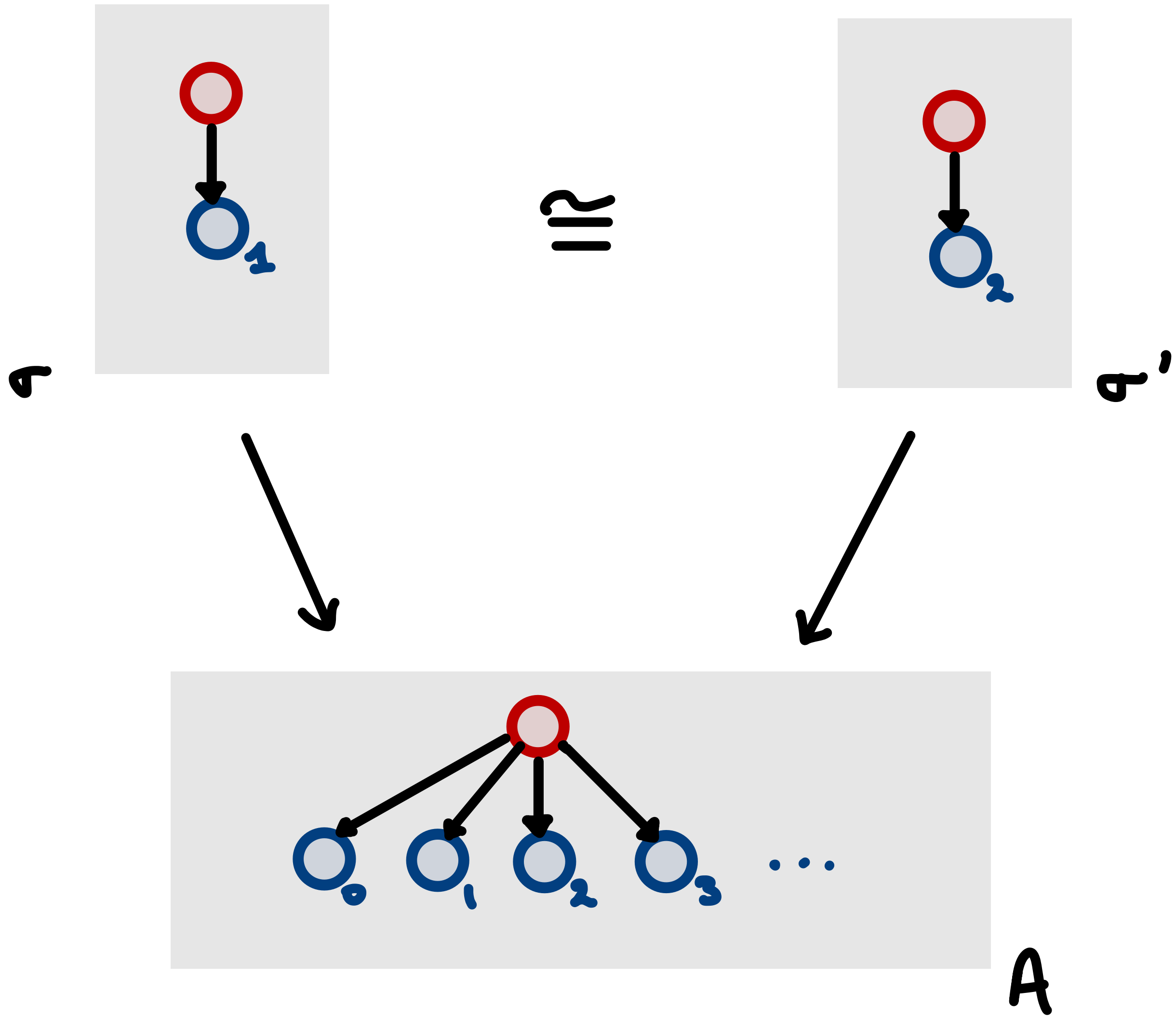




(event structures with symmetry)

Clairambault, Castellan, Winskel

Strategies playing symmetrically :



Strategies need to be bi-invariant:

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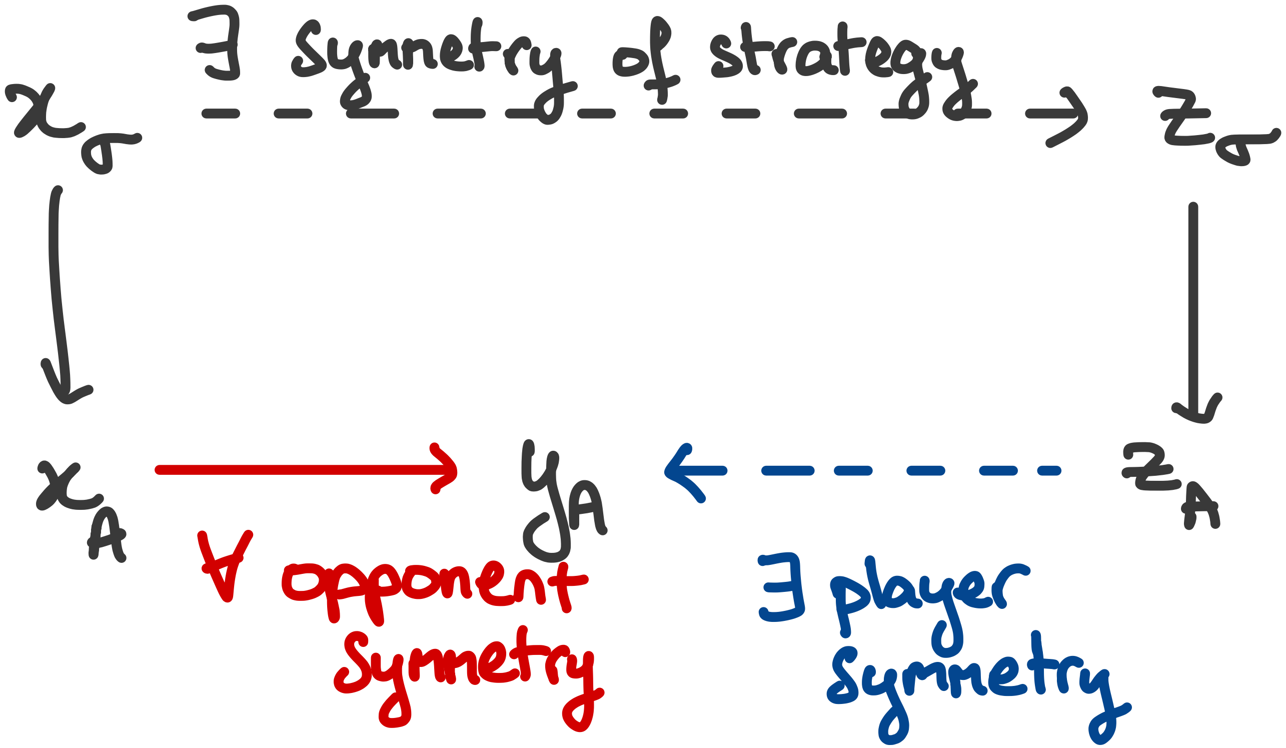
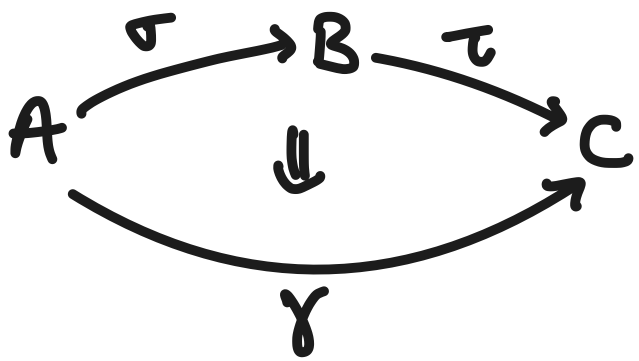
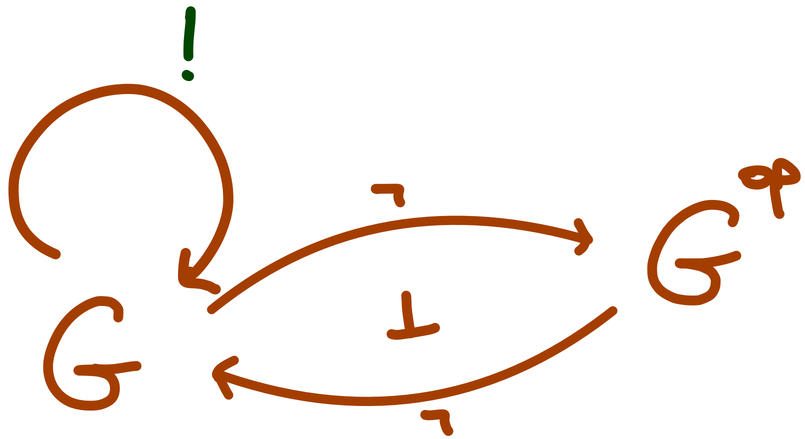




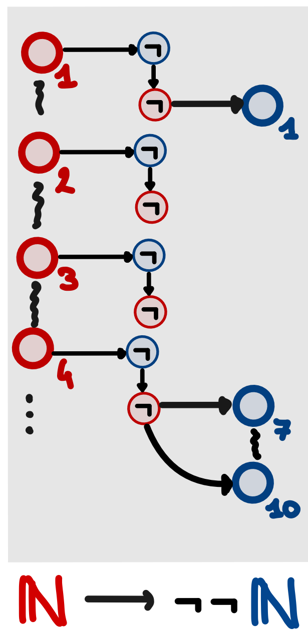
idea: send synchronized pair (x_σ, x_τ) up to symmetry to x_γ.

Thm. The virtual 2-category of games with symmetry, strategies, and multimaps is representable.

Summary



- Generalize the foundations of game semantics from categories to bicategories



- 2-dimensional setting: "proof-relevant" all the time

- Other applications of premonoidal bicategories: graded monads, "PARA" construction