Dissymetrical Linear Logics

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Introduction

- **Topics**: exponentials “!” and “?” of Linear Logic (the modalities dedicated to non linear effect in cut-elimination/computation)
- **Goals:**
  - to evaluate the (often heard) “slogan”:

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    “!” is the intuitionistic exponential, while “?” is the classical one.
  - to design “intermediate (computational) logics” between Intuitionistic Linear Logic (ILL) and Classical Linear Logic (CLL), in other words:
  - to extend the “intuitionistic part of the proofs dynamic” by a complementary “classical like” dynamic being weaker than the full classical one.

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- **Framework:** sequent calculus CLL for classical linear logic.
  - Version with **bilateral** sequents $\Gamma \vdash \Delta$ (Γ, Δ multi-sets of formulas).
  - Focus **only** on exponentials (! and ?). But all results fully compatible with adding multiplicatives, additives and $1^{\mathbf{st}}$ and $2^{\mathbf{d}}$ order quantifiers.

About LL's exponentials "!" and "?"

- **Statically:**
  - only exponentiated formulas can be subject to structural rules, more precisely:
    - formulas $!A$, on the left side of sequents;
    - formulas $?A$, on the right side.

Structural rules in CLL

$$\text{!-ctr}\ \frac{\Gamma, !A, !A \vdash \Delta}{\Gamma, !A \vdash \Delta} \qquad \text{!-w}\ \frac{\Gamma \vdash \Delta}{\Gamma, !A \vdash \Delta} \qquad\qquad \frac{\Gamma \vdash ?A, ?A, \Delta}{\Gamma \vdash ?A, \Delta}\ \text{?-ctr} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash ?A, \Delta}\ \text{?-w}$$

  - In Intuitionistic Linear Logic (ILL):
    - sequents are mono-conclusion, i.e. have shape: $\Gamma \vdash A$.
    - no room for right structural rules, hence no need for the "?" exponential in ILL.
  - Picture: "!" is devoted to left structural rules and "?" to right structural rules. Thus the slogan (often heard) :

    "!" is "the intuitionistic exponential", while "?" is "the classical one".
- **Dynamically:**
  - exponentials take in charge the "structural part" of the computational process
  - they decompose it in **four kind of actions/effects**.

Let us quickly present/recall the dynamic associated to exponentials, in the simpler case of ILL, Intuitionistic Linear Logic

Sequent calculus for Intuitionistic Linear Logic (ILL)

Sequent calculus ILL (exponential fragment)

Sequent calculus for Intuitionistic Linear Logic (ILL)

Sequent calculus ILL (exponential fragment)

Identity rules

$$\frac{}{A \vdash A}\,\textbf{ax} \qquad\qquad \frac{\Gamma \vdash A \qquad \Gamma', A \vdash C}{\Gamma, \Gamma' \vdash C}\,\textbf{cut}$$

!-structural rules

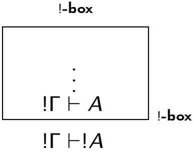
$$\textbf{!-ctr}\,\frac{\Gamma, !A, !A \vdash C}{\Gamma, !A \vdash C} \qquad\qquad \textbf{!-w}\,\frac{\Gamma \vdash C}{\Gamma, !A \vdash C}$$

Introduction rules for the !-exponential

$$\textbf{!-der}\,\frac{\Gamma, A \vdash C}{\Gamma, !A \vdash C} \qquad\qquad \frac{!\Gamma \vdash A}{!\Gamma \vdash !A}\,\textbf{!-box} \quad \text{(a.k.a. the } \textbf{!-promotion}\text{)}$$

To present or render visible the four kinds of actions/effects of exponentials during the cut-elimination process, **nothing compares to the proofnet syntax and its exponential boxes**.

Reason why the !-promotion $\frac{!\Gamma \vdash A}{!\Gamma \vdash !A}\,\textbf{!-box}$ will be noted



!-box

$$\vdots$$

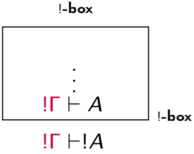
$!\Gamma \vdash A$ **!-box**

$!\Gamma \vdash !A$

Boxed notation: sequent calculus as a notation for proofnets

$$\frac{!\Gamma \vdash A}{!\Gamma \vdash !A} \ \textbf{!-box (a.k.a. !-promotion)}$$

Terminology:



!-box

$\vdots$

$!\Gamma \vdash A$

!-box

$!\Gamma \vdash !A$

"the doors" of the !-box

the main formula of the !-box

Advantages of the box notation:

- it renders properly visible that only subproofs enclosed into boxes will be subject to non linear effects : **duplications**, **erasing**.

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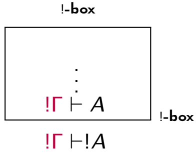

!-box

$$\vdots$$

$!\Gamma \vdash A$

!-box

$!\Gamma \vdash !A$

"the doors" of the !-box

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Advantages of the box notation:

- it renders properly visible that only subproofs enclosed into boxes will be subject to non linear effects : **duplications**, **erasing**.
- it induces a natural notion of depth of a formula (defined as the maximal number of boxes that "it has to cross" in order to "find the external world"). And the introduction rules for exponentials (to be presented) are **the (only) rules inducing depth's increasing** or **depth's decreasing** during the computation. As we will see.

1st kind of action: duplication of a box (content included)

Duplication of a !-box

(the $\overset{\textbf{!-ctr}}{\rightsquigarrow}$ elementary reduction step)

$$\dfrac{\begin{array}{c}\overset{\textbf{!-box}}{\boxed{\begin{array}{c}\vdots\\ !\Gamma \vdash A\end{array}}}\\ !\Gamma \vdash !A\end{array} \qquad \textbf{!-ctr}\ \dfrac{\begin{array}{c}\vdots\\ \Gamma', !A, !A \vdash C\end{array}}{\Gamma', !A \vdash C}}{!\Gamma, \Gamma' \vdash C}\ \textbf{cut}$$

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$$\overset{\textbf{!-ctr}}{\rightsquigarrow} \quad \textbf{!-ctr}\ \begin{array}{c}\dfrac{\begin{array}{c}\textbf{!-box}\\ \boxed{\begin{array}{c}\vdots\\ !\Gamma \vdash A\end{array}}\\ !\Gamma \vdash !A\end{array} \qquad \dfrac{\begin{array}{c}\textbf{!-box}\\ \boxed{\begin{array}{c}\vdots\\ !\Gamma \vdash A\end{array}}\\ !\Gamma \vdash !A\end{array} \qquad \begin{array}{c}\vdots\\ \Gamma', !A, !A \vdash C\end{array}}{!\Gamma, \Gamma', !A \vdash C}\ \textbf{cut}}{!\Gamma, !\Gamma, \Gamma' \vdash C}\ \textbf{cut}\\ \hline\hline !\Gamma, \Gamma' \vdash C\end{array}$$

2^{d} kind of action: erasure of a box (its content included)

Erasure of a !-box

(the $\overset{\textbf{!-w}}{\rightsquigarrow}$ elementary reduction step)

$$\dfrac{\begin{array}{c}\textbf{!-box}\\ \boxed{\begin{array}{c}\vdots\\ !\Gamma \vdash A\end{array}}\\ !\Gamma \vdash !A\end{array} \qquad \textbf{!-w}\;\dfrac{\begin{array}{c}\vdots\,\pi\\ \Gamma' \vdash C\end{array}}{!A, \Gamma' \vdash C}}{!\Gamma, \Gamma' \vdash C}\;\textbf{cut}$$

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3^d kind of action: swallowing of a box (content included)

Swallowing of a !-box by a !-box

(the $\overset{\textbf{!-door}}{\underset{\textbf{[!-box]}}{\rightsquigarrow}}$ elementary reduction step)

$$\dfrac{\begin{array}{c}\textbf{!-box (1)}\\ \boxed{\begin{array}{c}\vdots\,\pi\\ !\Gamma\vdash A\end{array}}\\ !\Gamma\vdash !A\end{array}\qquad\qquad \begin{array}{c}\textbf{!-box (2)}\\ \boxed{\begin{array}{c}\vdots\,\pi'\\ !A,!\Gamma'\vdash B\end{array}}\\ !A,!\Gamma'\vdash !B\end{array}}{!\Gamma,!\Gamma'\vdash !B}\,\textbf{cut}$$

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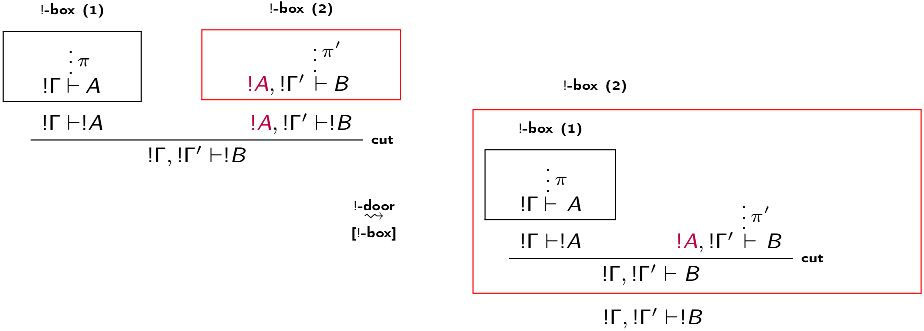

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$$\overset{\textbf{!-door}}{\underset{\textbf{[!-box]}}{\rightsquigarrow}}$$

$$\begin{array}{c}\textbf{!-box (2)}\\ \boxed{\dfrac{\begin{array}{c}\textbf{!-box (1)}\\ \boxed{\begin{array}{c}\vdots\,\pi\\ !\Gamma \vdash A\end{array}}\\ !\Gamma \vdash !A\end{array} \qquad \begin{array}{c}\vdots\,\pi'\\ !A, !\Gamma' \vdash B\end{array}}{!\Gamma, !\Gamma' \vdash B}\ \textbf{cut}}\\ !\Gamma, !\Gamma' \vdash !B\end{array}$$

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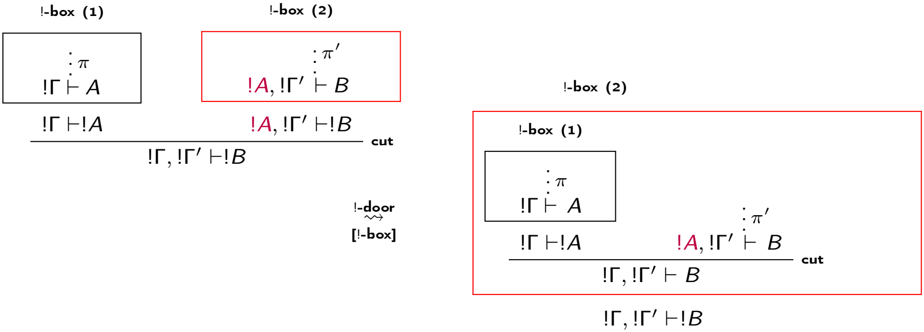

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Remark: that's the only step which entails **depth's increasing**

4th kind of action: undraw a box (not its content)

Undrawing a !-box

(the $\overset{\textbf{!-der}}{\rightsquigarrow}$ elementary reduction step)

$$\dfrac{\begin{array}{c}\textbf{!-box}\\ \boxed{\begin{array}{c}\vdots\,\pi\\ !\Gamma \vdash A\end{array}}\\ !\Gamma \vdash !A\end{array} \qquad \textbf{!-der}\;\dfrac{\begin{array}{c}\vdots\\ \Gamma',\, A \vdash C\end{array}}{\Gamma', !A \vdash C}}{!\Gamma, \Gamma' \vdash C}\;\textbf{cut}$$

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Remark: that's the only step which entails **depth's decreasing**.

Variants of “!” and “?” specialized in some of those 4 effects

NB: there exist variants of exponentials, dedicated to control only some of the 4 effects (duplication, erasing, depth-increasing, depth-decreasing). For example:

- **exponentials with a specialization in specific non-linear effects:**
  - a $\underset{c}{!}$ dedicated only to contraction (weakenings not allowed for $\underset{c}{!}A$ formulas)
  - a $\underset{w}{!}$ dedicated only to weakening (contractions not allowed for $\underset{w}{!}A$ formulas)
  - even a $\underset{\emptyset}{!}$ dedicated to none of the non-linear actions, hence controlling only depth-increasing/decreasing

  To get closure by cut-elimination, of course, the introduction rules have to be carefully designed (cf. Danos-Joinet-Schellinx's paper about exponentials).

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- **exponentials dedicated to the control of specific depth’s modifications:**
  - no depth increasing : functorial “!-promotion” alias functorial !-box (here intuitionistic case): $\dfrac{\Gamma \vdash A}{!\Gamma \vdash !A}$

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  - no depth increasing : functorial "!-promotion" alias functorial !-box (here intuitionistic case): $\dfrac{\Gamma \vdash A}{!\Gamma \vdash !A}$
  - no depth decreasing: when the standard dereliction is dropped (in presence of a functorial "promotion", for instance).

It's a toolkit: (a) an exponential may well combine both kinds of specializations; (b) all variants may well "live" together (no prejudice for cut-elimination and identity axioms expansion)

Back to Classical Linear Logic

Bilateral sequent calculus for Classic Linear Logic (CLL)

- Sequents are now multi-conclusions ones : $\Gamma \vdash \Delta$
- Right structural rules will be dealt with the "?" exponential
- NB: again, we skip the presentation of rules for multiplicative and additive connectives and for quantifiers.

Intuitionistic case: ILL **(where $\sharp(\Delta), \sharp(\Delta') = 1$)**

Identity rules

$$\frac{}{A \vdash A}\,\textbf{ax} \qquad \frac{\Gamma \vdash A \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta'}\,\textbf{cut}$$

Introduction rules for exponentials

$$\frac{!\Gamma \vdash A}{!\Gamma \vdash !A}\,\textbf{!-box} \qquad \textbf{!-der}\,\frac{\Gamma, A \vdash \Delta}{\Gamma, !A \vdash \Delta}$$

Structural rules

$$\textbf{!-ctr}\,\frac{\Gamma, !A, !A \vdash \Delta}{\Gamma, !A \vdash \Delta} \qquad \textbf{!-w}\,\frac{\Gamma \vdash \Delta}{\Gamma, !A \vdash \Delta}$$

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Structural rules

$$\mathbf{!\text{-}ctr}\,\frac{\Gamma, !A, !A \vdash \Delta}{\Gamma, !A \vdash \Delta} \qquad \mathbf{!\text{-}w}\,\frac{\Gamma \vdash \Delta}{\Gamma, !A \vdash \Delta} \qquad \frac{\Gamma \vdash ?A, ?A, \Delta}{\Gamma \vdash ?A, \Delta}\,\mathbf{?\text{-}ctr} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash ?A, \Delta}\,\mathbf{?\text{-}w}$$

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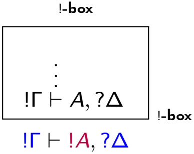
Structural rules

$$\mathbf{!\text{-}ctr}\,\frac{\Gamma, !A, !A \vdash \Delta}{\Gamma, !A \vdash \Delta} \qquad \mathbf{!\text{-}w}\,\frac{\Gamma \vdash \Delta}{\Gamma, !A \vdash \Delta} \qquad\qquad \frac{\Gamma \vdash ?A, ?A, \Delta}{\Gamma \vdash ?A, \Delta}\,\mathbf{?\text{-}ctr} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash ?A, \Delta}\,\mathbf{?\text{-}w}$$

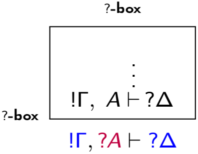
Boxed notation: two kinds of boxes

$$\frac{!\Gamma \vdash A, ?\Delta}{!\Gamma \vdash !A, ?\Delta}\ \text{!-}\mathbf{box} \qquad\qquad \text{?-}\mathbf{box}\ \frac{!\Gamma, A \vdash ?\Delta}{!\Gamma, ?A \vdash ?\Delta}$$



$$\boxed{\begin{matrix} \vdots \\ !\Gamma \vdash A, ?\Delta \end{matrix}}\ \text{!-}\mathbf{box}$$

$$!\Gamma \vdash !A, ?\Delta$$



$$\text{?-}\mathbf{box}\ \boxed{\begin{matrix} \vdots \\ !\Gamma, A \vdash ?\Delta \end{matrix}}$$

$$!\Gamma, ?A \vdash ?\Delta$$

doors of the !-box

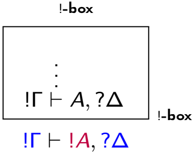
main formula of the !-box

doors of the ?-box

main formula of the ?-box

Boxed notation: two kinds of boxes

$$\frac{!\Gamma \vdash A, ?\Delta}{!\Gamma \vdash !A, ?\Delta}\ !\text{-}\mathbf{box} \qquad ?\text{-}\mathbf{box}\ \frac{!\Gamma, A \vdash ?\Delta}{!\Gamma, ?A \vdash ?\Delta}$$



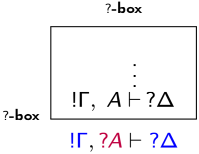
!-**box**

$$\vdots$$

$!\Gamma \vdash A, ?\Delta$

!-**box**

$!\Gamma \vdash !A, ?\Delta$



?-**box**

$$\vdots$$

$!\Gamma, A \vdash ?\Delta$

?-**box**

$!\Gamma, ?A \vdash ?\Delta$

doors of the !-box

main formula of the !-box

doors of the ?-box

main formula of the ?-box

Dynamically, the respective jobs of ! and ? seem well distinct:

- **!**-contractions/weakenings/derelictions only duplicate/erase/undraw **!**-boxes
- **?**-contractions/weakenings/derelictions only duplicate/erase/undraw **?**-boxes

For instance:

An example of the specialization of roles: boxes erasing

Erasure of a !-box by a (left) !-weakening

(the $\overset{\textbf{!-w}}{\leadsto}$ elementary reduction step)

$$\begin{array}{c}
\begin{array}{c}\textbf{!-box}\\ \boxed{\begin{array}{c}\vdots\\ !\Gamma\vdash ?\Delta, A\end{array}}\\ !\Gamma\vdash ?\Delta, !A\end{array}
\quad
\textbf{!-w}\ \begin{array}{c}\vdots\,\pi\\ \Gamma'\ \vdash\ \Delta'\\ \hline \Gamma', !A\vdash\Delta'\end{array}
\\ \hline
!\Gamma, \Gamma'\vdash ?\Delta, \Delta'
\end{array}\ \textbf{cut}
\qquad \overset{\textbf{!-w}}{\leadsto} \qquad
\textbf{!-w}\ \begin{array}{c}\vdots\,\pi\\ \Gamma'\ \vdash\ \Delta'\\ \hline\hline !\Gamma, \Gamma'\vdash ?\Delta, \Delta'\end{array}\ \textbf{?-w}$$

Erasure of a ?-box, by a (right) ?-weakening

(the $\overset{\textbf{?-w}}{\leadsto}$ elementary reduction step)

$$\begin{array}{c}
\begin{array}{c}\vdots\,\pi\\ \Gamma\ \vdash\ \Delta\\ \hline \Gamma\vdash\Delta, ?A\end{array}\ \textbf{?-w}
\quad
\begin{array}{c}\textbf{?-box}\\ \boxed{\begin{array}{c}\vdots\\ A, !\Gamma'\vdash ?\Delta'\end{array}}\\ ?A, !\Gamma'\vdash ?\Delta'\end{array}
\\ \hline
\Gamma, !\Gamma'\vdash\Delta, ?\Delta'
\end{array}\ \textbf{cut}
\qquad \overset{\textbf{?-w}}{\leadsto} \qquad
\textbf{!-w}\ \begin{array}{c}\vdots\,\pi\\ \Gamma\ \vdash\ \Delta\\ \hline\hline \Gamma, !\Gamma'\vdash\Delta, ?\Delta'\end{array}\ \textbf{?-w}$$

“?” as the classical exponential, “!” as the intuitionistic one

Because of that specialization, the general picture seems to be :

- Dynamically, ? is in charge of the “classical” non linear effects : “classical” duplication/erasing/undrawing = duplication/erasing/undrawing of classical boxes, namely the ?-boxes
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- Hence the slogan : “?” is “the classical exponential”
- The picture is wrong or confusing, however. Indeed, in CLL, neither of the two exponentials has an autonomous existence. They are statically anf thus dynamically interdependent:

In CLL, ! and ? are statically interdependent

From the statical viewpoint, the introduction rule for ! (resp. ?) **requires** the existence of introduction rules for ? (resp. !).

$$\frac{!\Gamma \vdash A, ?\Delta}{!\Gamma \vdash !A, ?\Delta}\ \textbf{!-box} \qquad\qquad \textbf{?-box}\ \frac{!\Gamma, A \vdash ?\Delta}{!\Gamma, ?A \vdash ?\Delta}$$

To introduce one of the two, the other one has to be already present (in general)

That “statical” interdependence entails a “dynamical” one

- This *static interdependency* also induces their *dynamic interdependency*:
- e.g. when a a !-contraction duplicates a box (hence a !-box), new ?-contractions are created which may themselves come to duplicate ?-boxes:

A (left) !-contraction duplicates a !-box

$$\dfrac{\dfrac{\overset{\textbf{!-box}}{\boxed{\begin{array}{c}\vdots\\ !\Gamma \vdash ?\Delta, A\end{array}}}}{!\Gamma \vdash ?\Delta, !A} \qquad !\text{-ctr}\,\dfrac{\begin{array}{c}\vdots\\ \Gamma', !A, !A \vdash \Delta'\end{array}}{\Gamma', !A \vdash \Delta'}}{!\Gamma, \Gamma' \vdash ?\Delta, \Delta'}\,\textbf{cut}$$

$$\overset{!\text{-ctr}}{\rightsquigarrow}\quad !\text{-ctr}\ \begin{array}{c}\dfrac{\dfrac{\overset{\textbf{!-box}}{\boxed{\begin{array}{c}\vdots\\ !\Gamma \vdash ?\Delta, A\end{array}}}}{!\Gamma \vdash ?\Delta, !A} \qquad \dfrac{\dfrac{\overset{\textbf{!-box}}{\boxed{\begin{array}{c}\vdots\\ !\Gamma \vdash ?\Delta, A\end{array}}}}{!\Gamma \vdash ?\Delta, !A} \qquad \begin{array}{c}\vdots\\ \Gamma', !A, !A \vdash \Delta'\end{array}}{!\Gamma, \Gamma', !A \vdash ?\Delta, \Delta'}\,\textbf{cut}}{!\Gamma, !\Gamma, \Gamma' \vdash ?\Delta, ?\Delta, \Delta'}\,\textbf{cut} \\ \hline\hline !\Gamma, \Gamma' \vdash ?\Delta, \Delta'\end{array}\ ?\text{-ctr}$$

That duplication generates ?-contractions
(which, themselves, may well come to duplicate ?-boxes)

Dynamical mix up of the roles of “!” and “?”

- So, in the chain of events, !-contractions generally induce non linear effects (duplication, erasure) over ?-boxes, i.e. the interdependency leads in a way to mix up the roles of ? and ! (which thus cannot be specific, differenciated).
- So, typically, it is not possible to choose to have, say, the ? exponential specialized in contraction, while the ! would remain standard (i.e. endowed with all structural roles: contraction and weakening). Both exponentials would then play a specific role, but such a sequent calculus would not be closed by cut-elimination.

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Another symptom of that “confusion of roles” : no door is “by essence” door of, say, a ?-box. Indeed, a door of a ?-box may well “become” door of a !-box, through the “shallowing” steps (So, typically, one cannot choose the functorial version for, say, the ?-box rule while the !-box would remain standard: both exponentials would then play a specific role, but such a system is not closed by cut-elimination).

Let us focus on that:

When a the !-door of a !-box “becomes” a !-door of a ?-box

In CLL, there are four kinds of swallowings: swallowing of a !-box by a !-box (the only one which could happen in ILL); swallowing of a !-box by a ?-box; swallowing of a ?-box by a ?-box; swallowing of a ?-box by a !-box.

The three new, similar elementary reduction steps are noted: $\overset{\text{!-door}}{\underset{\text{[?-box]}}{\rightsquigarrow}}$; $\overset{\text{?-door}}{\underset{\text{[?-box]}}{\rightsquigarrow}}$ and $\overset{\text{?-door}}{\underset{\text{[!-box]}}{\rightsquigarrow}}$.

Swallowing of a !-box by a ?-box

(the $\overset{\text{!-door}}{\underset{\text{[?-box]}}{\rightsquigarrow}}$ elementary reduction step)

$$\dfrac{\begin{array}{c}\text{!-box}\\ \boxed{\begin{array}{c}\vdots\,\pi\\ !\Gamma \vdash ?\Delta,\ A\end{array}}\\ !\Gamma \vdash ?\Delta, !A\end{array} \qquad \begin{array}{c}\text{?-box}\\ \boxed{\begin{array}{c}\vdots\,\pi'\\ !A, !\Gamma', B \vdash ?\Delta'\end{array}}\\ !A, !\Gamma', ?B \vdash ?\Delta'\end{array}}{!\Gamma, !\Gamma', ?B \vdash ?\Delta, ?\Delta'}\,\text{cut}$$

$$\overset{\text{!-door}}{\underset{\text{[?-box]}}{\rightsquigarrow}}$$

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The doors of the ?-box “become” doors of a !-box.

Confusion or redundancy?

- Seen through the eyes of duality (of ? and ! in CLL), the underlined "confusion" may be simply rather understood as "redundancy".
- Indeed, in monolateral sequent calculus for CLL (where the set of formulas is quotiented by *de Morgan* equivalences, where the sequents $\vdash \Gamma$ are single sided, where the *identity constraints* of "Identity rules" are replaced by *duality constraints*) there is no "confusion of roles". Roles are univoquely attributed:
  - all the boxes are !-boxes (having only ?-doors)
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  - all the boxes are !-boxes (having only ?-doors)
  - and all the contractions (weakenings, derelictions) are ?-contractions (?-weakenings, ?-derelictions).
- Such a way for obtaining "de-confusion" is however possible only when a perfect, full symmetry prevails: in Classical Linear Logic (where ? and ! are dual).

Toward specific, well differentiated roles for “?” and “!”

- **Goal:** making the symmetrical interdependency of “?” and “!” cease (hence *breaking the symmetry* of “?” and “!” rules, hence loosing the duality !/?) in such a way that “!” and “?” eventually play distinct, specific, well differentiated roles ?

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- ILL is of course itself a symmetry breaker sub-system of CLL. But with ILL, half of the baby is gone with the bathwater: there is no more interdependency, but just because only one exponential (namely "!") survives to the treatment.

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- The purpose is to break the symmetry of CLL’s exponentials in a less drastic way than ILL does, in order to design *computational fragments* of CLL where the ?/! *symmetric interdependency* proper to CLL does not prevail anymore (even if some non symmetric dependencies between them will subsist).

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- The resulting (no more dual) exponentials ? and ! will become independent enough to be able to play proper, well differentiated roles.

Dissymetrical Linear Logics

Dissymetrical Linear Logic (DLL)

Definition

Full 'Dissymetrical Linear Logic', DLL, is the system one got by replacing (in CLL) $\dfrac{!\Gamma \vdash A, ?\Delta}{!\Gamma \vdash !A, ?\Delta}\ \textbf{!-box}$ by $\dfrac{!\Gamma \vdash A}{!\Gamma \vdash !A}\ \textbf{!-box}$ (other CLL rules being unchanged).

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Proposition The system DLL:

- is closed by cut-elimination (the potentially problematic e.r.s. $\underset{\textbf{[!-box]}}{\overset{\textbf{?-door}}{\rightsquigarrow}}$ never applies);
- is closed by expansion of identity axioms;
- is thus a computational fragment of CLL (hence is strongly normalizing – and confluent for the additive free fragment).
- It proves the same *de Morgan* equivalences than CLL, but for the !/? duality ($\neg !X \vdash ?\neg X$ is not cut free provable in DLL);
- thus has a strictly weaker expressive power than CLL's one (at the provability level);
- is evidently at least as powerful (computationally) as System F (since ILL is a sub-system of DLL).

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2. That static autonomy results in a dynamic autonomy: a !-ctr (resp. a !-w) can only duplicate (resp. erase) !-boxes;
3. !-doors are not “proper” to !-boxes: during the cut-elimination process, a door of a !-box may well “become” door of a ?-box.

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2. That static dependence results in an absence of dynamic autonomy: a ?-ctr (resp. a ?-w) may “cause” the duplication (resp. erasing) of a ?-box, but also of a !-box (Observe in passing that the dependence ?/! is no more a symmetrical interdependence as the one prevailing in CLL).

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1. At the static level, ? is however not autonomous, it depends on ! (introducing ? on the left hand side of a sequent generally requires that ! have been introduced).
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3. However, doors of ?-boxes (be them prefixed by ? or by !) are proper to ?-boxes: during the cut-elimination process, a door of a ?-box, never “becomes” door of a !-box (contrary to what happens in CLL, as we saw).

In the next subsections, we deeply take advantage of this third property, to treat the ?-contexts in specific ways.

Reinforcing the germinal dissymmetry introduced by DLL

In the rest of the talk, we use our third observation about "?" to reinforce the germinal dissymmetry introduced by DLL.

We examine two main ways to perform such a reinforcement:

- the first one by considering what we call a semi-functorial ?-promotion/box rule (next section)
- the second by considering ?-box rules using "specialized" exponentials as ?-doors (à la Danos-Joinet-Schellinx then Lellmann, Olarte, Pimentel; see also a recent work of Laurent and Bauer).

Other dissymetrical Linear Logics

Recalling functorial versions of promotions/boxes

Let us recall (in the boxed notation) the (fully) functorial versions of promotions/boxes:

$$\dfrac{\overset{\textbf{!-fbox}}{\boxed{\begin{array}{c}\vdots\\ \Gamma \vdash A,\ \Delta\end{array}}}}{!\Gamma \vdash !A, ?\Delta}\ \textbf{!-fbox} \qquad\qquad \textbf{?-fbox}\ \dfrac{\overset{\textbf{?-fbox}}{\boxed{\begin{array}{c}\vdots\\ \Gamma,\ A \vdash \Delta\end{array}}}}{!\Gamma, ?A \vdash ?\Delta}$$

and that the system fCLL, for *functorial Classical Linear Logic*, obtained by replacing in CLL the rules !-box and ?-box by their functorial version !-fbox and ?-fbox is a computational fragment of CLL.

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By the way, this is still the case, when one moreover gets rid of !-der and ?-der.

Remark: it would not be possible, starting from CLL (neither from DLL, actually), to choose to have one of the two box-rules (e.g. the ?-box) being functorial while the other one (the !-box) would remain standard (this is typical of "?" and "!"'s lack of autonomy, which **renders us unable to assign them differentiated roles**) : indeed neither resulting system is closed through $\overset{\textbf{!-door}}{\underset{[\textbf{?-fbox}]}{\rightsquigarrow}}$.

Once the symmetry !/? is broken as in DLL, however, one finds a way to assign differentiated, specialized roles to ? and !.

Semi-functorial Dissymetrical Linear Logic (system sfDLL)

Definition

We call semi-functorial Dissymetrical Linear Logic (sfDLL), the system one gets by replacing, in DLL, the ?-box rule, by its semi-functorial version: $\text{?-sf-box}\ \dfrac{!\Gamma, A \vdash \Delta}{!\Gamma, ?A \vdash ?\Delta}$ (all other rules of DLL being kept).

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Proposition sfDLL and sfDLL without the ?-dereliction rule as well are :

- closed by cut-elimination and identity axioms expansion,
- strongly normalizing (and confluent for the additive free fragment) computational sub-systems of CLL,
- weaker than sfDLL from the provability point of view (e.g. $??X \vdash ?X$ is not provable in sfDLL; and if one moreover drops ?-der, one also looses $X \vdash ?X$).

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A second example: toward a dissymmetrically specialized Linear Logic. . .

Semi-specialized Dissymetrical Linear Logic (ssDLL)

Definition

"Semi-specialized Dissymmetrical Linear Logic" (ssDLL):

Introduction rules for !:

$$!\text{-der}\,\frac{\Gamma, A \vdash \Delta}{\Gamma, !A \vdash \Delta} \qquad !\text{-box}\,\frac{!\Gamma \vdash A}{!\Gamma \vdash !A}$$

Structural rules for !:

$$!\text{-ctr}\,\frac{\Gamma, !A, !A \vdash \Delta}{\Gamma, !A \vdash \Delta} \qquad !\text{-w}\,\frac{\Gamma \vdash \Delta}{\Gamma, !A \vdash \Delta}$$

Introduction rules for specialized ? ($\underset{w}{?}$ and $\underset{c}{?}$):

$$\underset{w}{?}\text{-box}\,\frac{!\Gamma, A \vdash \underset{w}{?}\Delta}{!\Gamma, \underset{w}{?}A \vdash \underset{w}{?}\Delta} \quad \frac{\Gamma \vdash A, \Delta}{\Gamma \vdash \underset{w}{?}A, \Delta}\,\underset{w}{?}\text{-der} \qquad \underset{c}{?}\text{-box}\,\frac{!\Gamma, A \vdash \underset{c}{?}\Delta}{!\Gamma, \underset{c}{?}A \vdash \underset{c}{?}\Delta} \quad \frac{\Gamma \vdash A, \Delta}{\Gamma \vdash \underset{c}{?}A, \Delta}\,\underset{c}{?}\text{-der}$$

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Proposition ssDLL is closed by cut-elimination and identity axioms expansion. It's a strongly normalizing (and confluent, for the additive free fragment) subsystem a computational “sub-system” of CLL.

It is a toolkit

The semi-functoriality feature used in sfDLL and the semi-specialization feature just presented are fully compatible.

For instance, one could well consider (without losing the cut-elimination property etc) a system where one would replace the exponential rules $\underset{w}{?}$ and $\underset{c}{?}$ above by:

$$\underset{w}{?}\ \frac{!\Gamma, A \vdash \Delta}{!\Gamma, \underset{w}{?}A \vdash \underset{w}{?}\Delta} \qquad\qquad \underset{c}{?}\ \frac{!\Gamma, A \vdash \Delta}{!\Gamma, \underset{c}{?}A \vdash \underset{c}{?}\Delta}$$

Note that, in that last case (actually, as in the case of fully functorial Classical Linear Logic or fully Functorial Dissymetrical Linear Logic or even sfDLL), adding the corresponding dereliction rules is not compulsory (they are not needed for expansion of identity axioms).

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It is a toolkit to design intermediate systems between ILL and CLL

Conclusion: applications and future work

The systems designed in the present paper are computational systems stronger than ILL but weaker than CLL.

This suggests the following kinds of applications for them.

1. As they all extend second order ILL (in which System F can be represented) by introducing the ?-modality in charge of right structural rules (a first step toward "classical logic"), it could be interesting to (try to) use Dissymetrical Linear Logics to capture or classify specific "classical algorithms" (weaker however than the full ones, in the spirit of implicit computational complexity approaches) or at least specific "classical" strategies.

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2. As they are intermediate linear logics "between ILL and CLL", it could be interesting to (try to) use Dissymetrical Linear Logics to embed/interpret in Linear Logic intermediate logics "between intuitionistic logic and classical logic" (or even to embed/interpret multi-conclusions formulations of intuitionistic logic).