Taylor expansion for the infinitary λ-calculus

Rémy Cerda, Aix-Marseille Université, I2M
(jww. Lionel Vaux Auclair)

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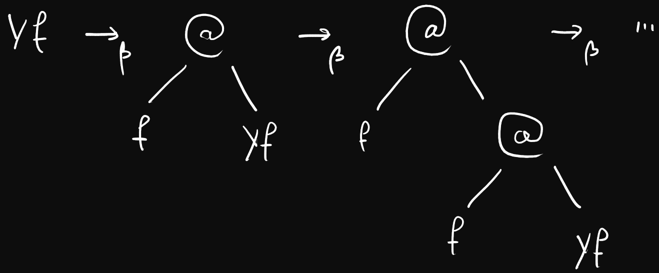
Outline

- The characters
  - Infinitary λ-calculi
  - The Taylor expansion
- The story
  - We have a nice (?) theorem
  - It tells about the Taylor expansion
  - It also tells about the infinitary λ-calculus
- Future adventures

The characters

The well known $Y = \lambda f.(\lambda x.(f)(x)x)\lambda x.(f)(x)x$ does not normalise, but still computes "something":



Infinitary λ-calculi?

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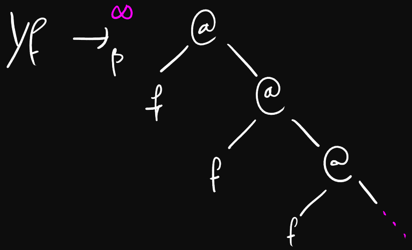

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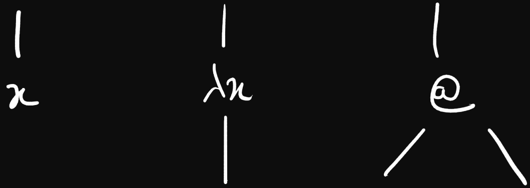
- Original definition: metric completion on the syntactic trees (**infinitary terms**) and strong notion of convergence (**infinitary reductions**).
- **Coinductive** reformulation in the 2010s (Endrullis and Polonsky 2013).

Our favorite infinitary λ-calculus: Λ_∞^{001}



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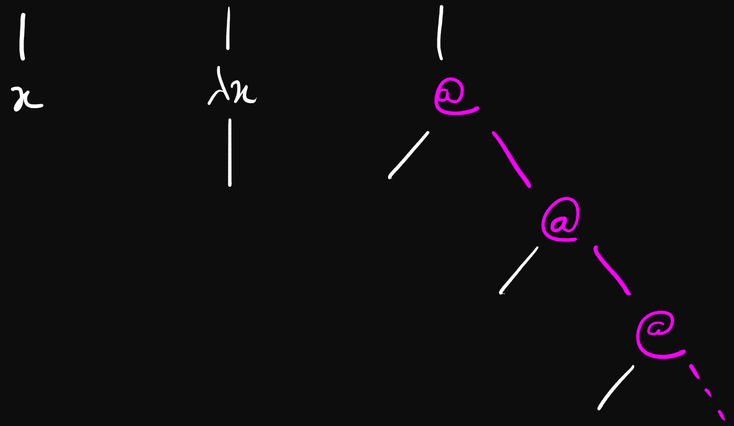

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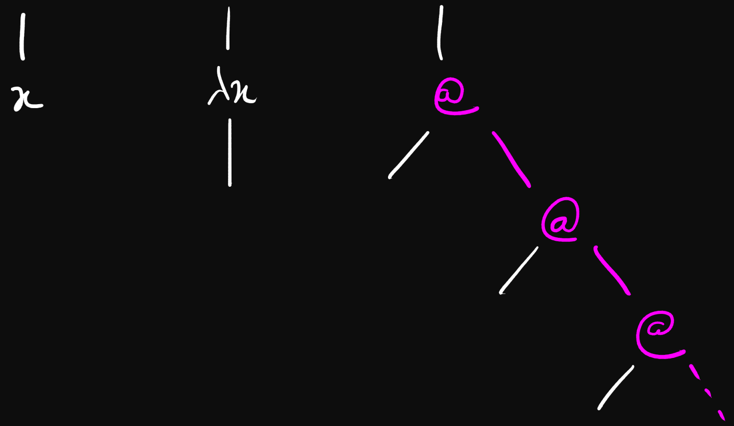
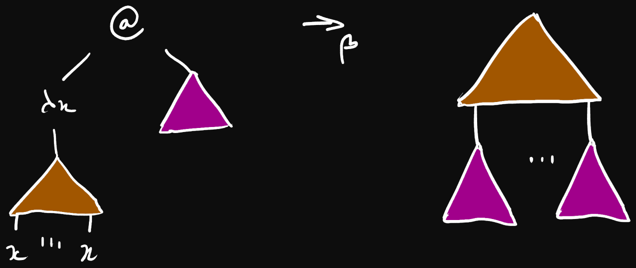


... and Λ_∞^{001} is endowed with a reduction $\longrightarrow_\beta^\infty$.

We would like to have a convenient framework to provide **finite approximations** of these infinite terms and reductions.

What is this thing called β-reduction?



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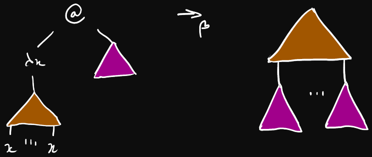

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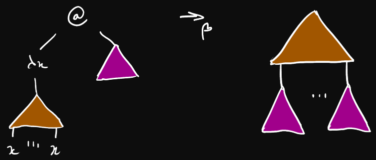

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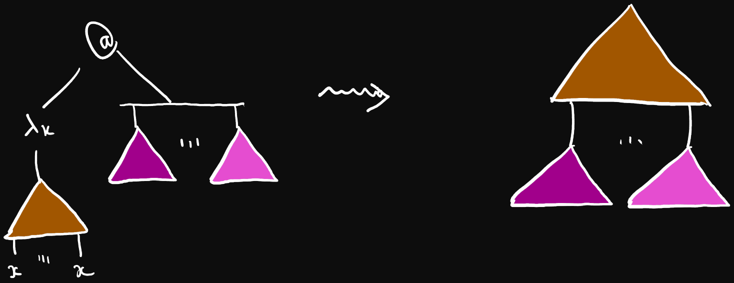

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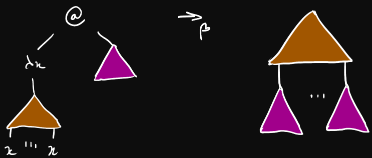

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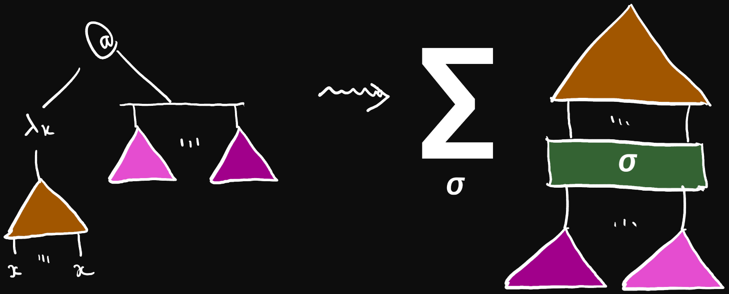
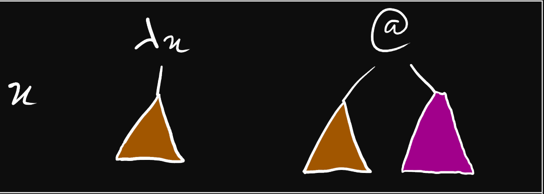
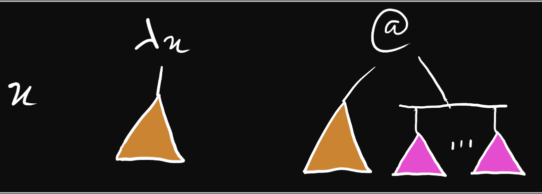
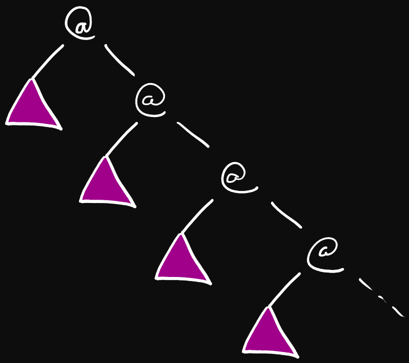


$\mathcal{T}(-)$ maps a term to the sum of its approximants.

Terms	
Approximants	

And for infinite terms?

Terms may look like this:



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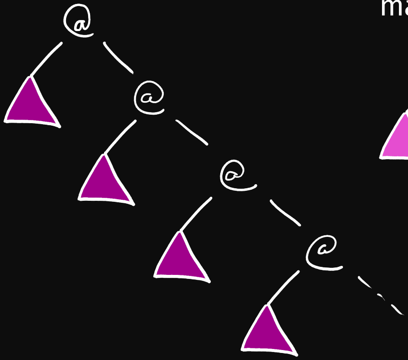
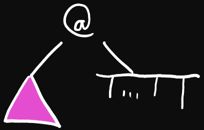


In which case they are approximated by terms like this:



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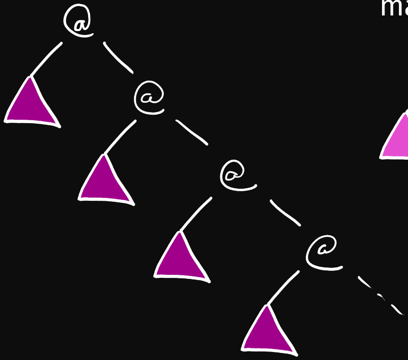

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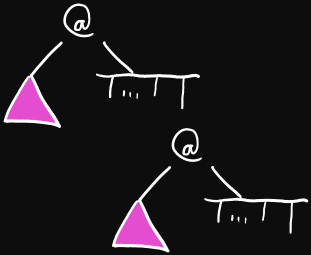

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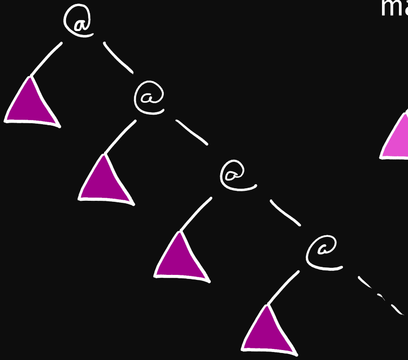

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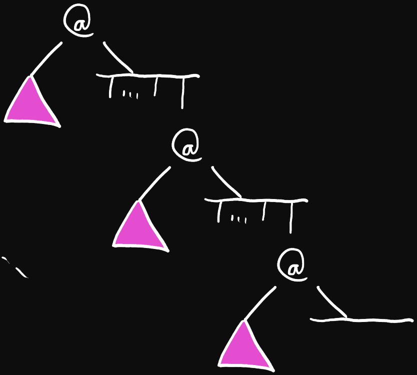

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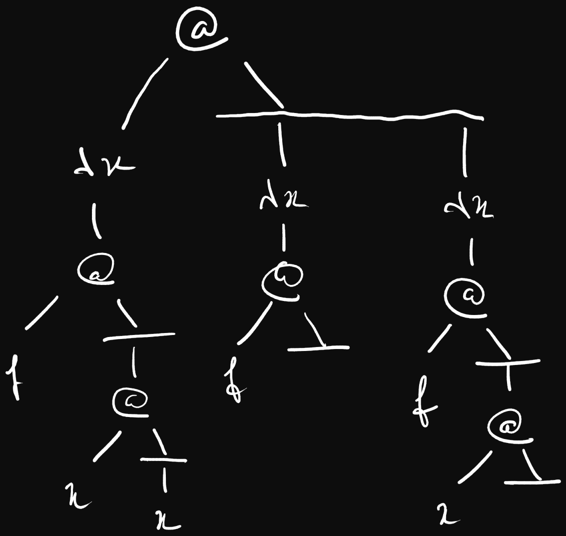

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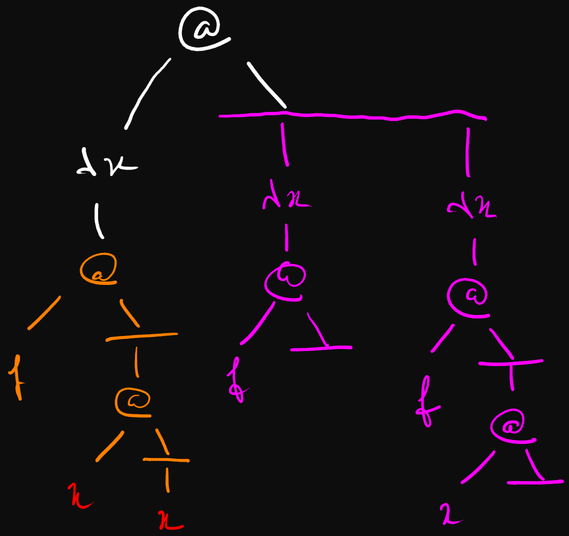

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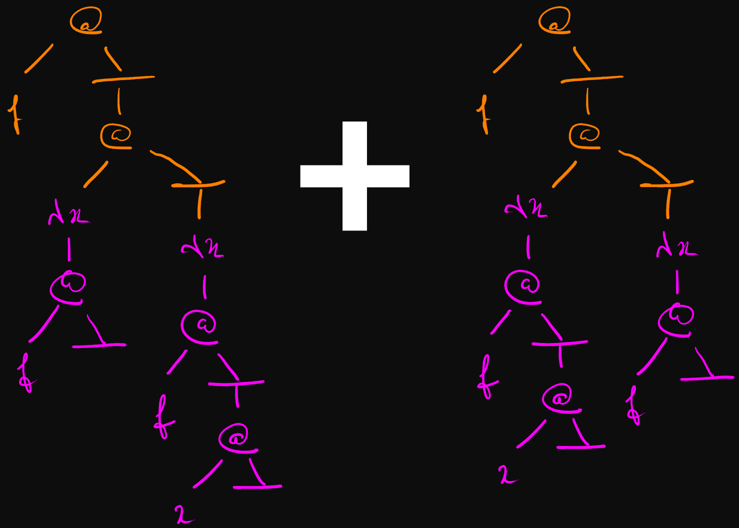

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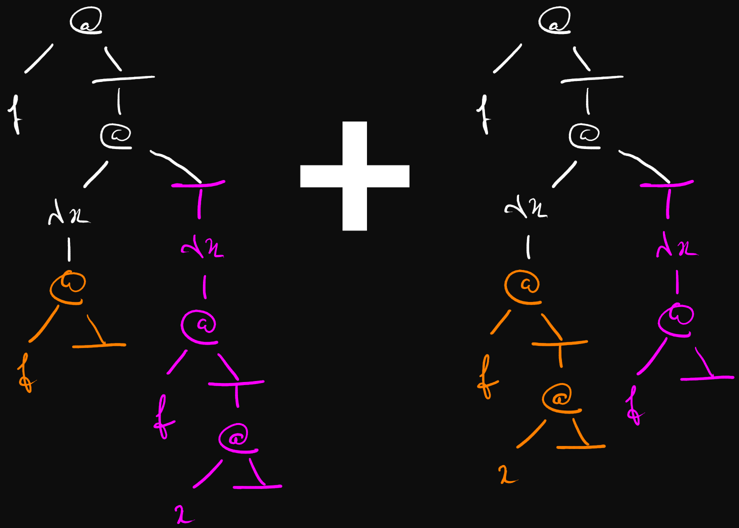

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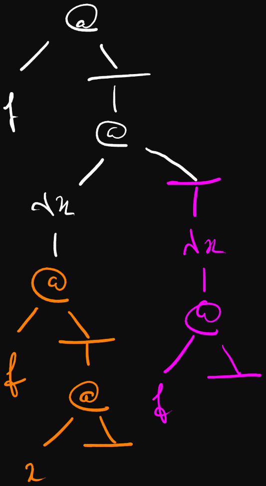

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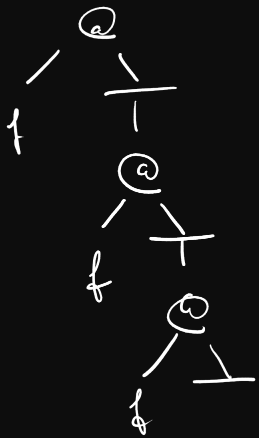

The big theorem about Taylor expansion of λ-terms:

Commutation theorem (Ehrhard and Regnier 2006)

Given a λ-term M, $\mathrm{nf}_r(\mathcal{T}(M)) = \mathcal{T}(\mathrm{BT}(M))$.

Motivation 2

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A bad (?) motivation: This formalism has been successfully applied to nondeterministic, probabilistic, CBV, CBPV (and more?) λ-calculi. Let's try another one: our Λ_∞^{001}.

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A good motivation: Λ_∞^{001} is a world where Böhm trees are "true" normal forms, this should be a nice setting to express the commutation property.

The story

We have a nice (?) theorem

Theorem (simulation)

For all $M, N \in \Lambda_{\infty}^{001}$, if $M \longrightarrow_{\beta}^{\infty} N$ then $\mathcal{T}(M) \rightsquigarrow_{r} \mathcal{T}(N)$.

Proof: some technicalities and a diagonal argument, see (Cerda and Vaux Auclair 2022).

Facts

- For all $M \in \Lambda_\infty^{001}$, $M \longrightarrow_{\beta\bot}^{\infty} \mathrm{BT}(M)$.
- For such an M, $\mathrm{BT}(M)$ is in normal form (for $\longrightarrow_{\beta\bot}^{\infty}$) and $\mathcal{T}(\mathrm{BT}(M))$ is in normal form too (for $\rightsquigarrow_r$).
- $\rightsquigarrow_r$ is confluent.

Corollary (Commutation theorem)

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Corollary (Commutation theorem)

For all $M \in \Lambda_\infty^{001}$, $\mathrm{nf}_r(\mathcal{T}(M)) = \mathcal{T}(\mathrm{nf}_{\beta\bot}(M))$.

Corollary (unicity of normal forms)

Let $M \in \Lambda_\infty^{001}$ be a term, then $BT(M)$ is its unique $\beta\bot$-normal form.

Corollary (confluence)

The reduction $\longrightarrow_{\beta\bot}^{\infty}$ is confluent.

These were the big results in (Kennaway, Klop, et al. 1997).

Theorem (characterisation of head-normalisables)

Let $M \in \Lambda_\infty^{001}$ be a term, then the following propositions are equivalent:

1. there exists $N \in \Lambda_\infty^{001}$ in HNF such that $M \longrightarrow_\beta^\infty N$,
2. there exists $s \in \mathcal{T}(M)$ such that $\mathrm{nf}_r(s) \neq 0$,
3. there exists $N \in \Lambda_\infty^{001}$ in HNF such that $M \longrightarrow_h^* N$.

Proof: Refinement of a folkore result, see (Olimpieri 2020).

We call a resource term *d-positive* if it has no occurrence of 1 at depth smaller than d.

Corollary (characterisation of normalisables)

Let $M \in \Lambda_\infty^{001}$ be a term, then the following propositions are equivalent:

1. there exists $N \in \Lambda_\infty^{001}$ in normal form such that $M \longrightarrow_\beta^\infty N$,
2. for any $d \in \mathbb{N}$, there exists $s \in \mathcal{T}(M)$ such that $\mathrm{nf}_r(s)$ contains a d-positive term.

An infinitary Genericity lemma

We define contexts: λ-terms with a "hole" (a constant $*$).

Theorem (Genericity)

Let $M \in \Lambda_{\infty}^{001}$ be unsolvable and $C(\!|*|\!)$ be a Λ_{∞}^{001}-context.
If $C(\!|M|\!)$ has a normal form C^*, then for any term $N \in \Lambda_{\infty}^{001}$,
$C(\!|N|\!) \longrightarrow_{\beta}^{\infty} C^*$.

There were versions of this in (Kennaway, Oostrom, and de Vries 1996; Salibra 2000), with different formalisms and proofs.

We're happy

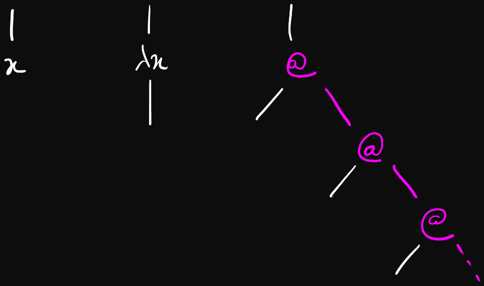
As we hoped:

- The Taylor expansion is a powerful tool to study Λ_∞^{001}.
- Λ_∞^{001} is a well-suited setting for defining the Taylor expansion.

Future adventures

What about other infinitary λ-calculi?

Two other interesting infinitary λ-calculi: Λ_∞^{101} (Lévy-Longo trees) and Λ_∞^{111} (Berarduci trees).



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What would a resource calculus and a Taylor expansion for these look like?

$$
\begin{array}{lcl}
\Lambda_r & := & \mathcal{V} \;\mid\; \lambda\mathcal{V}.\Lambda_r^{¿} \;\mid\; \left\langle \Lambda_r^{?} \right\rangle \Lambda_r^{!} \\
\Lambda_r^{¿} & := & \mathbb{p} \;\mid\; \Lambda_r \\
\Lambda_r^{?} & := & \mathbb{d} \;\mid\; \Lambda_r \\
\Lambda_r^{!} & := & 1 \;\mid\; \Lambda_r \cdot \Lambda_r^{!}
\end{array}
$$

What about the converse of the main theorem?

Conjecture (conservativity)

For all $M, N \in \Lambda_{\infty}^{001}$, if $\mathcal{T}(M) \rightsquigarrow_r \mathcal{T}(N)$ then $M \longrightarrow_{\beta}^{\infty} N$.

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Conjecture

This:

Conjecture (conservativity)
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is false.

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Thanks for your attention!