On non-commutative logic and process calculi

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Non commutative logic and process calculi

Process calculi:

Representing *concurrent systems*

Pomset logic:

Extension of MLL with a non-commutative and self-dual connective
In this presentation

We will talk about:

• A fragment of CCS
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- A fragment of \textit{CCS}
- Logic system \textit{NMAL}
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We will talk about:

• A fragment of CCS

• Logic system NMAL

• Correspondence between NMAL proofs and CCS processes executions
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• A fragment of **CCS**

• Logic system **NMAL**

• Correspondence between **NMAL** proofs and **CCS** processes executions

• A known problem - **catalyzers**
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• A fragment of CCS

• Logic system **NMAL**

• Correspondence between **NMAL** proofs and **CCS** processes executions

• A known problem - **catalyzers**

• Future work
A fragment of CCS

Given the sets $\text{Names} = \{a, b, \ldots\}$ and $\text{Conames} = \{\overline{a}, \overline{b}, \ldots\}$

$P, Q, G := \text{Nil}$ terminated process

$| P \mid Q$ parallel composition

$| P + Q$ sum - non determinism

$| a.P$ prefixing - execute $a$, then $P$

A structural equivalence:

$P \mid Q \equiv Q \mid P \quad P \mid (Q \mid R) \equiv (P \mid Q) \mid R \quad P + Q \equiv Q + P \quad P \mid \text{Nil} \equiv P$
Operational Semantic

interaction is synchronisation

choice between two behaviours

parallel computation

\[
\text{com:} \quad \alpha P \mid \overline{\alpha} Q \rightarrow P \mid Q
\]

\[
\text{sum:} \quad P \rightarrow P' \quad \text{sum:} \quad Q \rightarrow Q'
\]

\[
\text{Par:} \quad P \rightarrow P' \quad \text{Par:} \quad Q \rightarrow Q'
\]

\[
P \equiv P' \rightarrow Q' \equiv Q
\]

\[
P \rightarrow Q
\]
Examples

\[ P \mid Q \]

\[
\begin{align*}
(a \mid b) & \mid \overline{a}.b \\
& \downarrow \\
(b \mid b) & \\
& \downarrow \\
& \text{Nil}
\end{align*}
\]

\[ P \mid R \]

\[
\begin{align*}
(a \mid b) & \mid (\overline{a} \mid b) \\
& \downarrow \\
(b \mid b) & \\
& \downarrow \\
& \text{Nil}
\end{align*}
\]

\[
\begin{align*}
& \downarrow \\
(a \mid \overline{a}) & \\
& \downarrow \\
& \text{Nil}
\end{align*}
\]
Examples

\[(P | R) + S\]

\[\{(a | b) | (\bar{a} | \bar{b})\} + (c | \bar{c} | (a.\bar{b} | b.\bar{a}))\]

- \[a | \bar{a}\]
- \[b | \bar{b}\]
- \[a.\bar{b} | b.\bar{a}\]

Nil

Nil

Nil
Theorem (cut elimination)
Given a proof \( \pi \in \text{NMAL} \cup \{\text{cut}\} \) of conclusion \( \vdash \Gamma \) there exists a proof \( \pi' \in \text{NMAL} \) having the same conclusion \( \vdash \Gamma \).
NMAL: Non commutive, multiplicative additive logic

Theorem (cut elimination)

Given a proof $\pi \in \text{NMAL} \cup \{\text{cut}\}$ of conclusion $\vdash \Gamma$ there exists a proof $\pi' \in \text{NMAL}$ having the same conclusion $\vdash \Gamma$. 
Translation

We translate processes into formulas:

\[
[P | Q] = [P] \otimes [Q]
\]

\[
a.P = a \prec [P]
\]

\[
P + Q = [P] \oplus [Q]
\]
Some intuitions

Reduction steps and NMAL’s rules look alike

\[
\begin{align*}
com: & \quad \text{a}.P \mid \overline{a}.Q \rightarrow P \mid Q \\
\text{sum:} & \quad P \rightarrow P' \\
\rightarrow & \quad P + Q \rightarrow P'
\end{align*}
\]

\[
\begin{align*}
& \quad \vdash a, \overline{a} \vdash [P], [Q] \\
\Rightarrow & \quad \vdash a \triangleright [P], \overline{a} \triangleright [Q] \\
\overline{\Rightarrow} & \quad \vdash a \triangleright [P], \overline{a} \triangleright [Q]
\end{align*}
\]

\[
\begin{align*}
& \quad \vdash [P]' \\
\pi & \quad \vdash [P] \\
\oplus & \quad \vdash [P] \oplus [Q]
\end{align*}
\]
One reduction step : one linear implication

Theorem

If $P \rightarrow Q$ then $\vdash [Q] \triangleright [P]$ is provable in NMAL
One reduction step: one linear implication

Theorem

If $P \rightarrow Q$ then $\vdash [Q] \rightarrow [P]$ is provable in NMAL

Proof.

By induction. If $P \equiv a.P_1 \mid \overline{a}.P_2$ and $Q \equiv P_1 \mid P_2$ then

\[
\begin{array}{c}
\frac{\pi_1 \parallel [P], [P_1]}{[P_1, [P_1]] \otimes [P_2, [P_2]]} \\
\frac{\pi_2 \parallel [P_1, [P_1]] \otimes [P_2, [P_2]]}{[P_1, [P_1]] \otimes [P_2, [P_2]]} \\
\frac{a, \overline{a} \quad \otimes}{[P_1, [P_1]] \otimes [P_2, [P_2]], [P_1], [P_2]} \\
\frac{\overline{a} \quad \otimes}{[P_1, [P_1]] \otimes [P_2, [P_2]], a \prec [P_1], \overline{a} \prec [P_2]} \\
\frac{\overline{a} \quad \otimes}{[P_1, [P_1]] \otimes [P_2, [P_2]], a \prec [P_1], \overline{a} \prec [P_2]} \\
\end{array}
\]

□
Correspondence between executions and proofs

Theorem (Execution to proof)

If \( P_n \rightarrow \cdots \rightarrow P_0 \rightarrow \text{Nil} \), then \( \vdash \llbracket P_n \rrbracket \) is provable in NMAL.

Proof.

We define a proof \( \pi_n \) of \( P_n \) by induction:

• if \( n = 0 \), then \( P_0 \rightarrow \text{Nil} \) and \( P_0 = a | a \). Thus \( \pi_0 = ax a `, a ` \).

• if \( n > 0 \), then \( \pi_{n-1} \)IH \( \llbracket P_{n-1} \rrbracket \)Thm \( \llbracket P_{n-1} \rrbracket \) ⊸ \( \llbracket P_n \rrbracket \)cut \( \llbracket P_n \rrbracket \)⇝ cut-elim NMAL \( \llbracket P_n \rrbracket = \pi_n \).
Correspondence between executions and proofs

Theorem (Execution to proof)

If $P_n \rightarrow \cdots \rightarrow P_0 \rightarrow \text{Nil}$, then $\vdash [P_n]$ is provable in NMAL.

Proof.

We define a proof $\pi_n$ of $P_n$ by induction:

- if $n = 0$, then $P_0 \rightarrow \text{Nil}$ and $P_0 = a \mid \overline{a}$. Thus $\pi_0 = \frac{\text{ax}}{a, \overline{a}} \frac{\gamma}{a \gamma \overline{a}}$.

- if $n > 0$, then $\pi_{n-1} \triangledown \text{IH} \frac{[P_{n-1}]}{[P_n]} \overset{\text{Thm}}{\triangledown} \frac{[P_{n-1}]}{[P_n]} \overset{\text{cut-elim}}{\triangledown} \frac{[P_{n-1}]}{[P_n]} \overset{\text{cut-elim}}{\triangledown} \frac{[P_{n-1}]}{[P_n]}$.

\[\vdash_{\text{NMAL}} [P_n] = \pi_n\]
Correspondence between executions and proofs

Theorem (Execution to proof)

If $P_n \to \cdots \to P_0 \to \text{Nil}$, then $\vdash [P_n]$ is provable in NMAL.

Proof.

We define a proof $\pi_n$ of $P_n$ by induction:

- if $n = 0$, then $P_0 \to \text{Nil}$ and $P_0 = a \mid \overline{a}$. Thus $\pi_0 = \frac{\text{ax}}{a, \overline{a}} \frac{\overline{a}}{a \not\in \overline{a}}$.

- if $n > 0$, then

\[
\frac{\pi_{n-1} \parallel \text{IH}}{\parallel \text{Thm}} \frac{\parallel [P_{n-1}] \parallel [P_{n-1}] \to \parallel [P_n]}{\text{cut}} \frac{\parallel [P_n]}{\parallel [P_n]}
\]

$\rightsquigarrow$ cut-elim

$[[P_n]] = \pi_n$

□

Theorem (Proof to execution)

$[[P]]$ is provable in NMAL then $P \to^* \text{Nil}$
A natural question

Is $P$ safe? Waiting?
Nil is a catalyzer for $b$

How to find a catalyzer for $P$?
Yes we can

\[
\begin{align*}
\text{Com} & \quad \frac{C \vdash \Gamma}{C \vdash a, \bar{a}, \Gamma} \\
\text{Par} & \quad \frac{C \vdash \Gamma, A, B}{C \vdash \Gamma, A \otimes B} \\
\text{Plus} & \quad \frac{C \vdash \Gamma, A_i}{C \vdash \Gamma, A_1 \oplus A_2} \\
\text{Com} & \quad \frac{C \vdash \Gamma, A, B}{C \vdash \Gamma, a \triangleleft A, \bar{a} \triangleleft B} \\
\text{Release} & \quad \frac{C \triangleleft \bar{a} \vdash \Gamma, A}{C \vdash \Gamma, a \triangleleft A} \quad \text{if no other rule is applicable}
\end{align*}
\]
$P = a \overline{b} \mid b \overline{a}$

\[
\begin{align*}
\overline{a} & \triangleleft a \\
\text{Release} & \quad \overline{a} \vdash \overline{a} \\
\text{Com} & \quad \overline{a} \vdash b, b \triangleleft \overline{a} \\
\text{Release} & \quad \vdash a \triangleleft b, b \triangleleft \overline{a} \\
\text{Par} & \quad \vdash (a \triangleleft b) \otimes (b \triangleleft \overline{a})
\end{align*}
\]

$\overline{a}.a$ is a catalyzer for $P$
And for full CCS?

\[
P, Q, G \ := \ \begin{array}{l}
\text{Nil} & \text{terminated process} \\
| \ P \ | \ Q & \text{parallel composition} \\
| \ P + Q & \text{sum - non determinism} \\
| \ a.P & \text{prefixing - execute } a, \text{ then } P \\
| \ \nu a.P & \text{restriction - makes a private} \\
\end{array}
\]

Communication only allowed under \( \nu \)

\[
\begin{align*}
& \text{com} \\
\Rightarrow & \quad \nu a.(a.P \ | \ \bar{a}.Q) \rightarrow \nu a.(P \ | \ Q)
\end{align*}
\]

Not really a smooth operator...
To deadlock or not to deadlock?

Is $P$ safe, waiting or deadlocked?

And the catalyzer?
Work in progress

Deadlock war:

Enriching Session Types: Kobayashi [5]

Quest for catalyzers:

CCS: Bernardi et al. [1]

$\pi$-calculus: Montesi et al. [3]

IDEA: use NMAL to give a unified approach
Future work

\[
\begin{align*}
& a.c \mid \overline{a}.d \mid b.\overline{c} \mid \overline{b} \overline{d} \\
& c \mid d \mid b.\overline{c} \mid \overline{b} \overline{d} \\
& a.c \mid \overline{a}.d \mid \overline{c} \mid \overline{d} \\
& c \mid d \mid \overline{c} \mid \overline{d} \\
& \text{Nil}
\end{align*}
\]
Future work

\[
\begin{align*}
(a \land c) \Rightarrow (\lnot a \land d) & \Rightarrow ((b \land \lnot c) \Rightarrow (b \land \lnot d)) \\
(a \land c) \Rightarrow (\lnot a \land d) & \Rightarrow ((b \land \lnot c) \Rightarrow (b \land \lnot d))
\end{align*}
\]
Giovanni Bernardi and Adrian Francalanza.
Full-abstraction for client testing preorders.

Paola Bruscoli.
A purely logical account of sequentiality in proof search.

Marco Carbone, Ornela Dardha, and Fabrizio Montesi.
Progress as compositional lock-freedom.

Ross Horne and Alwen Tiu.
Towards proofs as successful executions of processes.
2016.

Naoki Kobayashi.
A new type system for deadlock-free processes.

Christian Retoré.