Generic Bidirectional Typing for Dependent Type Theories

Thiago Felicissimo

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Type annotations in dependent type theory

Dependent type theory suffers from verbosity of type annotations

$$\text{Application:} \quad t @_{A,x.B} u$$

$$\text{Dependent pair:} \quad \langle t, u \rangle_{A,x.B}$$

$$\text{Cons:} \quad t ::_A l$$

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Syntax so common that many don't realize that an omission is being made

Typechecking without annotations

Omission has a cost Knowing annotations is needed for typing

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma, x : A \vdash B \text{ type} \qquad \Gamma \vdash t : \Pi x : A.B \qquad \Gamma \vdash u : A}{\Gamma \vdash t\ u : B[u/x]}$$

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Bidirectional typing Decompose $t : A$ in modes check $t \Leftarrow A$ and infer $t \Rightarrow A$

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Allow specify flow of type information in typing rules, explain how to use them

$$\frac{\Gamma \vdash t \Rightarrow C \qquad C \longrightarrow^{*} \Pi x : A.B \qquad \Gamma \vdash u \Leftarrow A}{\Gamma \vdash t\ u \Rightarrow B[u/x]}$$

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Complements unannotated syntax very well, explains how to recover annotations

Contribution

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However, general guidelines have remained informal, no unified framework

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1. We give a general definition of type theories (or equivalently, a logical framework) supporting non-annotated syntaxes
2. For each theory, we define declarative and bidirectional type systems

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Roadmap

1. We give a general definition of type theories (or equivalently, a logical framework) supporting non-annotated syntaxes
2. For each theory, we define declarative and bidirectional type systems
3. We show, in a theory-independent fashion, their equivalence

The theories

One syntax for all!

$$
\begin{aligned}
t, u, T, U \ ::= \ & \mid x && \text{(variables)} \\
& \mid c(\vec{x}_1.u_1, \ldots, \vec{x}_k.u_k) && \text{(constructor application)} \\
& \mid d(t;\ \vec{x}_1.u_1, \ldots, \vec{x}_k.u_k) && \text{(destructor application)} \\
& \mid \mathsf{x}\{u_1, \ldots, u_k\} && \text{(metavariables)}
\end{aligned}
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In $d(t; \ldots)$, we call t the *principal argument*.

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Example

$$
\begin{aligned}
\Sigma_{\lambda\Pi} = \quad & \Pi(\mathsf{A}, \mathsf{B}\{x\}),\ \lambda(\mathsf{t}\{x\}),\ \mathrm{Ty},\ \mathrm{Tm}(\mathsf{A}), && \text{(constructors)} \\
& @(\mathsf{u}) && \text{(destructors)}
\end{aligned}
$$

$$
t, u, A, B ::= x \mid \mathsf{x}\{\vec{t}\} \mid \mathrm{Ty} \mid \mathrm{Tm}(A) \mid @(t; u) \mid \lambda(x.t) \mid \Pi(A, x.B)
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The theories

A *theory* $\mathbb{T}$ is made of *schematic typing rules* and *rewrite rules*.

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Sort rules Sorts are terms that can type other terms[1].

Used to define the *judgment forms* of the theory.

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3 schematic typing rules: *sort rules*, *constructor rules* and *destructor rules*

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Used to define the *judgment forms* of the theory.

Example: In MLTT, 2 judgment forms: $\square$ type and $\square : A$ for a type A.

$$\frac{}{\vdash \mathsf{Ty}\ \mathsf{sort}} \qquad \frac{\vdash \mathsf{A} : \mathsf{Ty}}{\vdash \mathsf{Tm}(\mathsf{A})\ \mathsf{sort}}$$

We can then write $A : \mathsf{Ty}$ for A type, and $t : \mathsf{Tm}(A)$ for $t : A$

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Constructor rules are bidirectionally typed in mode check

The sort of the rule is a pattern allowing to recover the omitted arguments

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Destructor rules are bidirectionally typed in mode infer

The sort of the *principal argument* $\mathtt{t} : T^{\mathsf{P}}$ should be a pattern allowing to recover the omitted arguments

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The theories

Rewrite rules Define the definitional equality (aka conversion) $\equiv$ of the theory.

$$@(\lambda(x.\mathtt{t}\{x\});\mathtt{u}) \longmapsto \mathtt{t}\{\mathtt{u}\}$$

In general, of the form $d(t^{\mathsf{P}}; \vec{x}_1.t_1^{\mathsf{P}}, \ldots, \vec{x}_k.t_k^{\mathsf{P}}) \longmapsto r$, with left-hand-side linear.

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Condition: no two left-hand sides unify.

Therefore, rewrite systems are orthogonal, hence confluent by construction!

Full example

Theory $\mathbb{T}_{\lambda\Pi}$, defining minimalistic Martin-Lof Type Theory.

$$\mathrm{Ty}(\cdot)\ \mathsf{sort}$$

$$\mathrm{Tm}(\mathsf{A} : \mathrm{Ty})\ \mathsf{sort}$$

$$\Pi(\cdot;\ \ \mathsf{A} : \mathrm{Ty},\ \mathsf{B}\{x : \mathrm{Tm}(\mathsf{A})\} : \mathrm{Ty}) : \mathrm{Ty}$$

$$\lambda(\mathsf{A} : \mathrm{Ty},\ \mathsf{B}\{x : \mathrm{Tm}(\mathsf{A})\} : \mathrm{Ty};\ \ \mathsf{t}\{x : \mathrm{Tm}(\mathsf{A})\} : \mathrm{Tm}(\mathsf{B}\{x\})) : \mathrm{Tm}(\Pi(\mathsf{A}, x.\mathsf{B}\{x\}))$$

$$@(\mathsf{A} : \mathrm{Ty},\ \mathsf{B}\{x : \mathrm{Tm}(\mathsf{A})\} : \mathrm{Ty};\ \ \mathsf{t} : \mathrm{Tm}(\Pi(\mathsf{A}, x.\mathsf{B}\{x\}));\ \ \mathsf{u} : \mathrm{Tm}(\mathsf{A})) : \mathrm{Tm}(\mathsf{B}\{\mathsf{u}\})$$

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Bidirectional typing

Matching modulo rewriting

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Given t^{P} and u, we define a matching judgment

$$t^{\mathsf{P}} \prec u \rightsquigarrow \vec{x}_1.t_1/\mathsf{x}_1, \dots, \vec{x}_k.t_k/\mathsf{x}_k$$

that tries to compute a metavariable substitution s.t. $t^{\mathsf{P}}[\vec{x}_1.t_1/\mathsf{x}_1, \dots, \vec{x}_k.t_k/\mathsf{x}_k] \equiv u$.

Inferable and checkable terms

Not all unannotated terms can be algorithmically typed

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Principal argument of a destructor can only be variable or another destructor.

For most theories: $t^{\mathsf{c}}, u^{\mathsf{c}}, \dots$ = normal forms, and $t^{\mathsf{i}}, u^{\mathsf{i}}, \dots$ = neutrals

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Completeness For t^{i} inferable, if $\Gamma \vdash t : T$ then $\Gamma \vdash t^{\mathsf{i}} \Rightarrow U$ with $T \equiv U$.
For t^{c} checkable, if $\Gamma \vdash t : T$ then $\Gamma \vdash t^{\mathsf{c}} \Leftarrow T$.

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For t^{c} checkable, if $\Gamma \vdash t : T$ then $\Gamma \vdash t^{\mathsf{c}} \Leftarrow T$.

Decidability If $\mathbb{T}$ weak normalizing, then inference is decidable for inferable terms, and checking is decidable for checkable terms.

More examples

Dependent sums

Extends $\mathbb{T}_{\lambda\Pi}$ with

$$\frac{\vdash \mathsf{A} : \mathbf{Ty} \qquad x : \mathbf{Tm}(\mathsf{A}) \vdash \mathsf{B} : \mathbf{Ty}}{\vdash \Sigma(\mathsf{A}, \mathsf{B}) \ : \mathbf{Ty}}$$

$$\frac{\vdash \mathsf{A} : \mathbf{Ty} \qquad x : \mathbf{Tm}(\mathsf{A}) \vdash \mathsf{B} : \mathbf{Ty} \qquad \vdash \mathsf{t} : \mathbf{Tm}(\mathsf{A}) \qquad \vdash \mathsf{u} : \mathbf{Tm}(\mathsf{B}\{\mathsf{t}\})}{\vdash \mathrm{pair}(\mathsf{t}, \mathsf{u}) \ : \mathbf{Tm}(\Sigma(\mathsf{A}, x.\mathsf{B}\{x\}))}$$

$$\frac{\vdash \mathsf{A} : \mathbf{Ty} \qquad x : \mathbf{Tm}(\mathsf{A}) \vdash \mathsf{B} : \mathbf{Ty} \qquad \vdash \mathsf{t} : \mathbf{Tm}(\Sigma(\mathsf{A}, x.\mathsf{B}\{x\}))}{\vdash \mathrm{proj}_1(\mathsf{t}; \cdot) \ : \mathbf{Tm}(\mathsf{A})}$$

$$\frac{\vdash \mathsf{A} : \mathbf{Ty} \qquad x : \mathbf{Tm}(\mathsf{A}) \vdash \mathsf{B} : \mathbf{Ty} \qquad \vdash \mathsf{t} : \mathbf{Tm}(\Sigma(\mathsf{A}, x.\mathsf{B}\{x\}))}{\vdash \mathrm{proj}_2(\mathsf{t}; \cdot) \ : \mathbf{Tm}(\mathsf{B}\{\mathrm{proj}_1(\mathsf{t})\})}$$

$$\mathrm{proj}_1(\mathrm{pair}(\mathsf{t}, \mathsf{u}); \varepsilon) \longmapsto \mathsf{t} \qquad \mathrm{proj}_2(\mathrm{pair}(\mathsf{t}, \mathsf{u}); \varepsilon) \longmapsto \mathsf{u}$$

Lists

Extends $\mathbb{T}_{\lambda\Pi}$ with

$$\frac{\vdash \mathsf{A} : \mathbf{Ty}}{\vdash \mathbf{List}(\mathsf{A}) : \mathbf{Ty}} \qquad \frac{\vdash \mathsf{A} : \mathbf{Ty}}{\vdash \mathbf{nil} : \mathbf{Tm}(\mathbf{List}(\mathsf{A}))} \qquad \frac{\vdash \mathsf{A} : \mathbf{Ty} \qquad \vdash \mathsf{x} : \mathbf{Tm}(\mathsf{A}) \qquad \vdash \mathsf{l} : \mathbf{Tm}(\mathbf{List}(\mathsf{A}))}{\vdash \mathbf{cons}(\mathsf{x}, \mathsf{l}) : \mathbf{Tm}(\mathbf{List}(\mathsf{A}))}$$

$$\frac{\vdash \mathsf{A} : \mathbf{Ty} \qquad \vdash \mathsf{l} : \mathbf{Tm}(\mathbf{List}(\mathsf{A})) \qquad x : \mathbf{Tm}(\mathbf{List}(\mathsf{A})) \vdash \mathsf{P} : \mathbf{Ty} \qquad \vdash \mathsf{pnil} : \mathbf{Tm}(\mathsf{P}\{\mathbf{nil}\}) \qquad x : \mathbf{Tm}(\mathsf{A}), y : \mathbf{Tm}(\mathbf{List}(\mathsf{A})), z : \mathbf{Tm}(\mathsf{P}\{y\}) \vdash \mathsf{pcons} : \mathbf{Tm}(\mathsf{P}\{\mathbf{cons}(x, y)\})}{\vdash \mathbf{ListRec}(\mathsf{l}; \mathsf{P}, \mathsf{pnil}, \mathsf{pcons}) : \mathbf{Tm}(\mathsf{P}\{\mathsf{l}\})}$$

$$\mathbf{ListRec}(\mathbf{nil}; x.\mathsf{P}\{x\}, \mathsf{pnil}, xyz.\mathsf{pcons}\{x, y, z\}) \longmapsto \mathsf{pnil}$$

$$\mathbf{ListRec}(\mathbf{cons}(\mathsf{x}, \mathsf{l}); x.\mathsf{P}\{x\}, \mathsf{pnil}, xyz.\mathsf{pcons}\{x, y, z\}) \longmapsto \mathsf{pcons}\{\mathsf{x}, \mathsf{l}, \mathbf{ListRec}(\mathsf{l}; x.\mathsf{P}\{x\}, \mathsf{pnil}, xyz.\mathsf{pcons}\{x, y, z\})\}$$

W types

Extends $\mathbb{T}_{\lambda\Pi}$ with

$$\frac{\vdash \mathsf{A} : \mathrm{Ty} \qquad x : \mathrm{Tm}(\mathsf{A}) \vdash \mathsf{B} : \mathrm{Ty}}{\vdash \mathrm{W}(\mathsf{A}, \mathsf{B}) : \mathrm{Ty}}$$

$$\frac{\begin{array}{c}\vdash \mathsf{A} : \mathrm{Ty} \qquad x : \mathrm{Tm}(\mathsf{A}) \vdash \mathsf{B} : \mathrm{Ty} \\ \vdash \mathsf{a} : \mathrm{Tm}(\mathsf{A}) \qquad \vdash \mathsf{f} : \mathrm{Tm}(\Pi(\mathsf{B}\{\mathsf{a}\}, x'.\mathrm{W}(\mathsf{A}, x.\mathsf{B}\{x\})))\end{array}}{\vdash \mathrm{sup}(\mathsf{a}, \mathsf{f}) : \mathrm{Tm}(\mathrm{W}(\mathsf{A}, x.\mathsf{B}\{x\}))}$$

$$\frac{\begin{array}{c}\vdash \mathsf{A} : \mathrm{Ty} \qquad x : \mathrm{Tm}(\mathsf{A}) \vdash \mathsf{B} : \mathrm{Ty} \qquad \vdash \mathsf{t} : \mathrm{Tm}(\mathrm{W}(\mathsf{A}, x.\mathsf{B}\{x\})) \qquad x : \mathrm{Tm}(\mathrm{W}(\mathsf{A}, x.\mathsf{B}\{x\})) \vdash \mathsf{P} : \mathrm{Ty} \\ x : \mathrm{Tm}(\mathsf{A}), y : \mathrm{Tm}(\Pi(\mathsf{B}\{x\}, x'.\mathrm{W}(\mathsf{A}, x.\mathsf{B}\{x\}))), z : \mathrm{Tm}(\Pi(\mathsf{B}\{x\}, x'.\mathsf{P}\{@(y, x')\})) \vdash \mathsf{p} : \mathrm{Tm}(\mathsf{P}\{\mathrm{sup}(x, y)\})\end{array}}{\vdash \mathrm{WRec}(\mathsf{t}; \mathsf{P}, \mathsf{p}) : \mathrm{Tm}(\mathsf{P}\{\mathsf{t}\})}$$

$$\mathrm{WRec}(\mathrm{sup}(\mathsf{a}, \mathsf{f}); x.\mathsf{P}\{x\}, xyz.\mathsf{p}\{x, y, z\}) \longmapsto \mathsf{p}\{\mathsf{a}, \mathsf{f}, \lambda(x.\mathrm{WRec}(@(\mathsf{f}, x); x.\mathsf{P}\{x\}, xyz.\mathsf{p}\{x, y, z\}))\}$$

Universes

Extends $\mathbb{T}_{\lambda\Pi}$ with

$$\frac{}{\vdash \mathrm{U}(\cdot) : \mathrm{Ty}} \qquad \frac{\vdash \mathsf{a} : \mathrm{Tm}(\mathrm{U})}{\vdash \mathrm{El}(\mathsf{a}; \cdot) : \mathrm{Ty}}$$

Tarski-style Adds codes for all types

$$\frac{}{\vdash \mathrm{u}(\cdot) : \mathrm{Tm}(\mathrm{U})} \qquad \mathrm{El}(\mathrm{u}; \varepsilon) \longmapsto \mathrm{U}$$

$$\frac{\vdash \mathsf{a} : \mathrm{Tm}(\mathrm{U}) \qquad x : \mathrm{Tm}(\mathrm{El}(\mathsf{a})) \vdash \mathsf{b} : \mathrm{Tm}(\mathrm{U})}{\vdash \pi(\mathsf{a}, \mathsf{b}) : \mathrm{Tm}(\mathrm{U})}$$

$$\mathrm{El}(\pi(\mathsf{a}, x.\mathsf{b}\{x\}); \varepsilon) \longmapsto \Pi(\mathrm{El}(\mathsf{a}; \varepsilon), x.\mathrm{El}(\mathsf{b}\{x\}; \varepsilon))$$

(Weak) Coquand-style

Adds a code constructor c

$$\frac{\vdash \mathsf{A} : \mathrm{Ty}}{\vdash \mathrm{c}(\mathsf{A}) : \mathrm{Tm}(\mathrm{U})}$$

$$\mathrm{El}(\mathrm{c}(\mathsf{A}); \varepsilon) \longmapsto \mathsf{A}$$

Conclusion

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Generic account of bidirectional typing for class of dependent type theories

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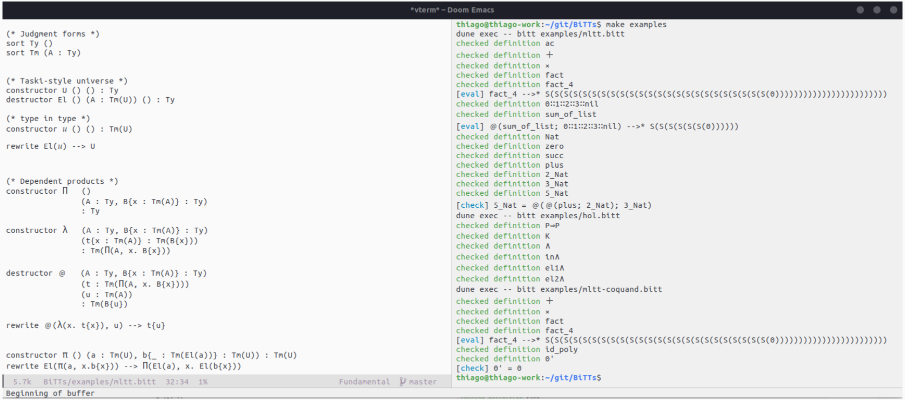
Bidirectional system implemented in a prototype, available at

https://github.com/thiagofelicissimo/BiTTs



Thank you for your attention!