A Functorial model of Differential Linear Logic

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Let's review: Models of LL

What is a model of linear logic ?

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What is a model of linear logic ? A possible categorical answer: Seely Categories

$$(\mathcal{C}, \times) \underset{U}{\overset{\mathcal{E}'}{\rightleftarrows}} (\mathcal{L}, \otimes)$$

A strong monoidal adjunction ($! \overset{\text{def}}{=} \mathcal{E}' \circ U$)

Between a monoidal category of linear morphisms

And a cartesian category of non linear morphisms

Let's review: Models of LL

How do we interpret a proof?

A proof π of conclusion $\Gamma \vdash A$ is interpreted as a morphism $[\![\Gamma]\!] \to [\![A]\!]$ compositionally:

$$\dfrac{\dfrac{\pi}{A \vdash \Gamma}}{!A \vdash \Gamma}$$

Use a natural transformation $d_A : !A \to A$

$$!A \overset{d}{\longrightarrow} A \overset{\pi}{\longrightarrow} \Gamma$$

Let's review: Models of LL

This is a compact version of requiring to have these natural transformations

Operator	Type	Intuition
w	$!A \multimap 1$	Create constant function
c	$!A \multimap !A \otimes !A$	From 2 to 1 parameter
d	$!A \multimap A$	Forget linearity
p	$!A \multimap !!A$	Higher order

Plus some commutative diagrams (to respect cut elimination)

What is Differential Linear Logic (DiLL) ?

DiLL is a variant of linear logic that was discovered by Ehrhard & Regnier in 2004 with a notion of differentiation:

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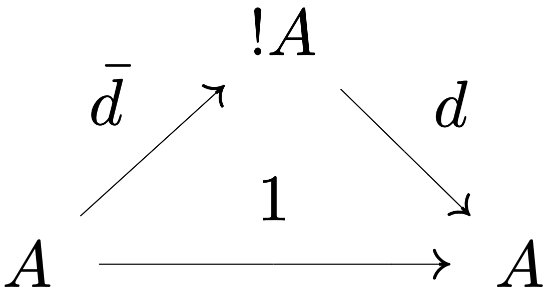
This amounts to adding these operator:

Operator	Type	Operator	Type	Intuition
w	$!A \multimap 1$	$\overline{\mathrm{w}}$	$1 \multimap !A$	Evaluation at 0
c	$!A \multimap !A \otimes !A$	$\overline{\mathrm{c}}$	$!A \otimes !A \multimap !A$	Convolution
d	$!A \multimap A$	$\overline{\mathrm{d}}$	$A \multimap !A$	Differentiation
p	$!A \multimap !!A$	x	x	x

Dbar diagrams



Dbar diagrams 2

$$
\begin{array}{ccccc}
A & \xrightarrow{\bar{d}} & !A & \xrightarrow{p} & !!A \\
{\scriptstyle \cong}\big\downarrow & & & & \big\uparrow{\scriptstyle m} \\
A \otimes I & \xrightarrow{\bar{d} \otimes w} & !A \otimes !A & \xrightarrow{\bar{d}_! \otimes p} & !!A \otimes !!A
\end{array}
$$

$$
\begin{array}{ccccc}
A \otimes !B & \xrightarrow{\bar{d} \otimes 1} & !A \otimes !B & \xrightarrow{\varphi} & !(A \otimes B) \\
& {\scriptstyle 1 \otimes d}\searrow & & \nearrow{\scriptstyle \bar{d}_{A \otimes B}} & \\
& & A \otimes B & &
\end{array}
$$

What is Differential Linear Logic (DiLL) ?

There is a reformulation of such models with a biproduct $\diamond$ which automatically gives most operators.

$$(\mathcal{C}, \times) \underset{U \,\approx\, d}{\overset{\mathcal{E}' \,\approx\, p}{\rightleftarrows}} (\mathcal{L}, \otimes, \diamond)$$

Type	Operator
$!A \multimap A$	$\bar{\mathsf{d}}$

But where would $\bar{\mathsf{d}}$ fit in such a setting ?

Our contribution: A model where $\overline{\mathrm{d}}$ is expressed as a functor.

(Purely functorial)

Why would we want that ?

- Express a modular transformation of program
  $\rightarrow$ (This is a key point of Differentiable Programming)
- Compactify definitions : Proving that something is a model becomes easier
- Makes link easier with Chiralities

Let's look at Differentiation

We want to capture the chain-rule in a functorial way:

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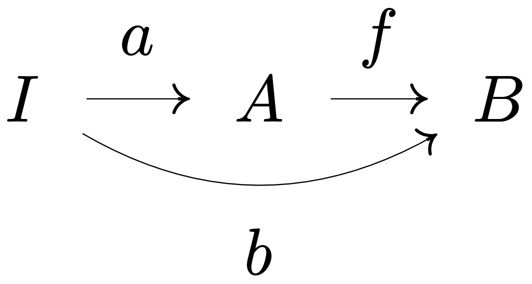
$$(A, a) \xrightarrow{\quad f \quad} (B, f(a)) \xrightarrow{\quad g \quad} (C, g(f(a)))$$

$\vec{D}$ should be a functor, with morally $\vec{D}(f) = D_a(f)$

The co-Slice Category

The category $I \downarrow \mathcal{C}$, the co-Slice of $\mathcal{C}$ is defined as follows:

- Objects: (A, a) with $a : I \to A$, intuitively, an element of A
- Arrows: $f : (A, a) \to (B, b)$ such that $f(a) = b$



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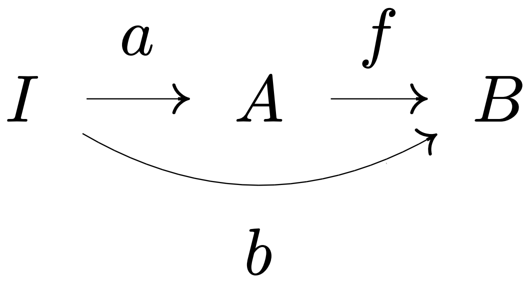


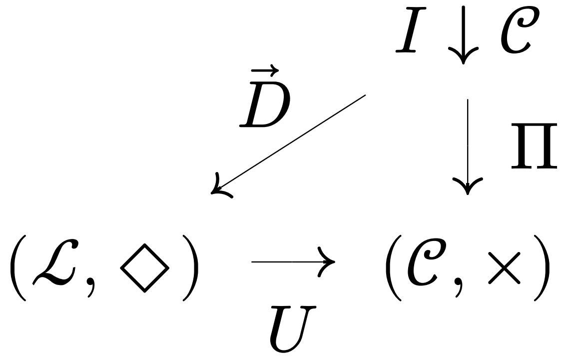
Now b:

$$D_a(g \circ f) = D(g) \circ D(f) = D_{f(a)}(g) \circ D_a(f)$$

Second thing

The differential (*ie*, best linear approximation) of a linear is itself.

Hence $\vec{D}$ should preserve linear morphisms: $\forall a, l : D_a(l) = l$



But...

$$\begin{array}{ccc} & & I \downarrow \mathcal{C} \\ & \swarrow^{\vec{D}} & \downarrow \Pi \\ (\mathcal{L}, \diamond) & \xrightarrow[U]{} & (\mathcal{C}, \times) \end{array}$$

We cannot go up from $\mathcal{C}$ to $I \downarrow \mathcal{C}$!

$\rightarrow$ Would require to choose a point

The category of Generalized Elements

Given a functor $U : \mathcal{L} \to \mathcal{C}$, the category $I \downarrow U$ of generalized elements over U is defined as:

- Objects: (A, a) with $a : I \to U(A)$
- Arrows: $l : A \to B$ such that $U(l)(a) = b$

In a sense, the linear part of the co-Slice.

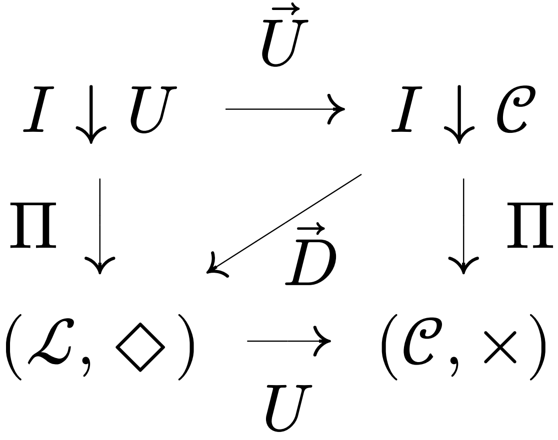
Definition of Functorial DiLL Model

A pre-model of DiLL,

$$\begin{array}{ccc} I \downarrow U & \xrightarrow{\vec{U}} & I \downarrow \mathcal{C} \\ \Pi \downarrow & & \downarrow \Pi \\ (\mathcal{L}, \otimes, \diamond) & \underset{U}{\to} & (\mathcal{C}, \times) \end{array}$$

Definition of Functorial DiLL Model

A pre-model of DiLL, plus a functor $\vec{D}$



Definition of Functorial DiLL Model

A pre-model of DiLL, plus a functor $\vec{D}$

$$
\begin{array}{ccc}
I \downarrow U & \xrightarrow{\vec{U}} & I \downarrow \mathcal{C} \\
\Pi \downarrow & \overset{\vec{D}}{\swarrow} & \downarrow \Pi \\
(\mathcal{L}, \diamond) & \xrightarrow[U]{} & (\mathcal{C}, \times)
\end{array}
$$

(And well pointedness relative to I ...)

Theorems

Theorem: Our functorial model is a model of DiLL.

Theorem: The converse is true for well pointed models.

What if my model isn't well pointed?

A pre-model of DiLL, plus a functor $\vec{D}$

$$
\begin{array}{ccc}
U \downarrow U & \xrightarrow{\vec{U}} & U \downarrow \mathcal{C} \\
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\Pi \downarrow & \swarrow \vec{D} & \downarrow \Pi \\
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\end{array}
$$

But we have no model/intuition of what is going on here...

What are Chiralities ?

A categorical framework by Melliès which refines *-autonomous categories into models of Polarized MLL.

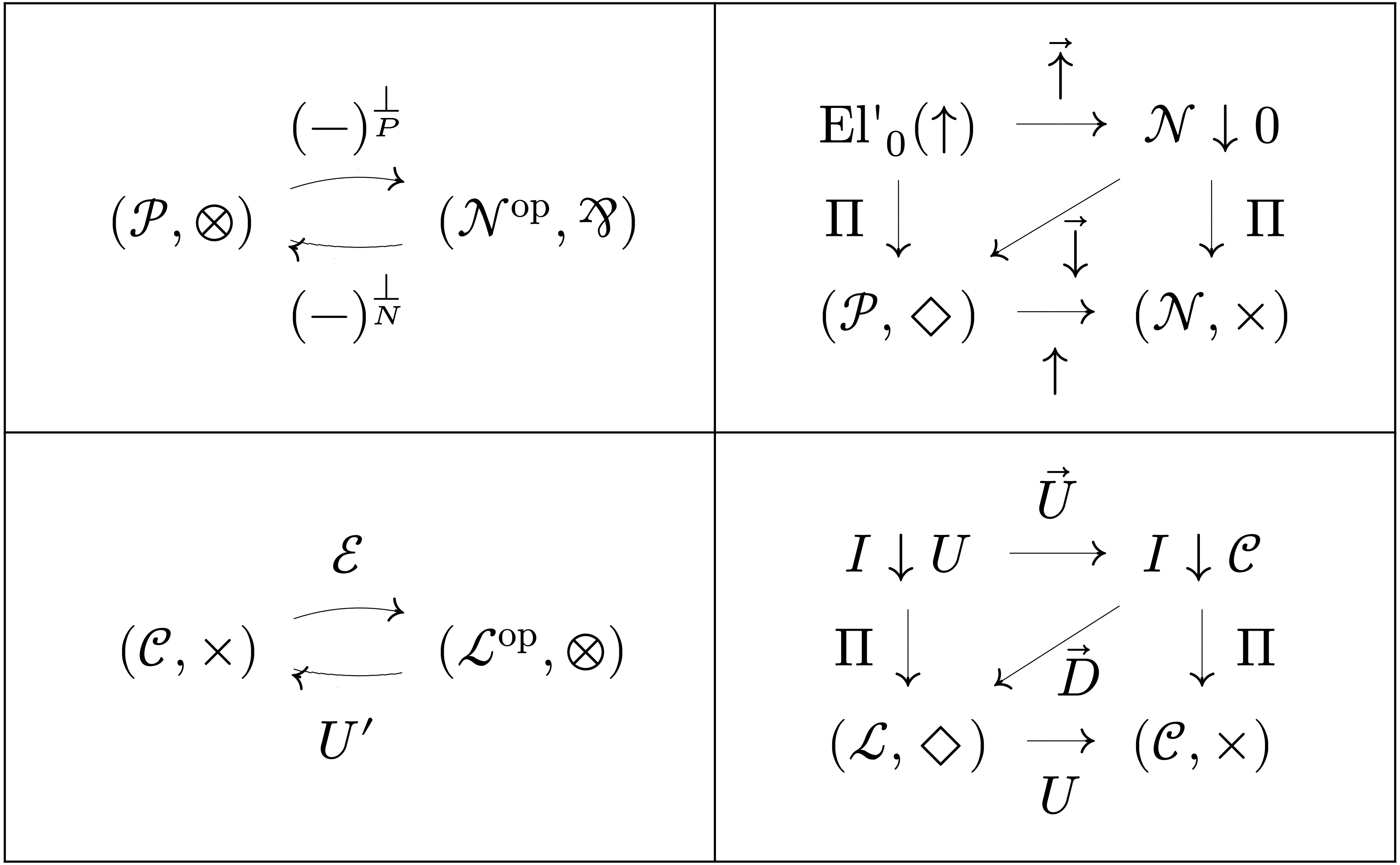
$$(\mathcal{P}, \otimes, 1) \underset{(-)^{\perp}_{N}}{\overset{(-)^{\perp}_{P}}{\rightleftarrows}} (\mathcal{N}^{\mathrm{op}}, \parr, \perp) \qquad\qquad \mathcal{P} \underset{\downarrow}{\overset{\uparrow}{\rightleftarrows}} \mathcal{N}$$

With the left being strong monoidal, and $\downarrow \circ \uparrow = \mathrm{Id}$

Plus some extra conditions on the adjunction

Appears a lot for “smooth” models of DiLL (in Functional Analysis)

A parallel to be made


$(-)^{\perp}_{P}$
$(\mathcal{P}, \otimes)$
$(\mathcal{N}^{op}, \parr)$
$(-)^{\perp}_{N}$
$\vec{\uparrow}$
$\mathrm{El}'_0(\uparrow)$
$\mathcal{N} \downarrow 0$
Π
$\vec{\downarrow}$
Π
$(\mathcal{P}, \diamond)$
$(\mathcal{N}, \times)$
$\uparrow$
$\mathcal{E}$
$(\mathcal{C}, \times)$
$(\mathcal{L}^{op}, \otimes)$
U'
$\vec{U}$
$I \downarrow U$
$I \downarrow \mathcal{C}$
Π
$\vec{D}$
Π
$(\mathcal{L}, \diamond)$
$(\mathcal{C}, \times)$
U

Last but not least: a funny remark

When $\mathcal{L}$ is a calculus category (so with integration on top), we have an relative $! \otimes \mathrm{Id}$-adjunction:

$$I \downarrow \mathcal{C} \underset{\overline{U}}{\overset{\vec{D}}{\rightleftarrows}} (\mathcal{L}, \otimes)$$

$$(I \downarrow \mathcal{C})((a, A), (b, B)) \simeq \mathcal{L}(!A \otimes A, B)$$

With $\overline{U}(l) = (U(l), U(u_A))$

This adjunction corresponds mathematically to the fundamental theorem of calculus !

Future Work

- Pursue similarities and extending our framework to Polarized LL with Mellie's **Chiralities**
- New model of Kerjean & Pacaud-Lemay with co-promotion should have a symmetric behavior: a functorial **co-chain rule** ?
- Investigate the **dependent** flavour

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Thank you for listening !

Any questions ? 😁