Plurimetric Fuzz: A Linear Type System for Bounding the Sensitivity of Vector Functions

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Formal Structures for Computation and Proofs

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1. Introduction

What is the sensitivity of a function?
What is differential privacy?

Setting: privacy-preserving queries to a database

An analyst makes queries to a database.

Example

- What is the average salary of employees?
- How many patients have the flu?
- How old is Mr. Doe?
- What is the salary of Mrs. Smith?

How to protect the privacy of individuals?

Anonymising the data is not enough

```
let query (db : database) : float =
    let dupont = filter
        (fun p -> p.birthyear = 1970
                  && p.postcode = 75000
                  && p.weight >= 70)
        db |> head
    in dupont.salary
```

One can find an individual by cross-referencing information.

What is differential privacy?

Let

- D be a database,
- x be an individual,
- q be a query.

We want $q(D)$ and $q\left(D \cup \{x\}\right)$ to be indistinguishable.

Formal definition of differential privacy (Dwork et al. 2006)

Let

- $\mathcal{D}$ be the set of databases,
- X be an arbitrary space (usually R^n),
- $f : \mathcal{D} \to X$ be a probabilistic algorithm.

Definition

f is (ε, δ)-*differentially private* if for all $D, D' \in \mathcal{D}$ such that $d(D, D') \leq 1$ and all $S \subseteq X$:

$$\Pr\big(f(D) \in S\big) \leq e^{\epsilon} \Pr\big(f(D') \in S\big) + \delta$$

The distance between two databases is the number of rows that differ between them.

Query sensitivity

Definition

A function $f\colon X \to Y$ between two metric spaces (X, d_X) and (Y, d_Y) is said to be *k-sensitive* if for all $x, x' \in X$, we have $d_Y\big(f(x), f(x')\big) \le k \cdot d_X(x, x')$.

It is a measure of the *specificity*, that is of the *continuity* of the function.

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Remark

The sensitivity of a query depends on the metric chosen on the source and target spaces.

Adding noise

In order to guarantee differential privacy *automatically*, one can add noise to the result of the query.

Theorem (Dwork and Roth 2014)

Let $f : X \to \mathbb{R}^k$ *be a* k*-sensitive function for the* L^1 *distance. The function* $\mathcal{M}_{Laplace}(f, \epsilon)$ *defined by:*

$$\mathcal{M}_{Laplace}(f, \epsilon)(x) = \left(\pi_1(f(x)) + Y_1, \ldots, \pi_n(f(x)) + Y_n\right)$$

where the Y_i *are i.i.d. according to a Laplace distribution of parameter* k/ϵ *is* $(\epsilon, 0)$*-DP.*

Reminder. $d_1(x, y) = \sum_{i=1}^{k} |x_i - y_i|$.

The role of logic

Program on a database

↓ **Type system**

Program whose sensitivity is known

↓ Noise addition mechanism

Differentially private program

2. The Fuzz functional language and its extensions

Sensitivity is an affine resource
which can be tracked by a type system.

Linear logic (Girard 1987)

Linear logic allows to reason about resources, state changes, etc.

Example

The $\otimes$ connective allows to model the simultaneous presence of two resources, and $\multimap$ their transformation.

$$H_2 \otimes H_2 \otimes O_2 \multimap H_2O \otimes H_2O$$

Deduction rules model consumption, distribution, etc.

Fuzz (Reed and Pierce 2010) is a functional language with affine types to reason about the sensitivity of programs.

Example

$$\frac{[x : A]_{s_1} \vdash b : B \quad [x : A]_{s_2} \vdash c : C}{[x : A]_{s_1+s_2} \vdash (b, c) : B \otimes C}$$

If $x \mapsto f(x)$ is s_1-sensitive and $x \mapsto g(x)$ is s_2-sensitive, then $x \mapsto \big(f(x), g(x)\big)$ is $(s_1 + s_2)$-sensitive.

Semantic soundness theorem (Azevedo de Amorim et al. 2017)

Types are interpreted as metric spaces,

Interpretation of types

- $A \otimes B$ is interpreted by the space $[\![A]\!] \times [\![B]\!]$ endowed with the metric d_1;
- $A \multimap B$ is interpreted by the space of 1-sensitive functions from $[\![A]\!]$ to $[\![B]\!]$ endowed with some metric $d_{\multimap}$.

and terms as functions between them.

Theorem

If $[x : A]_s \vdash e : B$ is derivable, then the following function is s-sensitive:

$$\begin{array}{rccc} [\![e]\!] & : & [\![A]\!] & \longrightarrow & [\![B]\!] \\ & & x & \longmapsto & [\![e]\!](x) \end{array}$$

Motivation for extending Fuzz

Motivation. Measure and track sensitivities for arbitrary L^p metrics, in particular to add noise according to different distributions:

- Laplace distribution for L^1,
- Gaussian distribution for L^2.

Reminder. For all $x, y \in \mathrm{R}^k$, $d_p(x, y) = \sqrt[p]{\sum_{i=1}^{k} |x_i - y_i|^p}$. For $p = 2$, this is the Euclidean metric.

Bunched Fuzz (Wunder et al. 2023)

The main idea is to introduce type constructors $\otimes_p$ and $\multimap_p$ for every parameter $p \in [1, +\infty]$. If $(a, b), (a', b') \in A \otimes_p B$, then

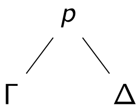
$$d_{\otimes_p}((a, b), (a', b')) = \sqrt[p]{d_A(a, a')^p + d_B(b, b')^p}.$$

Example

Semantically, we have rotate : Real $\otimes_2$ Real $\multimap_2$ Real $\otimes_2$ Real
but we do *not* have rotate : Real $\otimes_1$ Real $\multimap_1$ Real $\otimes_1$ Real

Bunched Fuzz contexts

The contexts are no longer lists of annotated variables, but trees.
For example, $\Gamma ,_p \Delta$ is the following tree:



and it is interpreted as $[\![\Gamma]\!] \otimes_p [\![\Delta]\!]$.

Example (Tensor introduction rule)

$$\frac{\Gamma \vdash a : A \qquad \Delta \vdash b : B}{\Gamma ,_p \Delta \vdash (a, b) : A \otimes_p B} \otimes I$$

Subject reduction property

Let $\downarrow$ be the evaluation relation for a standard big-step operational semantics.

~~Property~~ (subject reduction)

If $\emptyset \vdash a : A$ and $a \downarrow v$, then $\emptyset \vdash v : A$.

Bunched Fuzz does not satisfy the subject reduction property.

3. Plurimetric Fuzz

We extend Fuzz to L^p metrics using ordinary contexts to recover the subject reduction property.

Plurimetric Fuzz Types

Plurimetric Fuzz has:

- primitives types: Unit, Real, etc.;
- sum types: $A \oplus B$;
- product types: $A \otimes_p B$ for $p \in [1, +\infty]$;
- function types: $A \multimap_p B$ for $p \in [1, +\infty]$;
- exponential types: $!_s A$ for $s \in (0, +\infty]$;

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- distribution types: $\bigcirc A$;
- recursive types: $\mu\alpha.A$.

Plurimetric Fuzz Contexts

Idea. We only allow one parameter per context, so these are lists again, but with an additional annotation: $(p)\ \Gamma$.
We lose some flexibility, but we gain the subject reduction property.

Example

Tensor introduction rule

$$\frac{(p)\ \Gamma \vdash a : A \qquad (p)\ \Delta \vdash b : B}{(p)\ \mathrm{Contr}\,(p; \Gamma; \Delta) \vdash (a, b) : A \otimes_p B} \otimes I$$

Definition

$$\mathrm{Contr}\left(p; [x : A]_{s1}; [x : A]_{s2}\right) = [x : A]_{\sqrt[p]{s_1^p + s_2^p}}$$

Weakening rules

Two rules allow for changing the parameter of a context:

$$\frac{(p)\ \Gamma \vdash a : A \quad \Gamma \leq \Delta \quad p \geq q}{(q)\ \Delta \vdash a : A} \geq W \qquad \frac{(p)\ \Gamma \vdash a : A \quad \Gamma \leq \Delta \quad p \leq q}{(q)\ 2^{1/p - 1/q} \cdot \Delta \vdash a : A} \leq W$$

Lemma (Subtyping)

For all types A, B and parameters $p \leq q$, there exists two terms le *and* ge *such that*

$$\emptyset \vdash \mathsf{le} : A \otimes_p B \multimap_p A \otimes_q B \qquad \textit{and} \qquad \emptyset \vdash \mathsf{ge} : !_{2^{1/p-1/q}}(A \otimes_q B) \multimap_q A \otimes_p B$$

Examples

- lists, defined as $\mu\alpha.\mathsf{Unit} \oplus (A \otimes_p \alpha)$ generalising the list type constructor in Fuzz, and functions on them (map, fold, take, etc.);
- matrices, defined as $(A \otimes_p \ldots \otimes_p A) \otimes_1 \ldots \otimes_1 (A \otimes_p \ldots \otimes_p A)$ and functions on them;
- some machine-learning algorithms (gradient descent, k-means, k-nn clustering, etc.).

Translation of Fuzz judgements

Theorem

For all parameters p, the image by the mapping $P(p)$ of a (minimal) derivable judgement in Fuzz is a (minimal) derivable judgement in Plurimetric Fuzz.

Example

$$[n : \mathsf{Nat}]_2 \vdash (1, n, n) : \mathsf{Nat} \otimes \mathsf{Nat} \otimes \mathsf{Nat}$$

translates to the following judgement for $p = 2$:

$$(2)\ [n : \mathsf{Nat}]_{\sqrt{2}} \vdash (1, n, n) : \mathsf{Nat} \otimes_2 \mathsf{Nat} \otimes_2 \mathsf{Nat}$$

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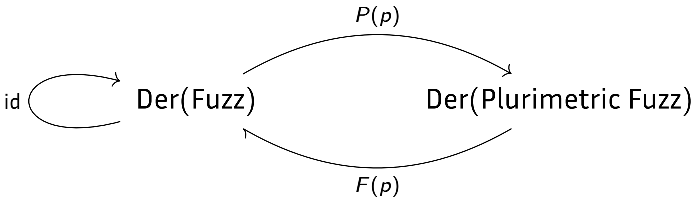
Remark

Not all Plurimetric Fuzz judgements can be obtained this way.

For all parameters p, the following diagram commutes:



Translation of primitive operations

The translation of the logical connectives does not extend to the primitive operations.

Example

For $p = 2$,

$$\frac{[x : \mathsf{Real}]_1 \vdash x : \mathsf{Real} \qquad [y : \mathsf{Real}]_1 \vdash y : \mathsf{Real}}{[x : \mathsf{Real}]_1, [y : \mathsf{Real}]_1 \vdash x + y : \mathsf{Real}} +$$

does not soundly translate to

$$\frac{(2)\ [x : \mathsf{Real}]_1 \vdash x : \mathsf{Real} \qquad (2)\ [y : \mathsf{Real}]_1 \vdash y : \mathsf{Real}}{(2)\ [x : \mathsf{Real}]_{\sqrt{1}}, [y : \mathsf{Real}]_{\sqrt{1}} \vdash x + y : \mathsf{Real}} +$$

Type safety

Proposition (substitution)

If we have

- $(p)\ \Delta \vdash a : A$;
- $(p)\ \Gamma, [x : A]_s \vdash b : B$;

then we have

$$(p)\ \text{Contr}\,(p; \Gamma; s\Delta) \vdash b[x \mapsto a] : B$$

Theorem (subject reduction)

If $(p)\ \emptyset \vdash a : A$ and $a \downarrow v$, then $(p)\ \emptyset \vdash v : A$.

Conclusion

Contribution: a standard (i.e., not bunched) type system that extends Fuzz to L^p metrics:

- such that the subject reduction property is satisfied
- with recursive types and a form of subtyping;
- that can be translated from and to Fuzz.

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- such that the subject reduction property is satisfied
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Future work may include:

- generalisation to other metrics on distributions;
- automatic type checking and type inference;

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