>>> Bounded Henkin quantifiers and the exponential time hierarchy

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Lille, GT-Scalp

>>> Plan of talk

Henkin quantifiers

>>> Plan of talk

Henkin quantifiers + Bounded quantifiers

>>> Plan of talk

Henkin quantifiers + Bounded quantifiers = Bounded Henkin quantifiers

Henkin quantifiers

>>> Hintikka's sentence

‘‘Some relative of each villager and some relative of each townsman hate each other.’’

$$(V(x_1) \wedge T(x_2)) \; / \; (R(x_1; y_1) \wedge R(x_2; y_2) \wedge H(y_1; y_2))$$

>>> Hintikka's sentence

‘‘Some relative of each villager and some relative of each townsman hate each other.’’

$$\begin{matrix} \forall x_1 \exists y_1 \\ \forall x_2 \exists y_2 \end{matrix} \quad (V(x_1) \wedge T(x_2)) \rightarrow (R(x_1, y_1) \wedge R(x_2, y_2) \wedge H(y_1, y_2))$$

>>> Definition by examples

$$H_2^2 = \begin{matrix} \forall x_1 \exists y_1 \\ \forall x_2 \exists y_2 \end{matrix} = \begin{matrix} \forall x_1 & & \forall x_2 \\ \vert & & \vert \\ \exists y_1 & & \exists y_2 \end{matrix}$$

>>> Definition by examples

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Henkin quantifier

A triple $Q = (A; E; d)$ such that $d \quad A \quad E$. A Hekin quantifier is called standard if it can be written like a matrix.

>>> Interpreting Henkin quantifiers

Interpret by Skolemisation:

$$\exists f \exists g \forall x_1 \forall x_2 \, (V(x_1) \wedge T(x_2)) \;!\; (R(x_1; f(x_1)) \wedge R(x_2; g(x_2)) \wedge H(f(x_1); g(x_2))$$

$L(H)$

Language of first-order logic extended by Henkin quantifiers

* $L(H)$ cannot be recursively axiomatised (Erhenfeucht)
* $L(H)$ equivalent to existential second-order logic (Enderton-Walkoe)
* Over finite structures, $L(H)$ can express exactly NP predicates (Blass-Gurevich)

Positive formula

An $L(H)$ formula where Henkin quantifiers occur under an even number of negations.

Proposition

Let be a positive $L(H)$ formula. There exists an H-formula Q such that (i) Q is standard (ii) is a quantifier-free, and (iii) and Q are equivalent.

Proposition

Every $L(H)$ formula is equivalent to an $L(H)$ formula of the form $R : Q_0 : Q_1 : : : : Q_n$ where R is either $: Q$ or Q, and $Q; Q_0; : : : ; Q_n$ are standard Henkin quantifiers.

Bounded Quantifiers and Computational Complexity

Terms

$$t; t' ::= x \; 2 \; Var \; j \; 0 \; j \; S(t) \; j \; t + t' \; j \; t \quad t' \; j \; jtj \; j \; t]t' \; j \; bt{=}2c$$

where jtj is intended to be interpreted as the number of digits in the binary representation of t and $t]t'$ as $2^{|t||t'|}$.

Formulas

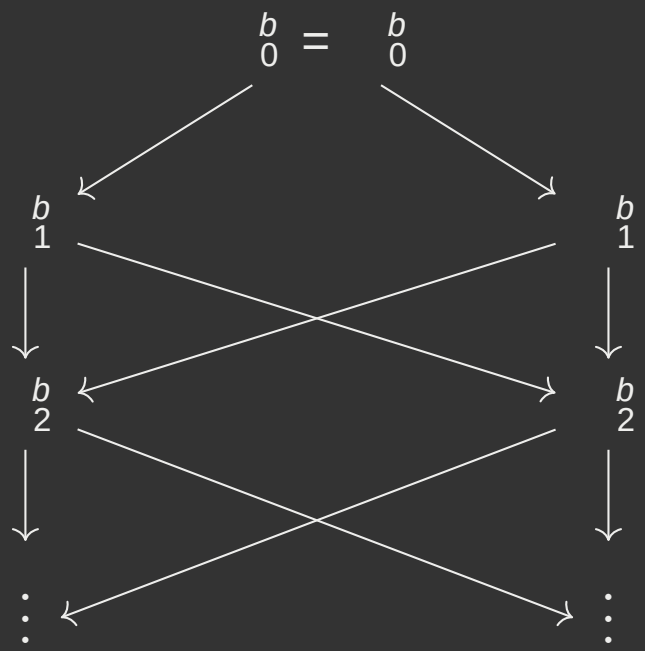
$$; \quad ::= s \quad t \; j \; : \quad j \quad \_ \quad j \quad \wedge \quad j \; 9x \quad s \quad j \; 8x \quad s$$

Quantifiers of the form $Qx \quad jsj$ are said to be sharply bounded. A formula is sharply bounded if all its quantifiers are sharply bounded.

>>> The Bounded Arithmetic Hierarchy

* $\Sigma^b_0 = \Pi^b_0$ are the set of sharply bounded formulas.
* $\Sigma^b_{i+1} = \{\exists x \leq s : j \leq 2 \mid \Pi^b_i\}$ modulo prenex operations.
* $\Pi^b_{i+1} = \{\forall x \leq s : j \leq 2 \mid \Sigma^b_i\}$ modulo prenex operations.

[Arbitrary sharply bounded quantifiers allowed in the 2nd and 3rd case]



>>> Capturing complexity classes

Proposition

Predicate $R \subseteq \mathbb{N}^k$ definable by a Σ^b_0 formula $\Leftrightarrow$ $R \in P$.

Theorem (Kent-Hodgson'82)

Predicate $R \subseteq \mathbb{N}^k$ definable by a Σ^b_1 formula $\Leftrightarrow$ $R \in NP$.

Corollary

Bounded Arithmetic Hierarchy corresponds to PH.

* Starting point of uniform proof complexity
* Consider weak sub-theories of PA in this language
* Characterise complexity classes in the sense that a function is provably total in a theory iff it belongs to a given complexity class.

>>> Second-order Bounded Arithmetic

* Language with second-order bounded quantification
* Captures the exponential hierarchy
* In particular, $\Sigma^{1;b}_1(\mathrm{N}) = NEXP$

Bounded Henkin Quantifiers

>>> Bounded Henkin quantifiers

Pretty much does what it says on the tin:

$$\begin{array}{l} 8x_1 9y_1 \\ 8x_2 9y_2 \end{array}$$

>>> Bounded Henkin quantifiers

Pretty much does what it says on the tin:

$$\begin{array}{cccc} 8x_1 & s_1 & 9y_1 & t_1 \\ 8x_2 & s_2 & 9y_2 & t_2 \end{array}$$

H-formulas

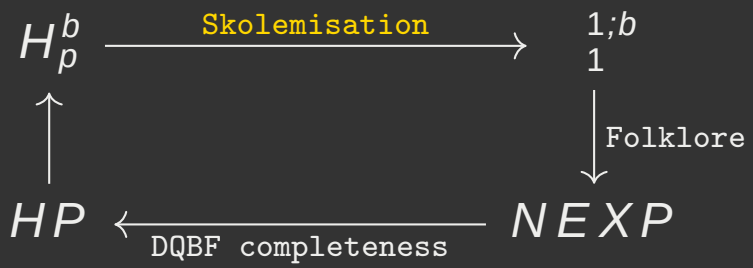
Formulas in the language of bounded arithmetic with bounded Henkin quantifiers.

Main result

Predicate $R \; 2 \; \mathsf{N}^k$ definable by a positive H-formula $( ) \;\; R \; 2 \; NEXP$

>>> Proof technique

$H_p^b := fR \quad \mathrm{N}^k \, j \, R$ definable by a positive H-formula$g$



Few points about Skolemisation

* There is polynomial bounded Gödel encoding of pairs:

$$\mathrm{p}hm; niq := \cancel{2^m 3^n} \; \mathrm{j} \frac{m+n}{2} \mathrm{k} \; m+n$$

* Bounded arithmetic can be bootstrapped with pairing function .

$$(i; \mathrm{p}ha_1; :::; a_k iq) = \begin{cases} n & \text{if } i = 0; \\ a_i & \text{if } 1 \quad i \quad k \end{cases}$$

* Therefore, Skolem functions can be replaced by polynomially bounded predicates.