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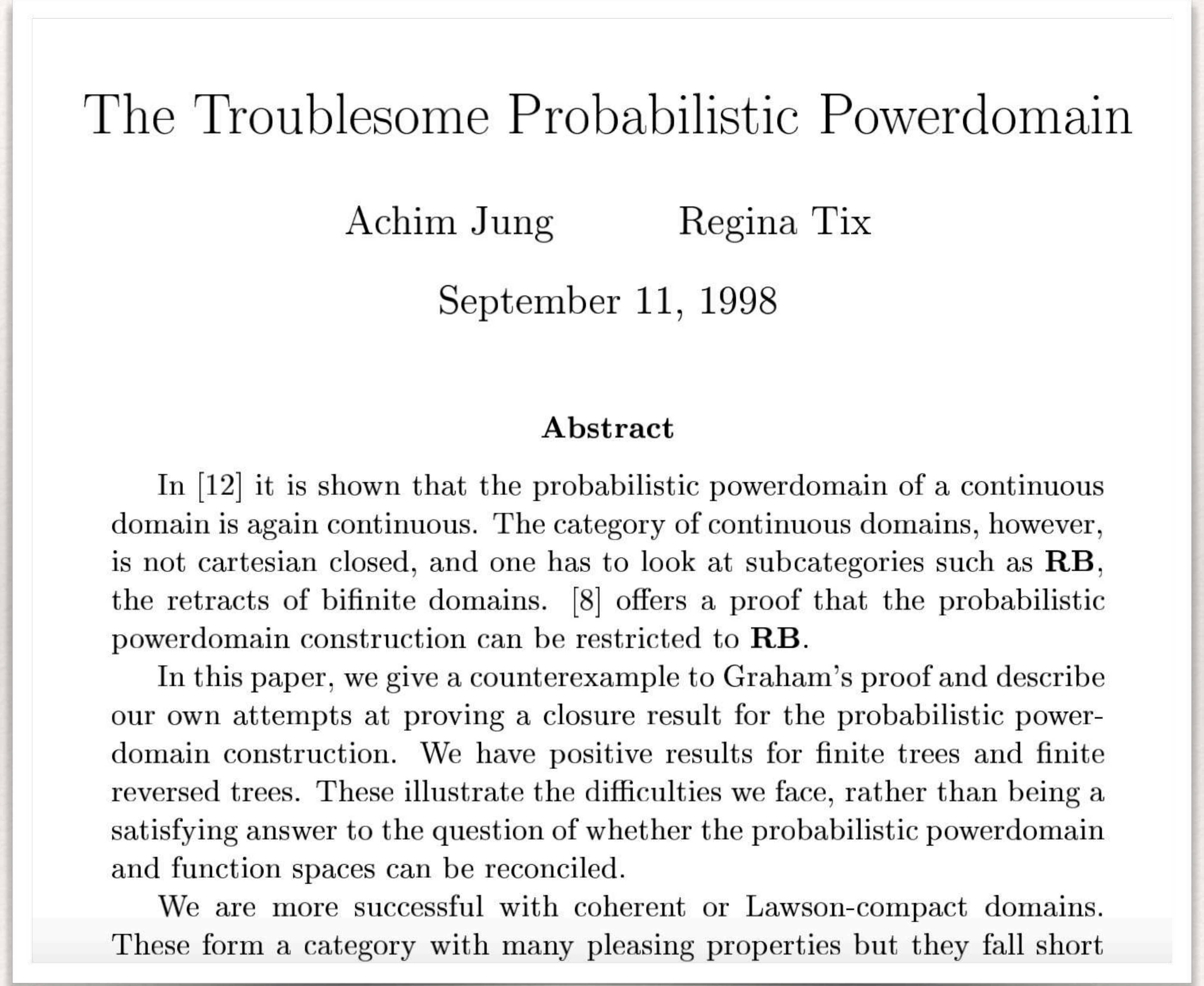
Is there any trouble with the probabilistic powerdomain monad?

GT Scalp, Lille, 18 novembre 2024

- A pun on the title of a famous paper by A. Jung and R. Tix
- To spoil the end of the talk:
  no, there is **no problem**
  with the probabilistic powerdomain
- … but there are many interesting questions



The Troublesome Probabilistic Powerdomain

Achim Jung Regina Tix

September 11, 1998

Abstract

In [12] it is shown that the probabilistic powerdomain of a continuous domain is again continuous. The category of continuous domains, however, is not cartesian closed, and one has to look at subcategories such as **RB**, the retracts of bifinite domains. [8] offers a proof that the probabilistic powerdomain construction can be restricted to **RB**.

In this paper, we give a counterexample to Graham's proof and describe our own attempts at proving a closure result for the probabilistic powerdomain construction. We have positive results for finite trees and finite reversed trees. These illustrate the difficulties we face, rather than being a satisfying answer to the question of whether the probabilistic powerdomain and function spaces can be reconciled.

We are more successful with coherent or Lawson-compact domains. These form a category with many pleasing properties but they fall short

Part I: domain theory and semantics

Bugs, and verification

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- E.g., do the following two programs compute the same thing?

  $$\mathbf{do}\ x \leftarrow \mathbf{rand3};\ (\mathbf{do}\ y \leftarrow \mathbf{ret}\ 0 \oplus \mathbf{ret}\ 1;\ \mathbf{ret}\ (x-y))$$
  $$\mathbf{do}\ y \leftarrow \mathbf{ret}\ 0 \oplus \mathbf{ret}\ 1;\ (\mathbf{do}\ x \leftarrow \mathbf{rand3};\ \mathbf{ret}\ (x-y))$$

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- That would seem obvious, right?
  The only difference is the order in which x and y are drawn at random.

Denotational semantics

- In order to settle the question,
  one needs to know **what** programs compute
- This is the role of **denotational semantics**,
  defining the **value** $[\![M]\!]$ of each program M:

$$[\![MN]\!] = [\![M]\!]([\![N]\!])$$
$$[\![\lambda x\,.\,M]\!] = (x \mapsto [\![M]\!])$$
$$[\ldots]$$
$$[\![\mathbf{rec}\ M]\!] = \text{least fixed point } \sup_{n \in \mathbb{N}} [\![M]\!]^n(\bot)$$
$$[\![M \oplus N]\!] = \tfrac{1}{2}\,[\![M]\!] + \tfrac{1}{2}\,[\![N]\!]$$
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  - Dcpos and denotational semantics
  - Continuous valuations, and the problem
  - A solution

Dcpos, partial values, and the order of information

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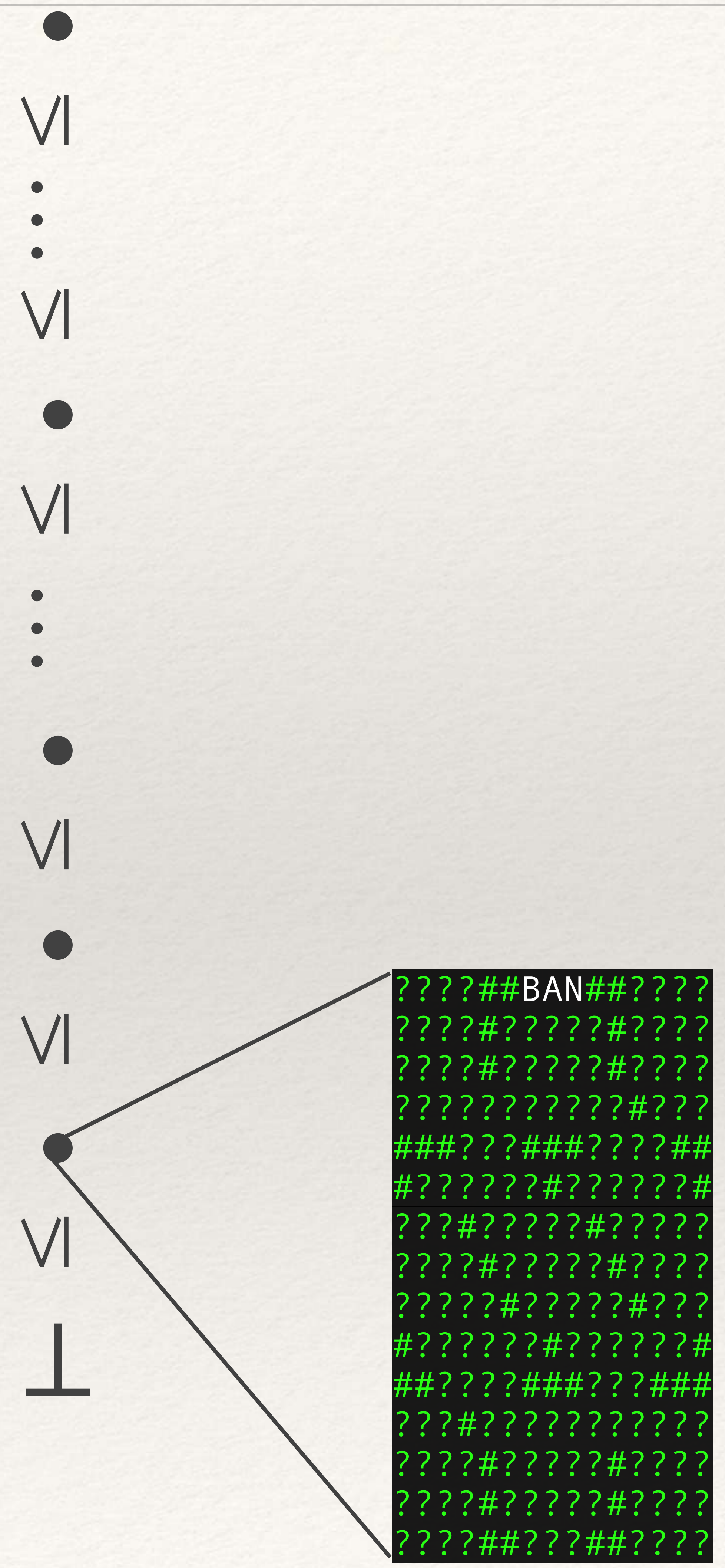

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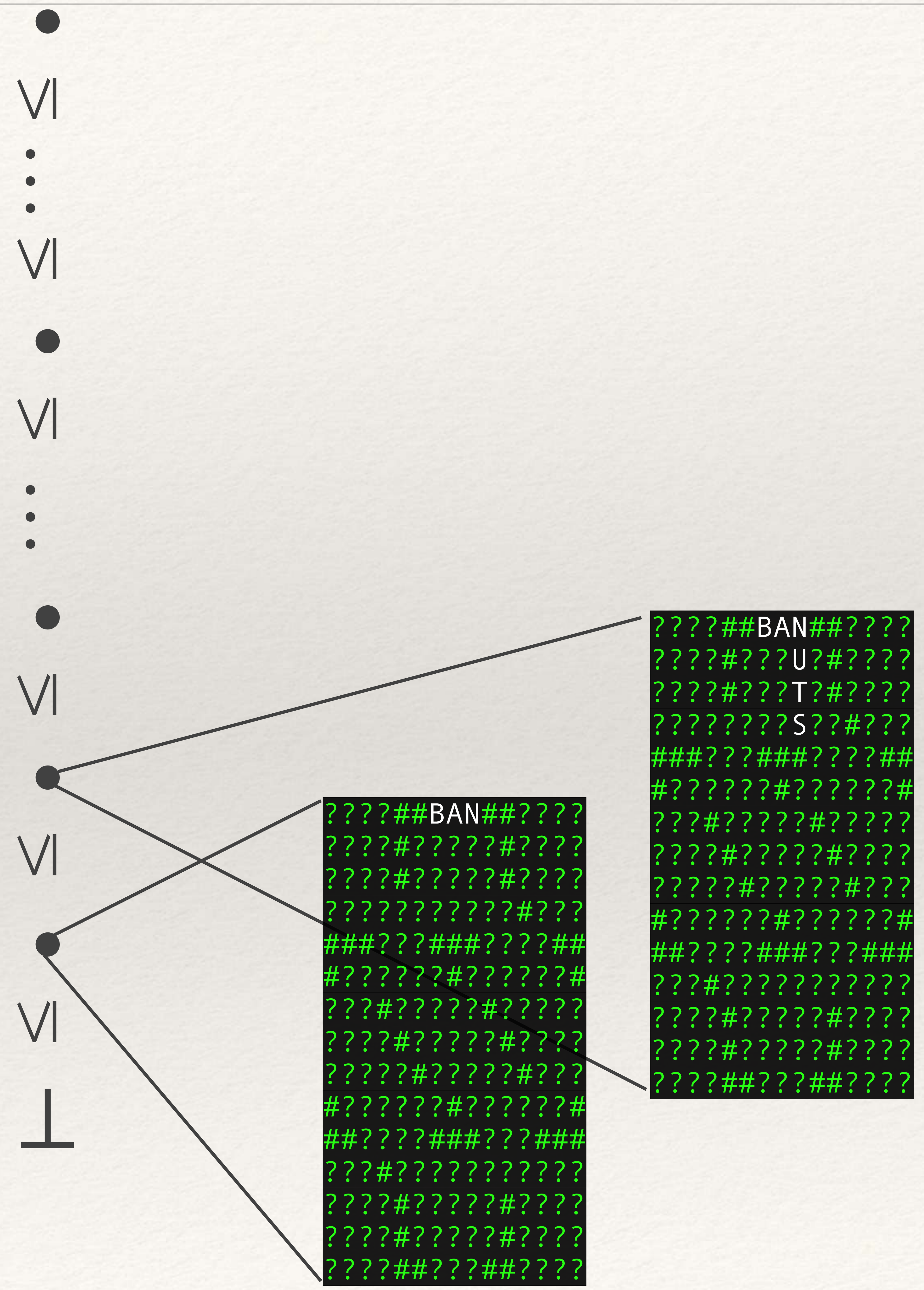

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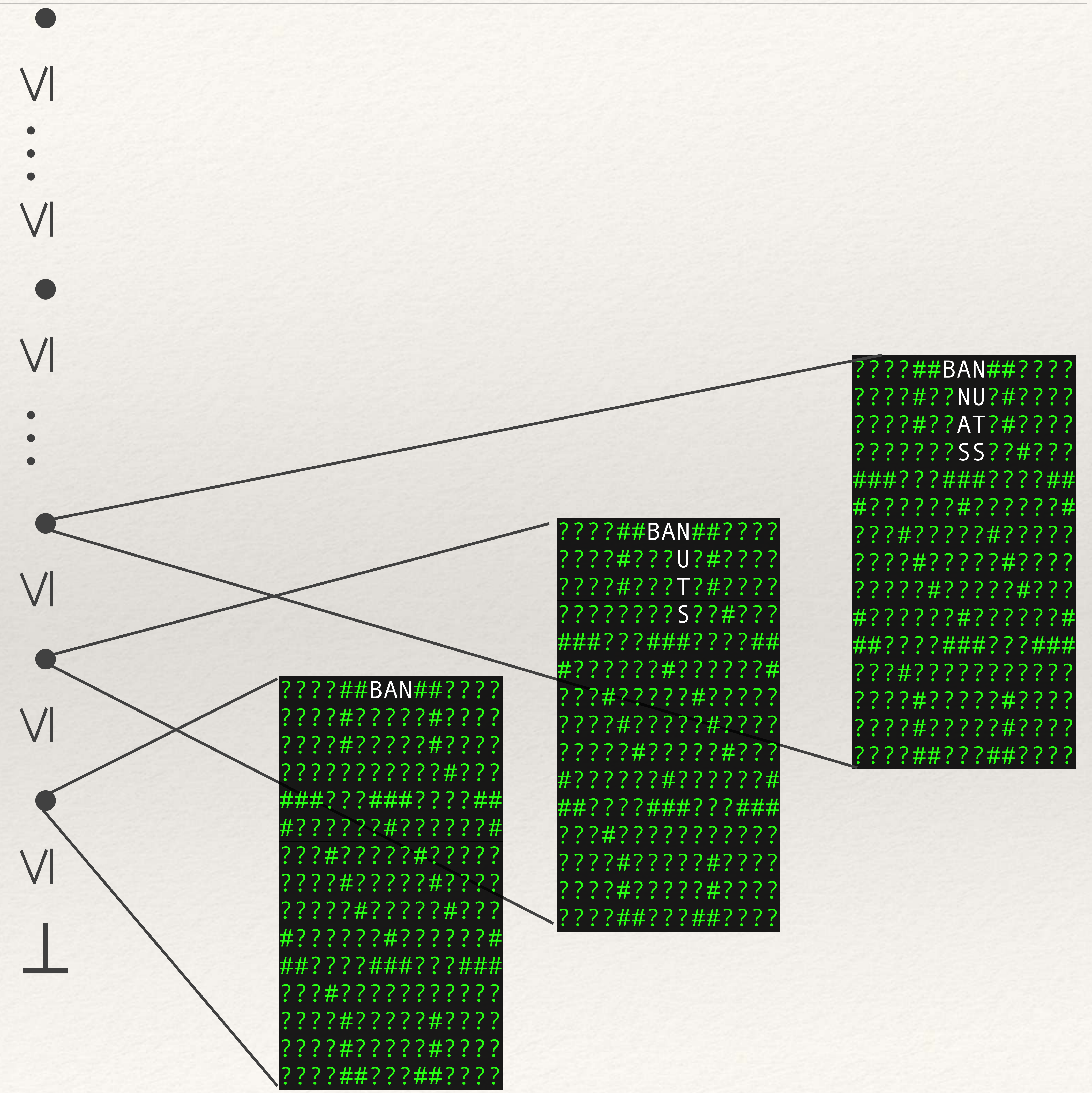

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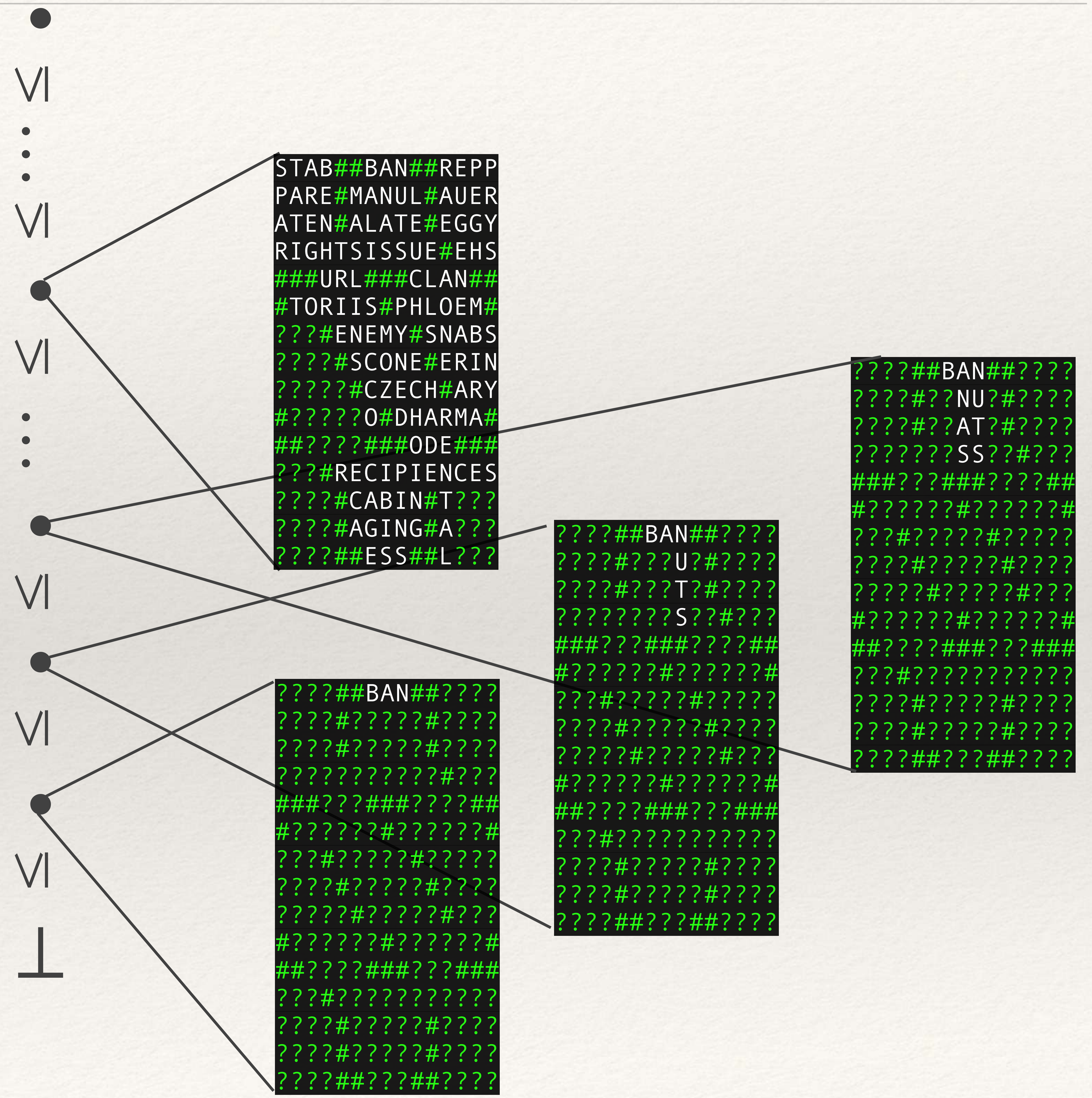

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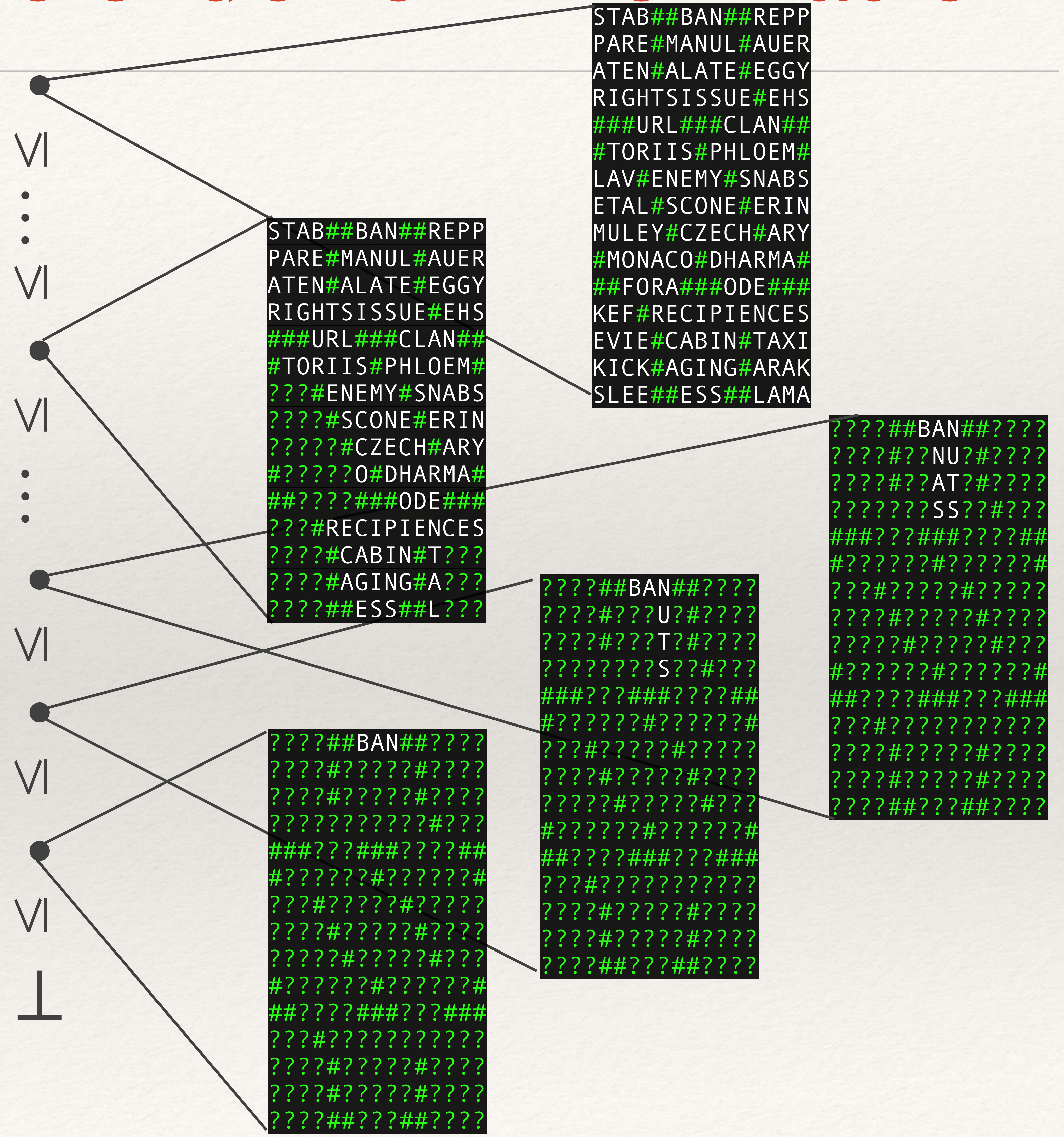

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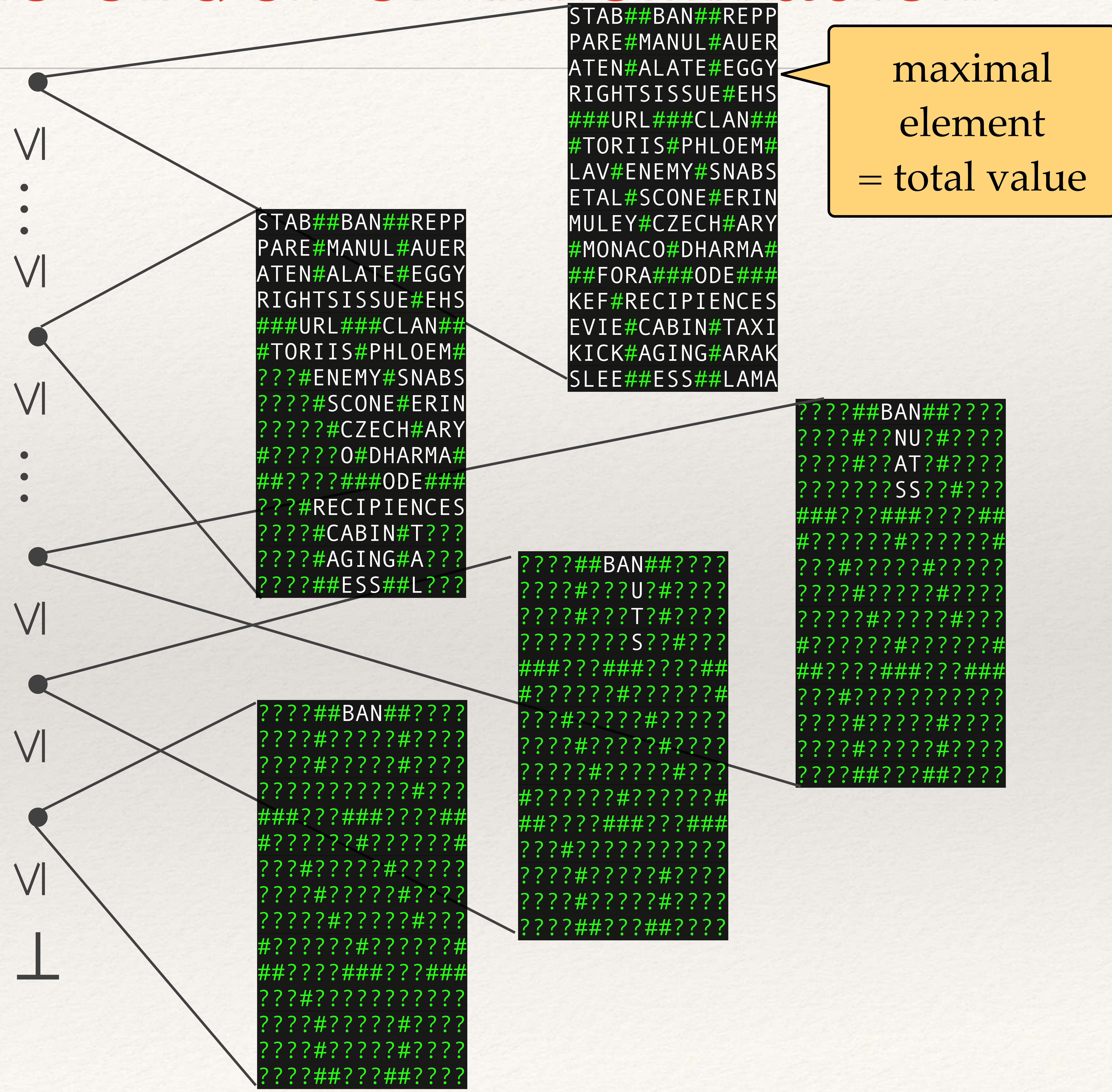

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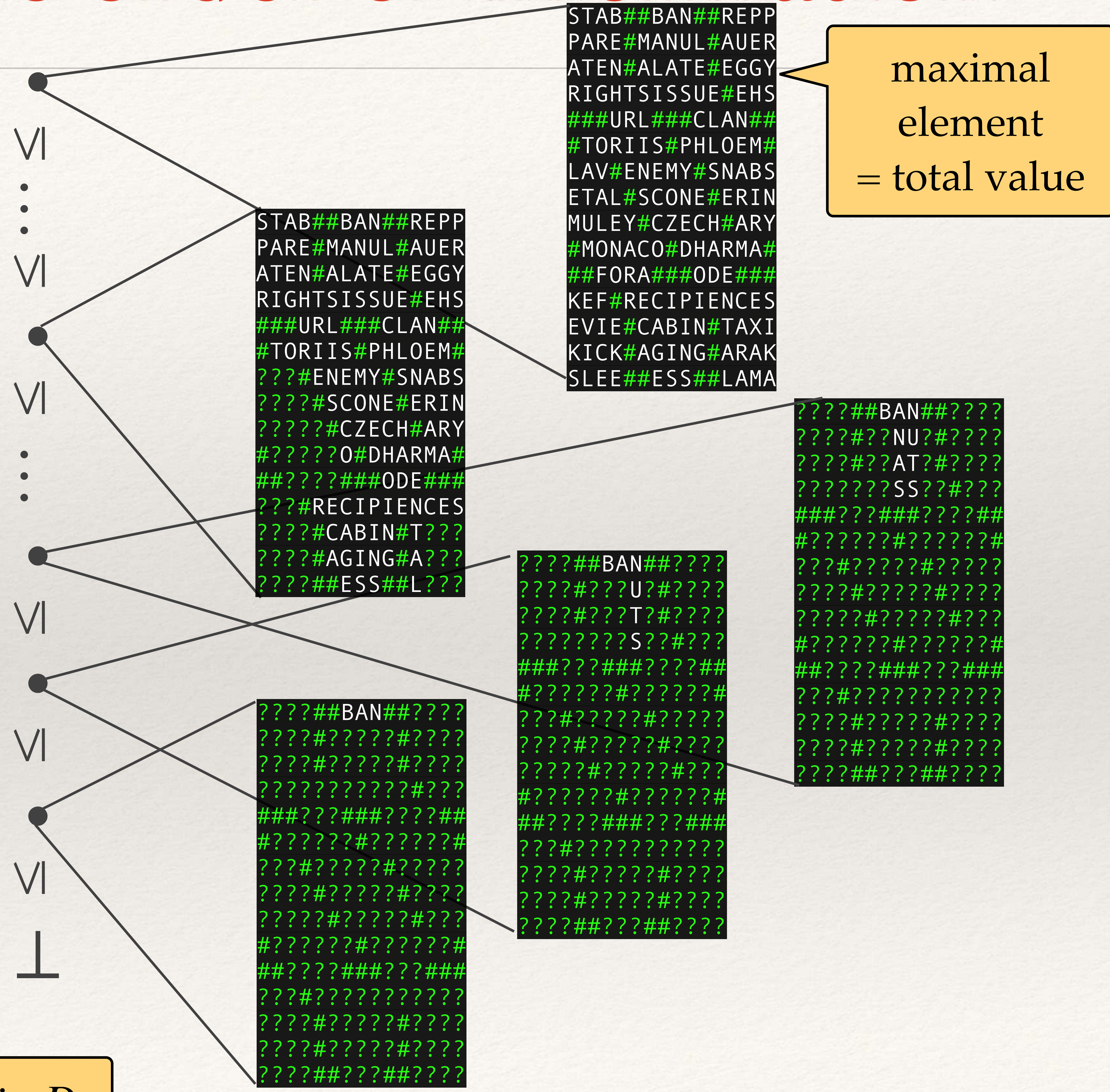
non-empty + every pair of elements of D has an upper bound in D



Basic dcpo constructions

- Given dcpos X, Y, the following are dcpos:
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- On a **pointed** dcpo X, every Scott-continuous map $f : X \to X$
  has a **least fixed point** $\mathrm{lfp}(f) = \sup_{n \in \mathbb{N}} f^n(\perp)$

and to recursion

A first, simple application: PCF

- Consider the higher-order, functional programming language PCF [Plotkin 77]

$M, N, P, \ldots ::= x, y, z, \ldots$	variables
$\mid MN$	application
$\mid \lambda x_\sigma . M$	abstraction
$\mid \mathbf{rec}(M)$	recursion
$\mid 0 \mid 1 \mid 2 \mid \ldots$	natural numbers
$\mid \mathbf{s}(M)$	successor
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elementary operations on type **nat**

Types

- PCF terms are **typed**: $\sigma, \tau, \ldots ::= \mathbf{nat} \mid \sigma \to \tau$

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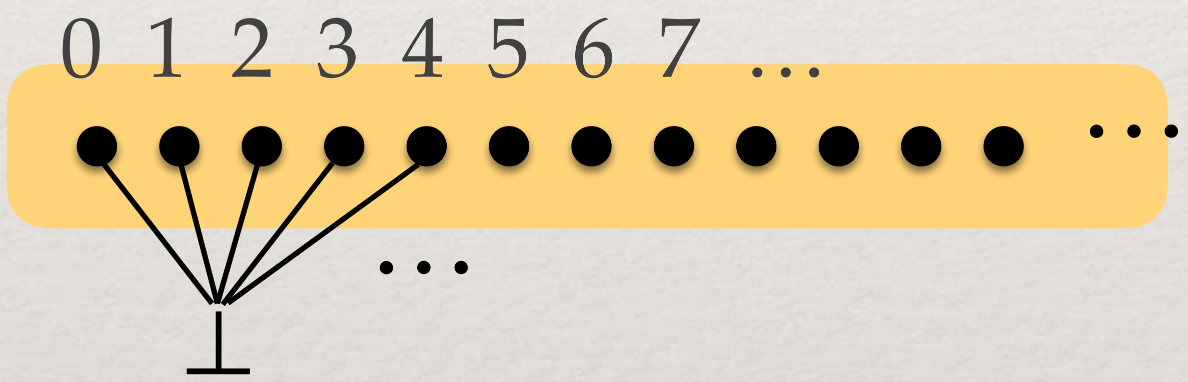

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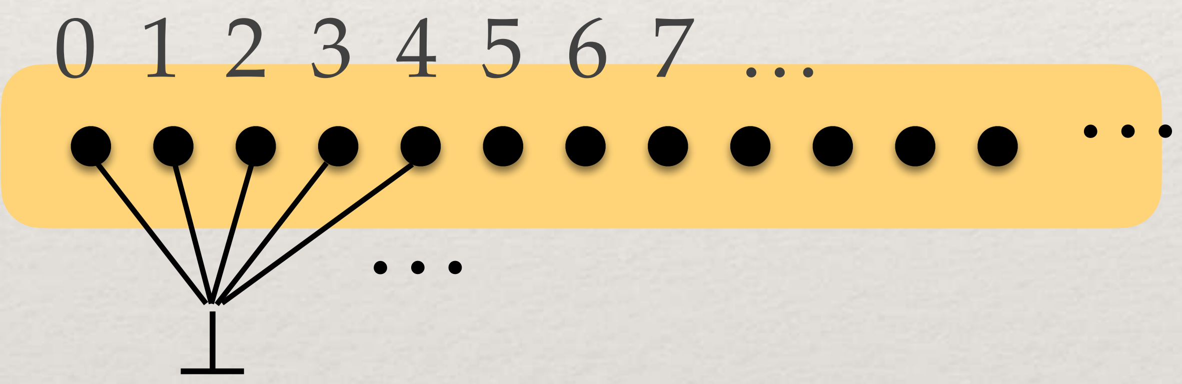

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A denotational semantics for PCF

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On **nat**:

- $[\![0]\!]\rho \triangleq 0$, $[\![1]\!]\rho \triangleq 1$, etc.
- $[\![\mathbf{s}(M)]\!]\rho \triangleq [\![M]\!]\rho+1$, $[\![\mathbf{p}(M)]\!]\rho \triangleq [\![M]\!]\rho-1$ if $[\![M]\!]\rho\neq 0, \perp$
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Lambda-calculus:

- $[\![x]\!]\rho \triangleq \rho(x)$
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Theorem 1. On a pointed dcpo X,
every Scott-continuous map
$f : X \to X$ has a least fixed point.

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- $[\![MN]\!]\rho \mathrel{\hat=} [\![M]\!]\rho([\![N]\!]\rho)$
- $[\![\lambda x\,.\,M]\!]\rho \mathrel{\hat=} (V \mapsto [\![M]\!](\rho[x \mapsto V]))$
- $[\![\mathbf{rec}(M)]\!]\rho \mathrel{\hat=} \mathit{lfp}([\![M]\!]\rho)$

Theorem 1. On a pointed dcpo X, every Scott-continuous map $f : X \to X$ has a least fixed point.

- Expressions have **transparent** semantics (functions are functions, application is application, etc.)
- **compositional** semantics: $[\![M]\!]\rho$ defined from the semantics of immediate **subterms** of M
- No **execution** mechanism involved

An operational semantics for PCF

- An **abstract machine** (à la Krivine) = a transition relation between **configurations** C, M

Contexts $C ::= \_ \mid C[\_N] \mid C[\mathbf{s}(\_)] \mid C[\mathbf{p}(\_)] \mid \mid C[\mathbf{if}\ \_ = 0\ \mathbf{then}\ N\ \mathbf{else}\ P]$

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Exploration rules (looking for redexes)

$$C, MN \to C[\_N], M$$
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Computation rules

$$
\begin{aligned}
C[\_N], \lambda x\,.\,M &\to C, M[x := N] \\
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C[\mathbf{p}(\_)], n+1 &\to C, n \\
C[\mathbf{if}\ \_ = 0\ \mathbf{then}\ N\ \mathbf{else}\ P], 0 &\to C, N \\
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- Proof through **logical relations** [Plotkin 77].

Probabilistic PCF

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\begin{array}{lr}
M, N, P, \ldots ::= \ldots & \text{(as in PCF)} \\
\quad \mid M \oplus N & \text{probabilistic choice} \\
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Probabilistic PCF

- $M, N, P, \ldots ::= \ldots$ (as in PCF)
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- Types: $\sigma, \tau, \ldots ::= \mathbf{nat} \mid \mathbf{unit} \mid \sigma \to \tau \mid \mathbf{T}\tau$

Tτ = type of **(first-class)** distributions

$\mathbf{T}\tau$: monadic types [Moggi 91]

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- New operational rules:

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Denotational semantics for probabilistic PCF

- Introduced in Claire Jones' PhD thesis [Jones 90]
- $[\![\mathbf{T}\tau]\!] \triangleq \mathbf{V}_{\leq 1}([\![\tau]\!])$ dcpo of **subprobability valuations** on $[\![\tau]\!]$
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Let us define all that first!

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Continuous valuations

- First studied by [SahebDjahromi 80]: gives mass to **Scott-open** subsets
- Makes sense on **every topological space**—in particular, dcpos with the Scott topology
- Let $\mathcal{O}X$ denote the lattice of open subsets of a space X
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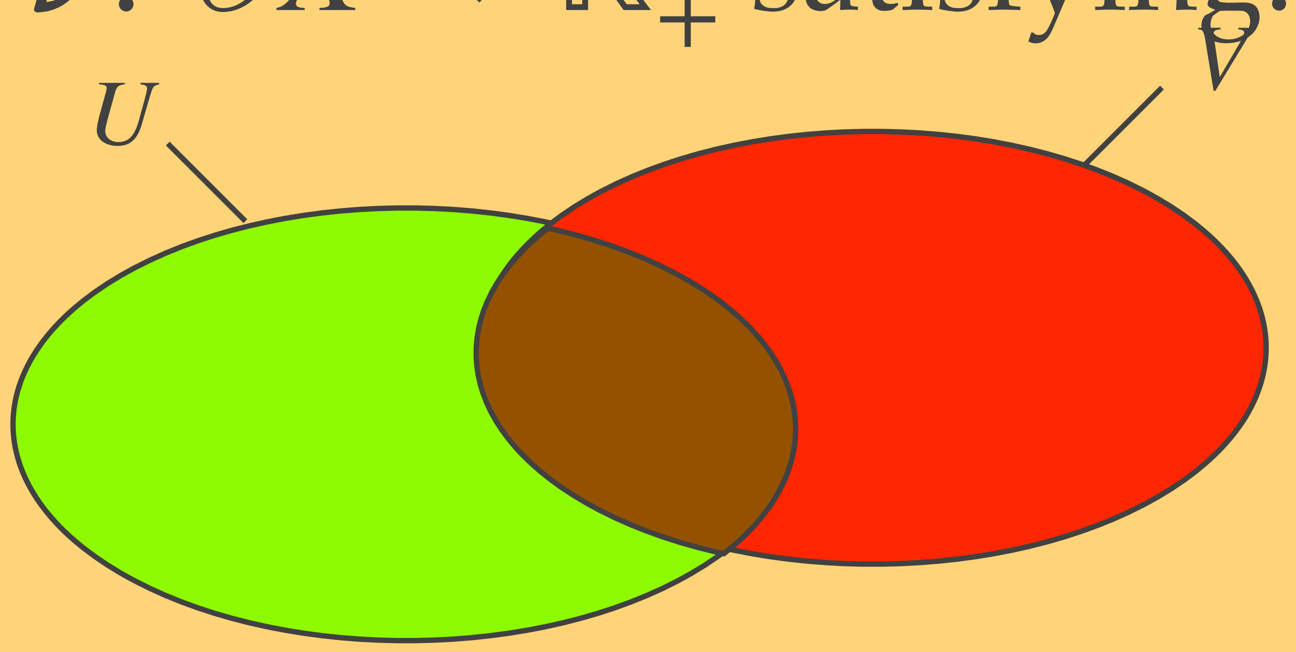

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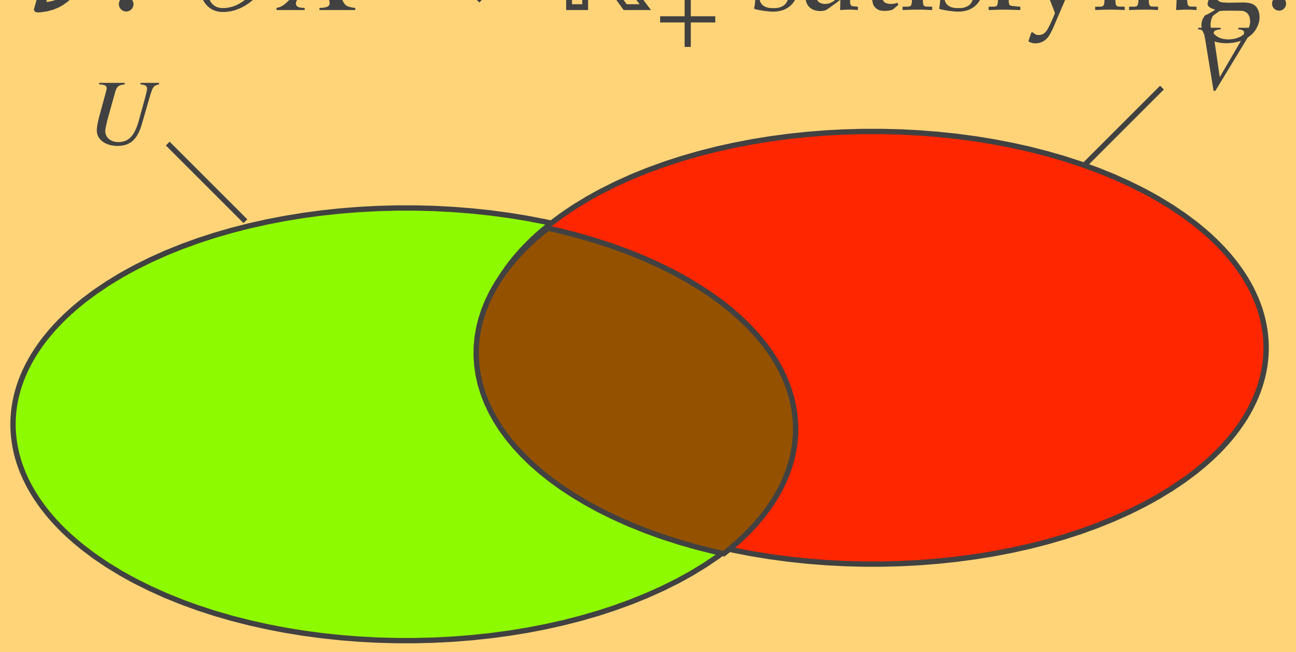

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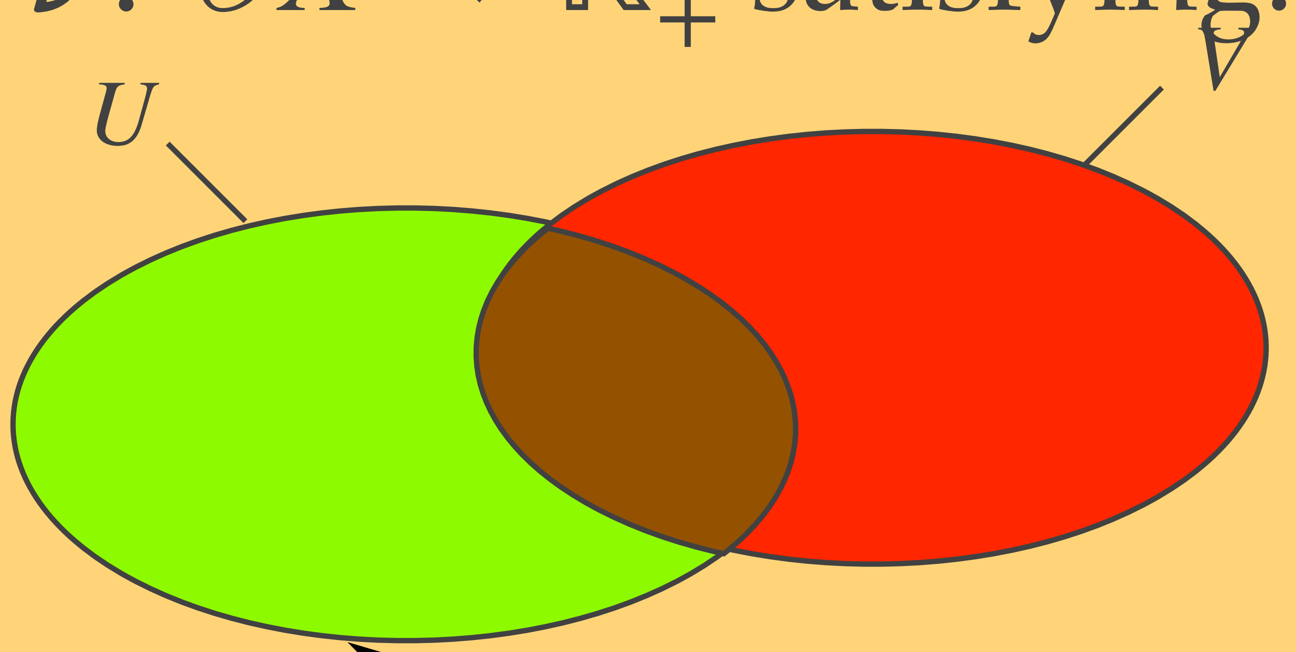

I will concentrate on **subprobability** valuations: $\nu(X) \leq 1$

Simple valuations

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$$\delta_x(U) \triangleq \begin{cases} 1 & \text{if } x \in U \\ 0 & \text{otherwise} \end{cases}$$
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Really defines a **strong monad** on **Dcpo**

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- Do the following two programs compute the same thing?
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There is a very subtle issue here … as Fubini-Tonelli holds on the **larger** category **Top** 😖

Fubini-Tonelli theorems

Fubini-Tonelli for continuous valuations

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(dcpo $X \times Y$)

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More generally:
— if X_σ or Y_σ is core-compact [Gierz,Hofmann,Keimel,Lawson,Mislove 03]
— if X_σ and Y_σ are first-countable [de Brecht, priv. comm., 19]
— if X and Y are lc_ω-dcpos [Lawson, Xu 24]

Continuous dcpos, a.k.a. domains

- Motto: the continuous dcpos are the **nice** dcpos,
  where (almost) every property you wish for is true

Let us skip that.

Continuous dcpos, a.k.a. domains

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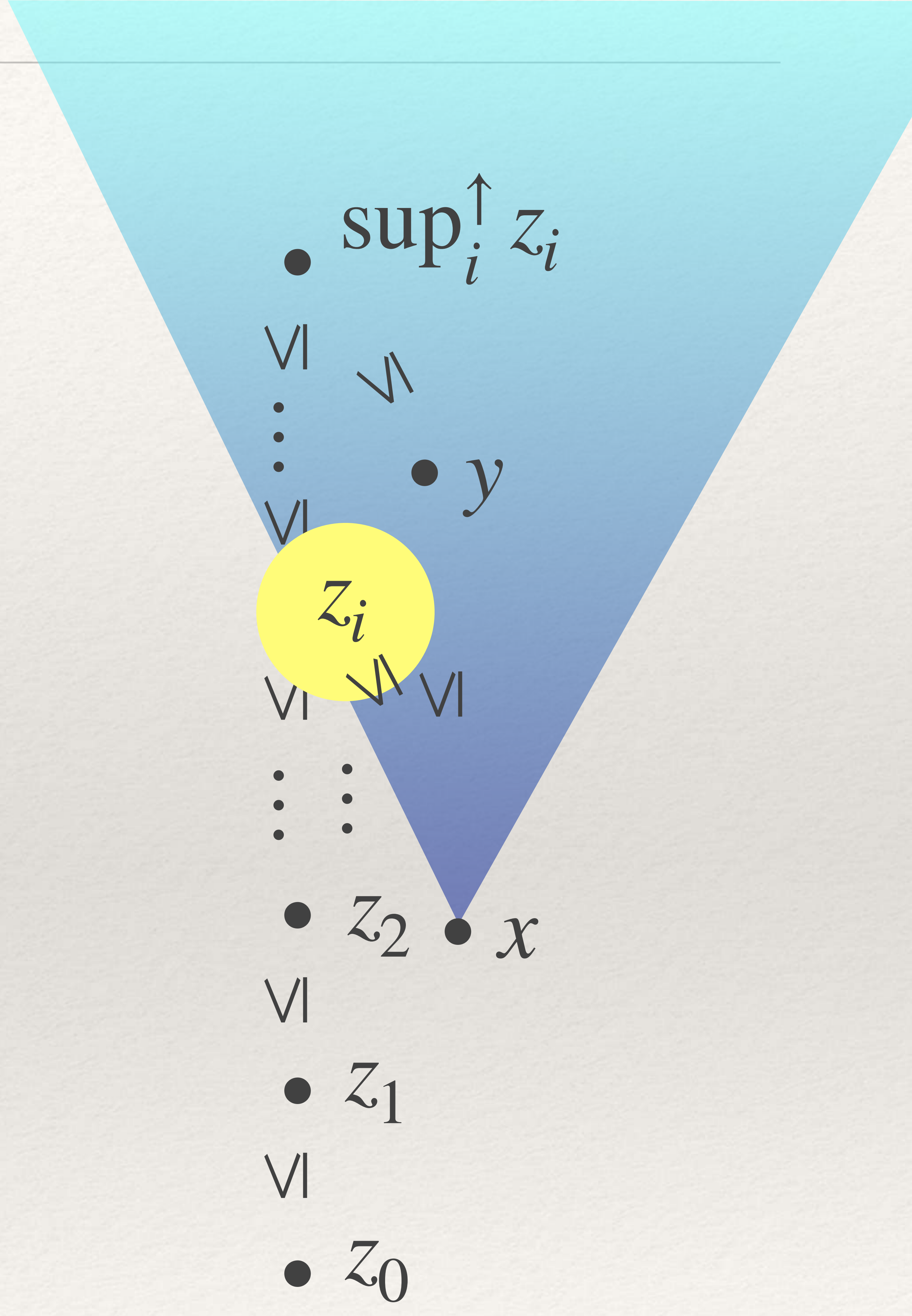
- Let $x \ll y$ (x **way-**below y) iff
  $y \leq \sup_i^\uparrow z_i$ implies $\exists i, x \leq z_i$

Let us skip that.

- **Definition.** A dcpo is **continuous** iff
  every point x is the supremum
  of some directed family of points **way-below** x.



Fubini-Tonelli for continuous valuations

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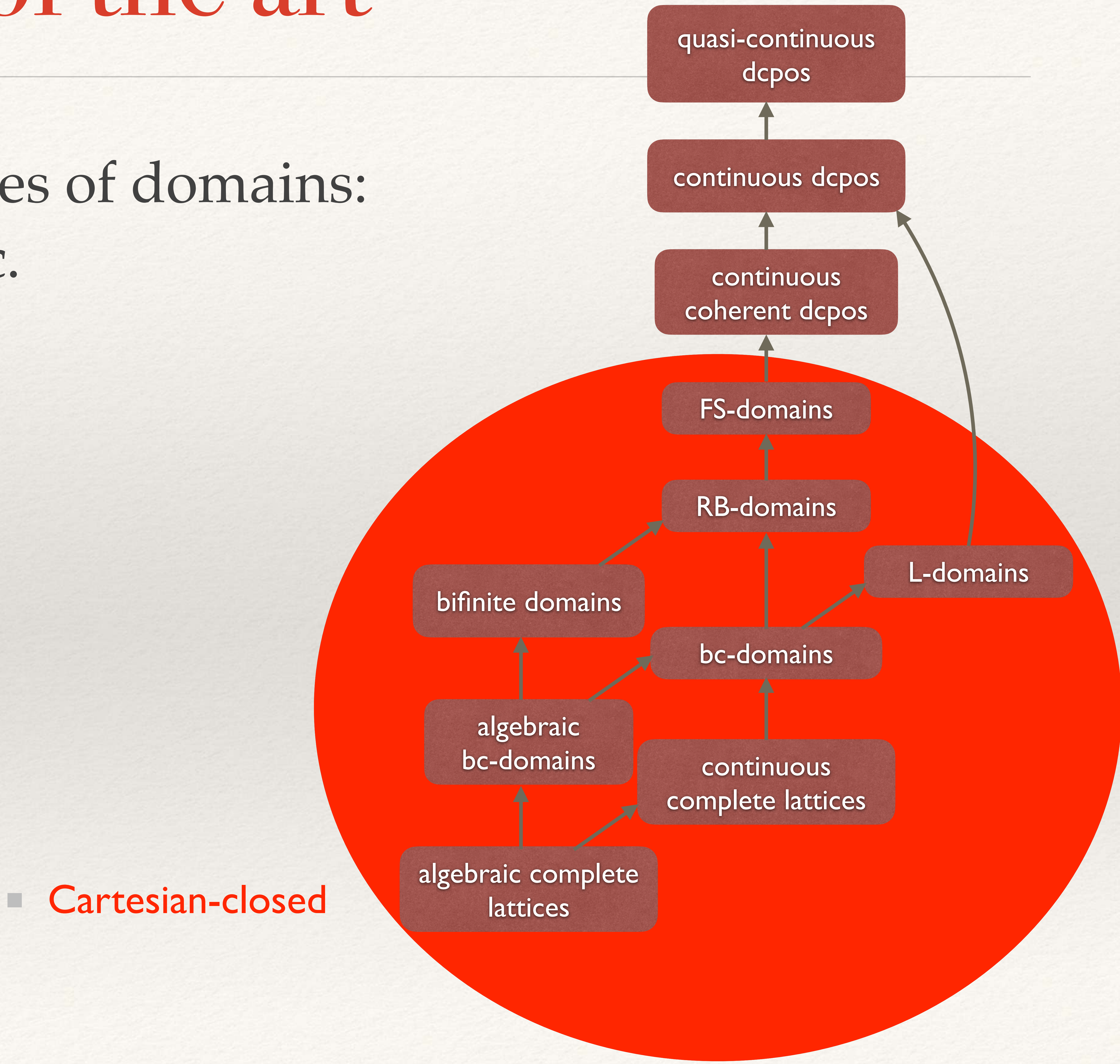
- Good news [Jones89]: $\mathbf{V}_{\leq 1}$ restricts to a (commutative) monad on **Cont**
- Bad news: **Cont** is not Cartesian-closed (had been known for a long time)
- Research took the path of looking for Cartesian-closed subcategories of **Cont** on which $\mathbf{V}_{\leq 1}$ restricts

The state of the art

- There are other Cartesian-closed categories of domains: L-domains, RB-domains, FS-domains, etc.



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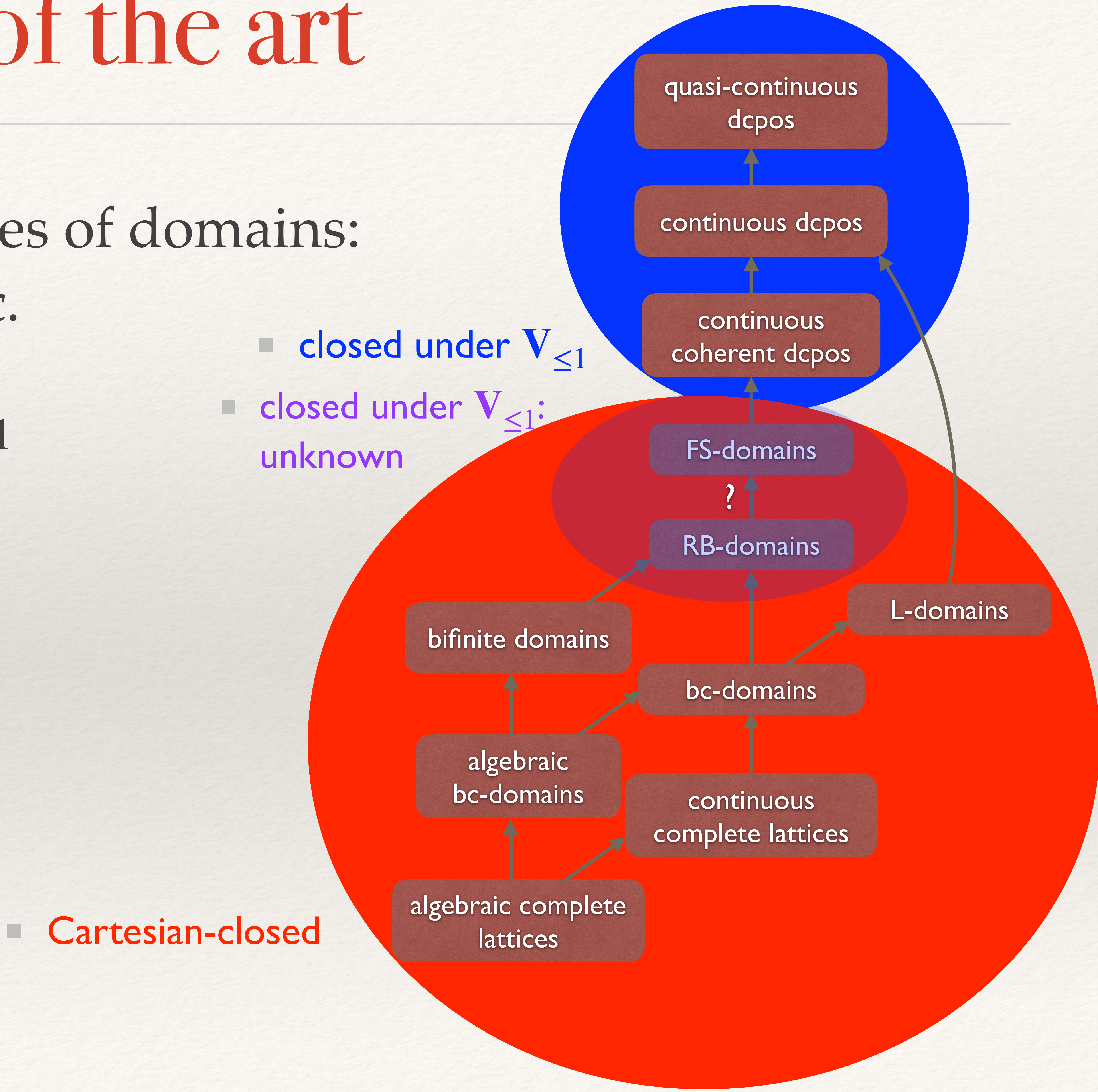

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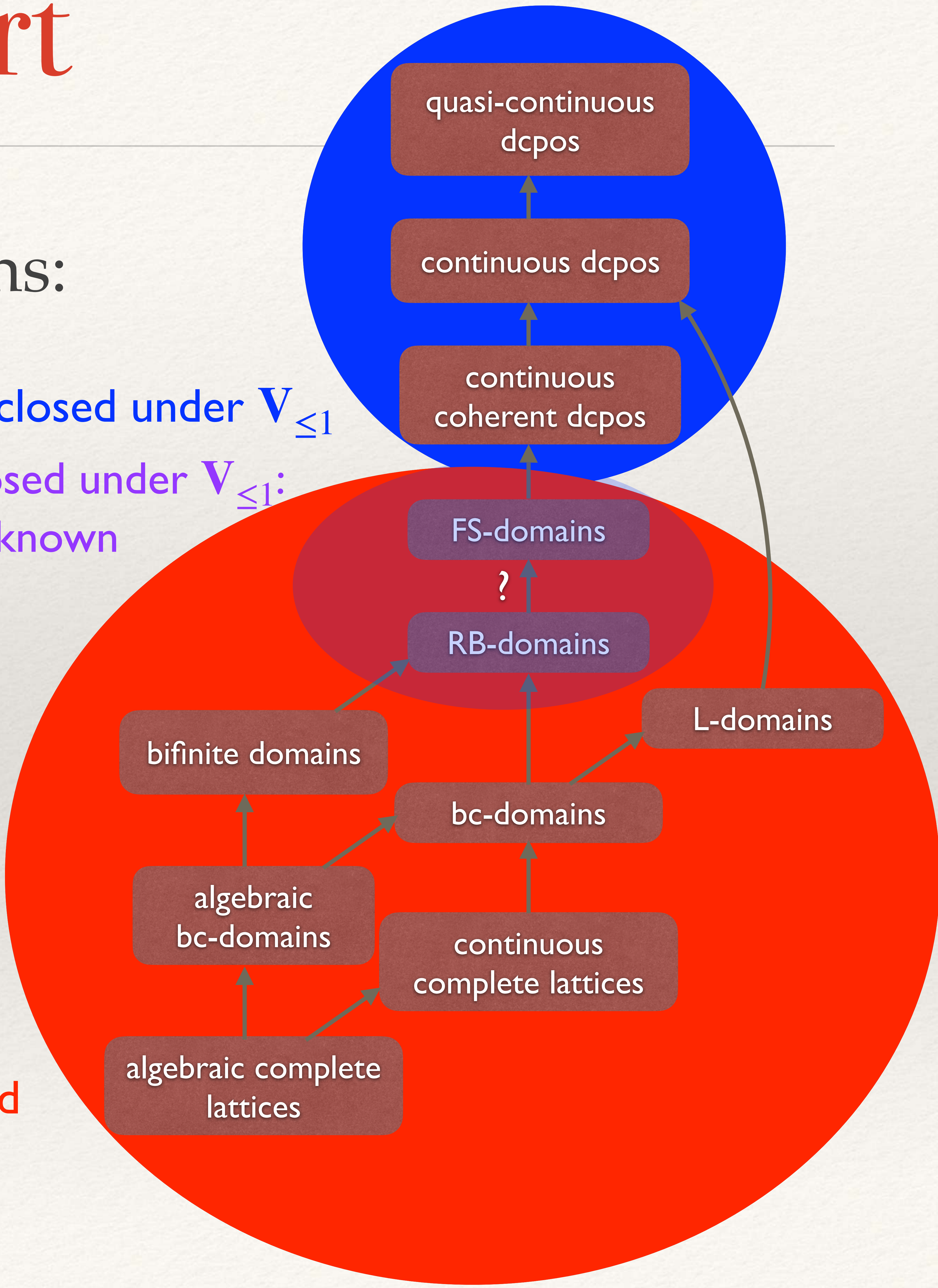
- 😕 None is known to be closed under $\mathbf{V}_{\leq 1}$
- As of 2023, the best results are still those of [Jung,Tix 98]

apart from [JGL 12] ($\mathbf{V}_{\leq 1}$(QRB-domain) is a QRB-domain)
or [Mislove 20] ($\mathbf{V}_{\leq 1}$(chain) is a continuous lattice)
or [JGL 22] ($\mathbf{V}_{\leq 1}$(quasi-cont. dcpo) is quasi-continuous)

- closed under $\mathbf{V}_{\leq 1}$
- closed under $\mathbf{V}_{\leq 1}$: unknown
- Cartesian-closed



A solution to the problem

All **commutative monads**
on the Cartesian-closed category **Dcpo**

- Replace $\mathbf{V}_{\leq 1}$ by appropriate **submonads:**
  - **Minimal** valuations [JLMZ 21; JGL, Jia 23]

    $\subseteq$
  - **Point-continuous** valuations [Heckmann 97; JLMZ 21]

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  - In general, **K**-valuations [JLMZ=Jia,Lindenhovius,Mislove,Zamdzhiev 21]

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Let me concentrate on these

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Continuous valuations

Minimal valuations

- **Definition.** A **simple valuation** is $\sum_{i=1}^{n} a_i \delta_{x_i}$, where $a_i \in \mathbb{R}_+$
- … draws each x_i with probability a_i (assuming x_i pairwise distinct)

- Let $\mathbf{V}_{\mathrm{fin}} X \stackrel{\mathrm{def}}{=} \{\text{simple valuations in } \mathbf{V}_{\leq 1} X\}$
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- **Prop [Jia,Lindenhovius,Mislove,Zamdzhiev 21; JGL, Jia 23].**
  Fubini-Tonelli holds on **Dcpo** if one of the valuations is minimal.
- *Proof sketch*: Integration commutes with directed suprema.
  This reduces the question to the case of simple valuations,
  where commutation is easy.

Minimal valuations are enough for semantics

- We (re)define:
  — $[\![\mathbf{T}\tau]\!] \mathrel{\hat{=}} \mathbf{M}[\![\tau]\!])$ minimal subprobability distributions
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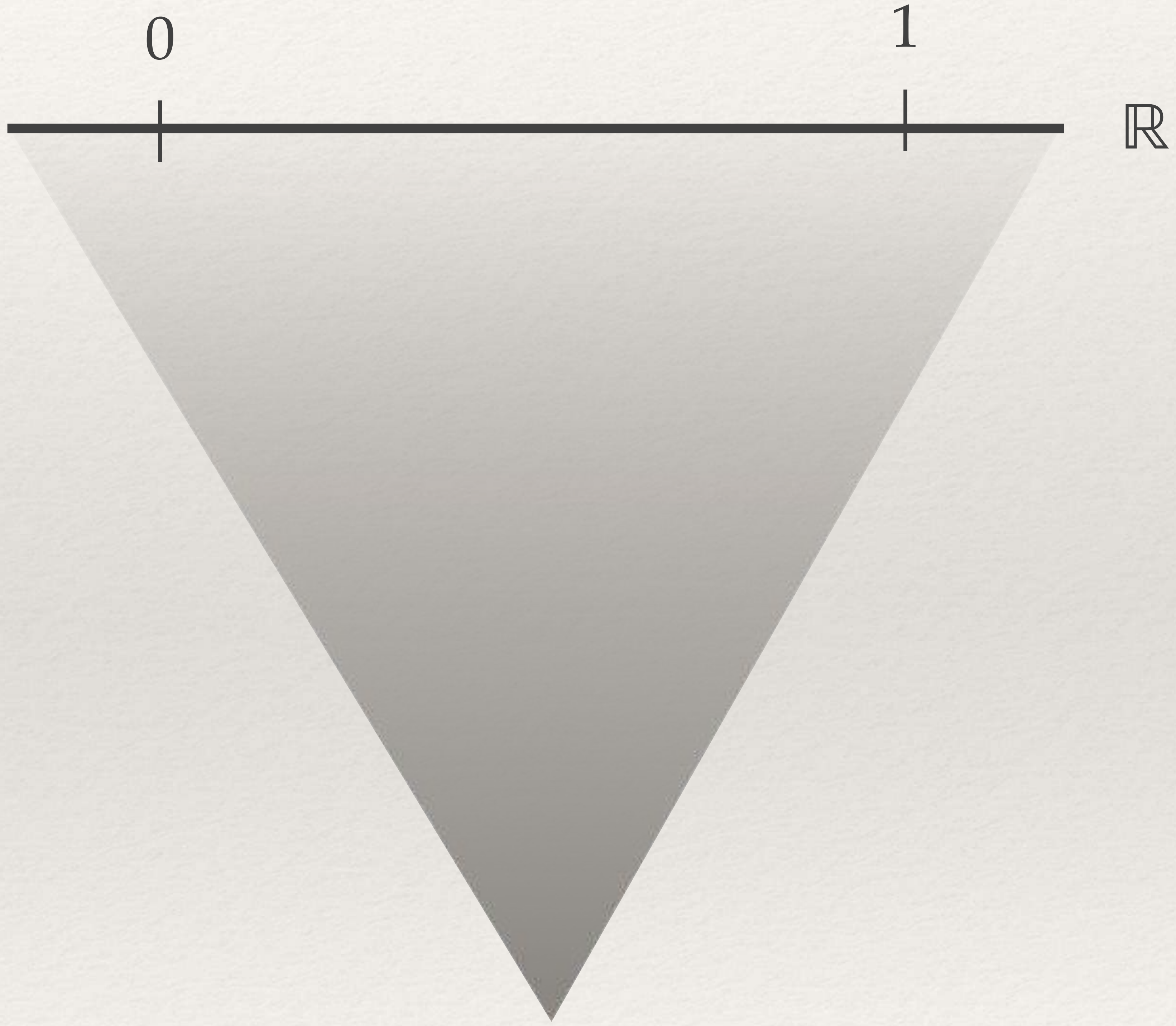
These constructions **preserve minimality**

- Even accommodates continuous distributions
  e.g., Lebesgue measure on **exact real numbers** [JGL, Jia 23], leading to
  ISPCF = PCF + exact real numbers + continuous distributions + soft conditioning

Beyond simple valuations

- Example: Lebesgue measure on $\mathbb{R}$,
through embedding into $\mathbf{I}\mathbb{R} \stackrel{\text{def}}{=} \{\text{intervals } [a, b]\}, \supseteq$
a classical dcpo for **exact real arithmetic**



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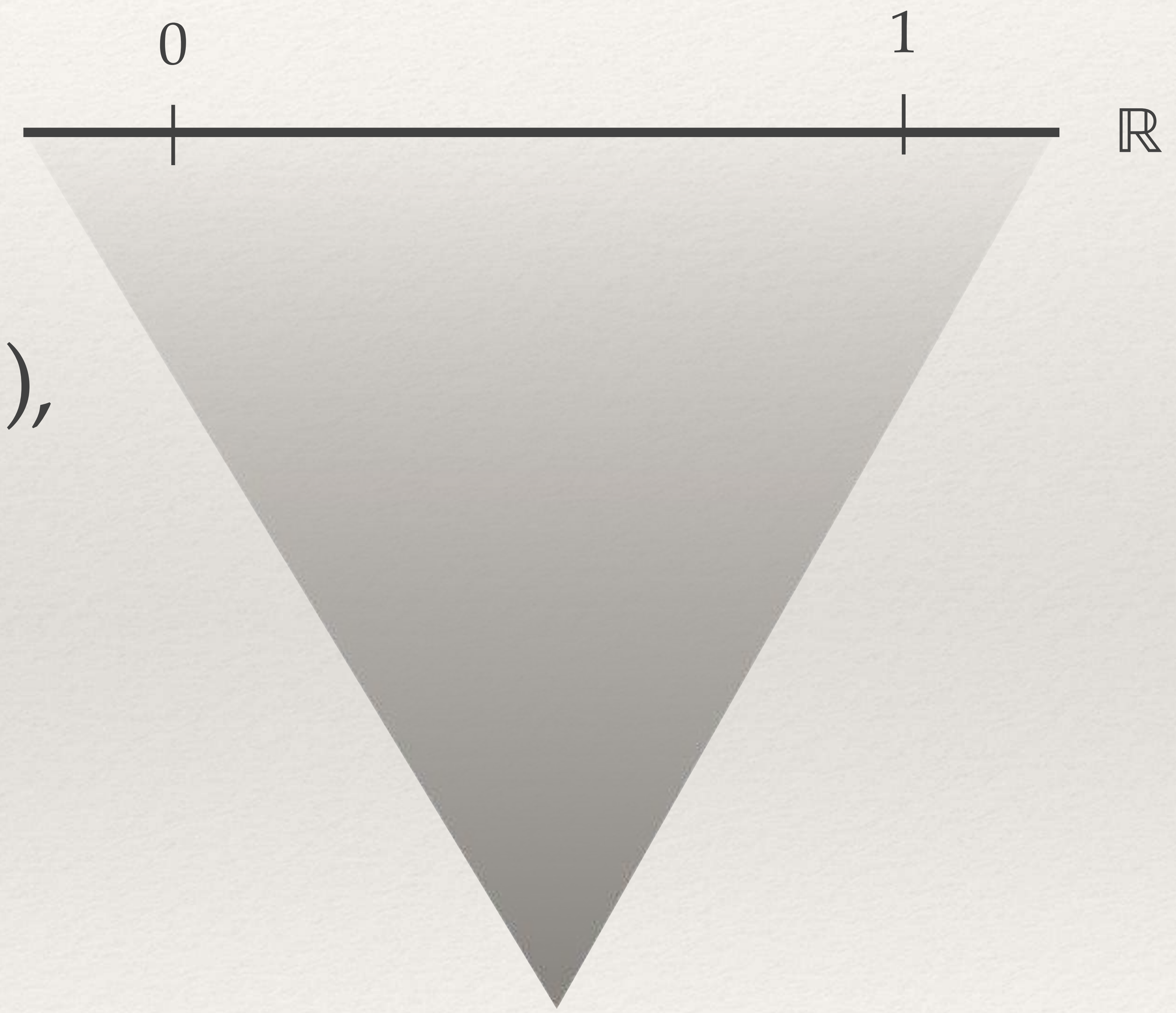

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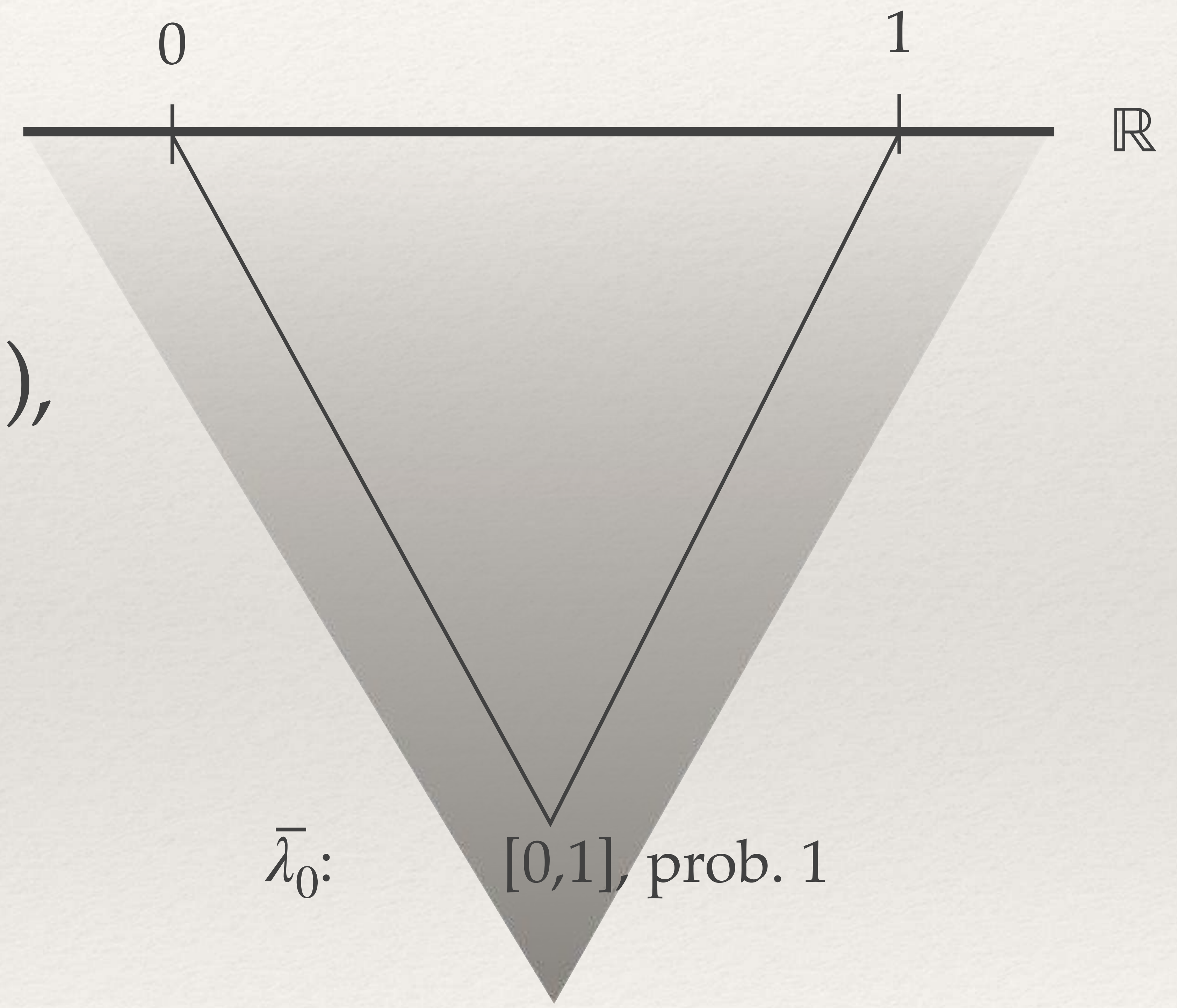

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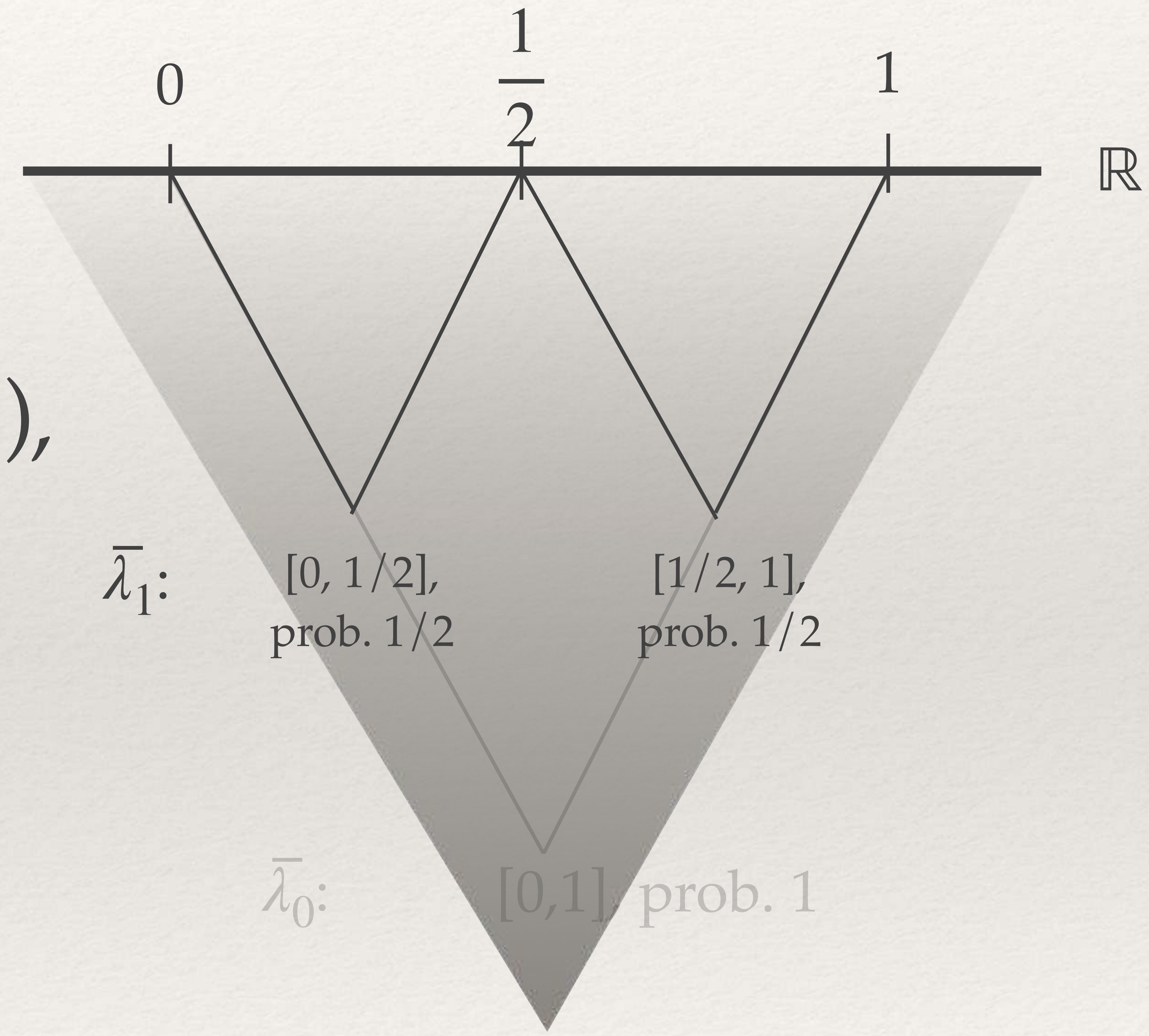

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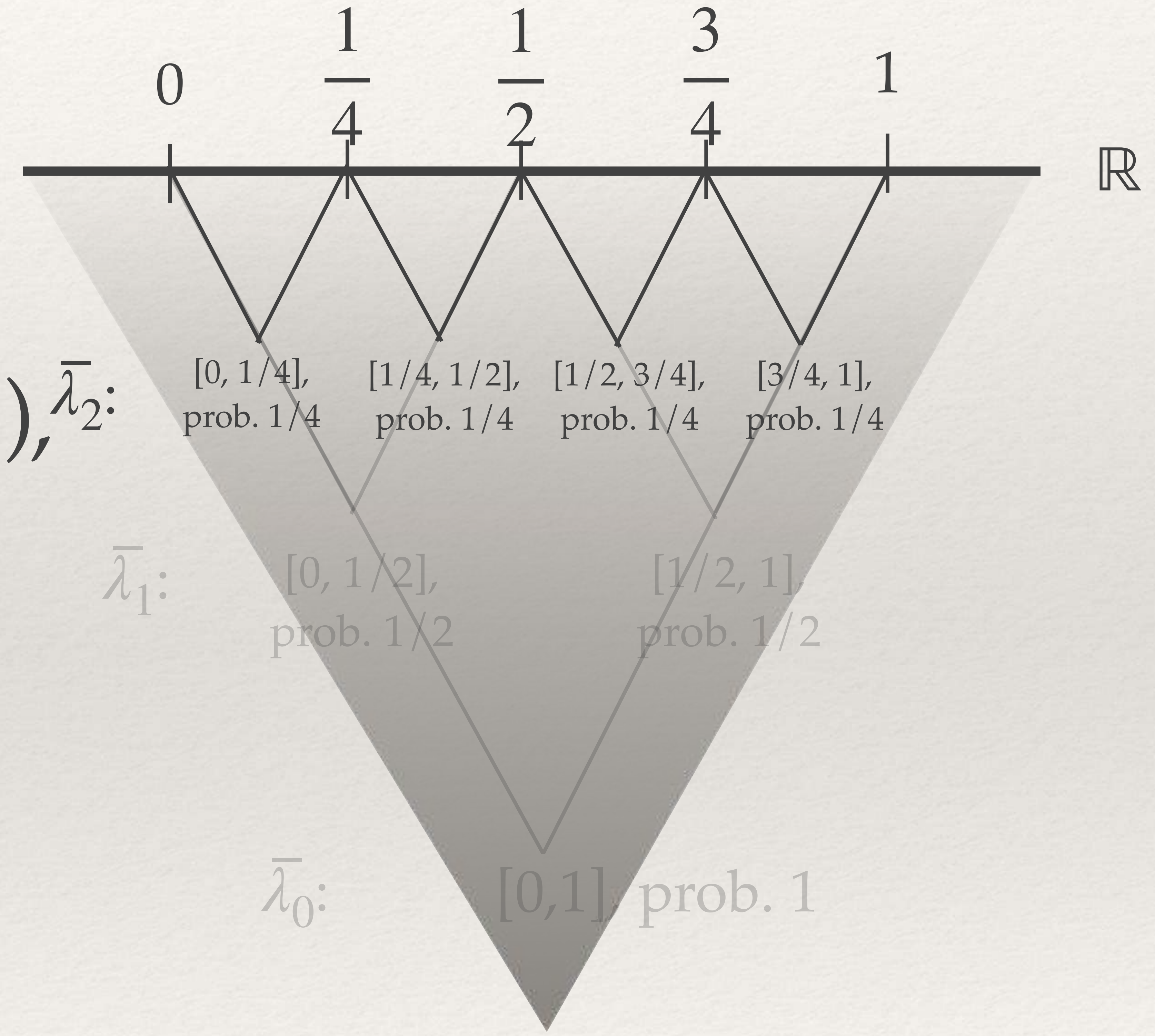

Interval Statistical PCF (ISPCF)

- $$
\begin{array}{lll}
M, N, P, \ldots ::= \ldots & & \text{(as in PCF)} \\
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\quad \cancel{\mid M \oplus N} & \cancel{\text{probabilistic choice}} & \text{(subsumed by } \mathbf{sample}[0,1]\text{)} \\
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\quad \mid \underline{r} & & \text{(real constants, } r \in \mathbb{R}\text{)} \\
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(subsumed by **sample**[0,1])

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Exact real arithmetic

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ISPCF, v2

- $[\![\mathbf{nat}]\!] \mathrel{\hat{=}} \mathbb{N}_\perp$, $[\![\mathbf{unit}]\!] \mathrel{\hat{=}} \{ \perp, * \}$, $[\![\sigma \to \tau]\!] \mathrel{\hat{=}} [[\![\sigma]\!] \to [\![\tau]\!]]$, $[\![\mathbf{real}]\!] \mathrel{\hat{=}} \mathbf{I}\mathbb{R}_\perp$

  $[\![\mathbf{T}\tau]\!] \mathrel{\hat{=}} \mathbf{M}[\![\tau]\!]$

  Instead of $\mathbf{V}_{\leq 1}[\![\tau]\!]$
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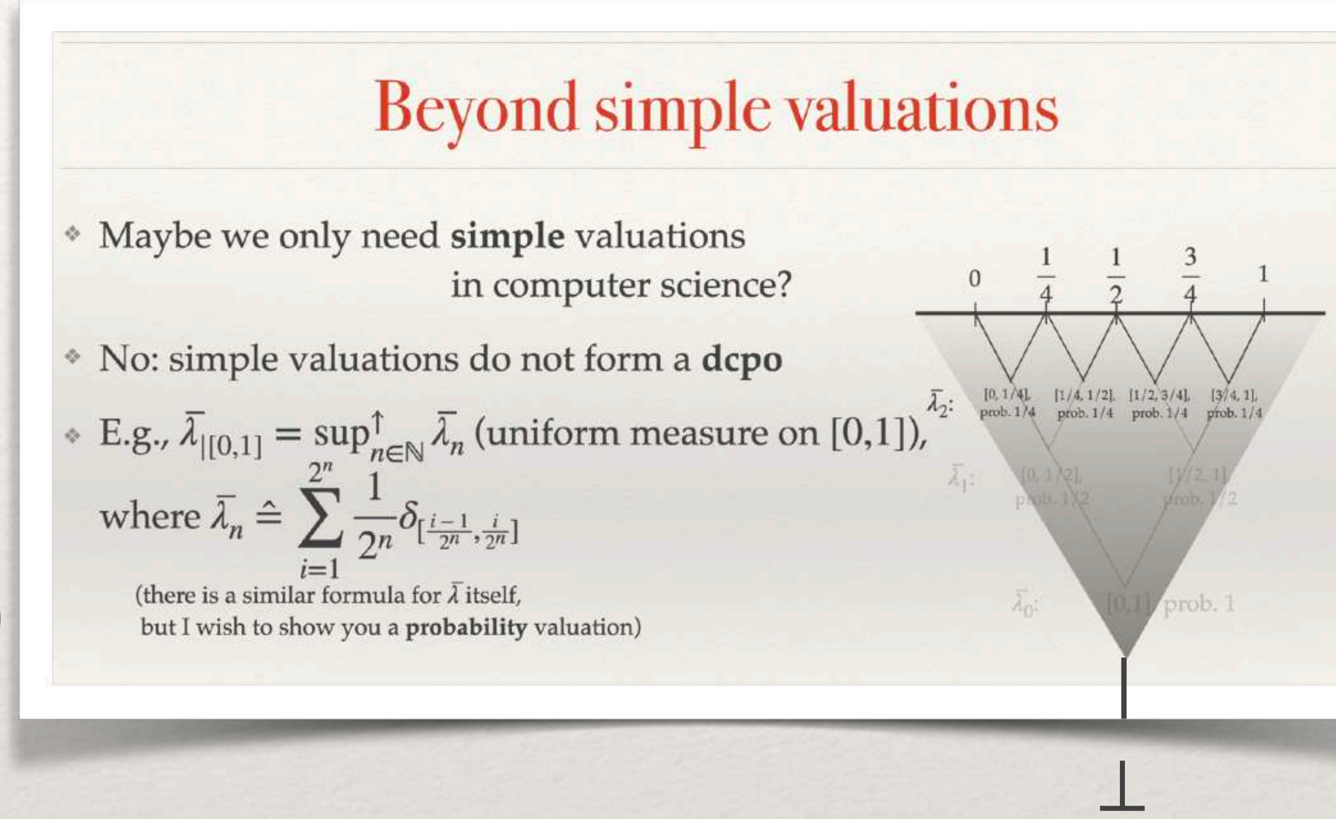

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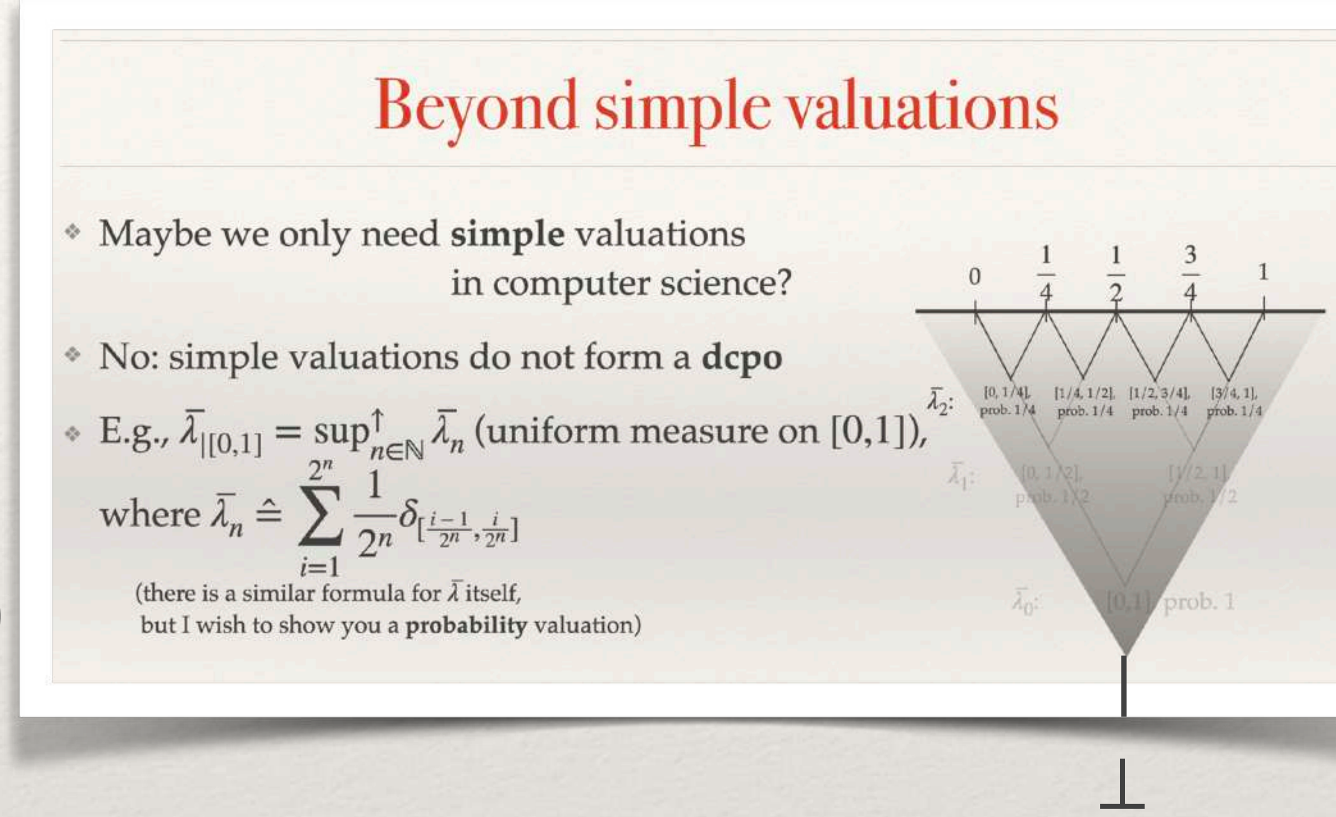

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  $\Pr[C, M \downarrow n] = [\![C[M]]\!]\rho(\{n\})$
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Operational semantics
is **unchanged**

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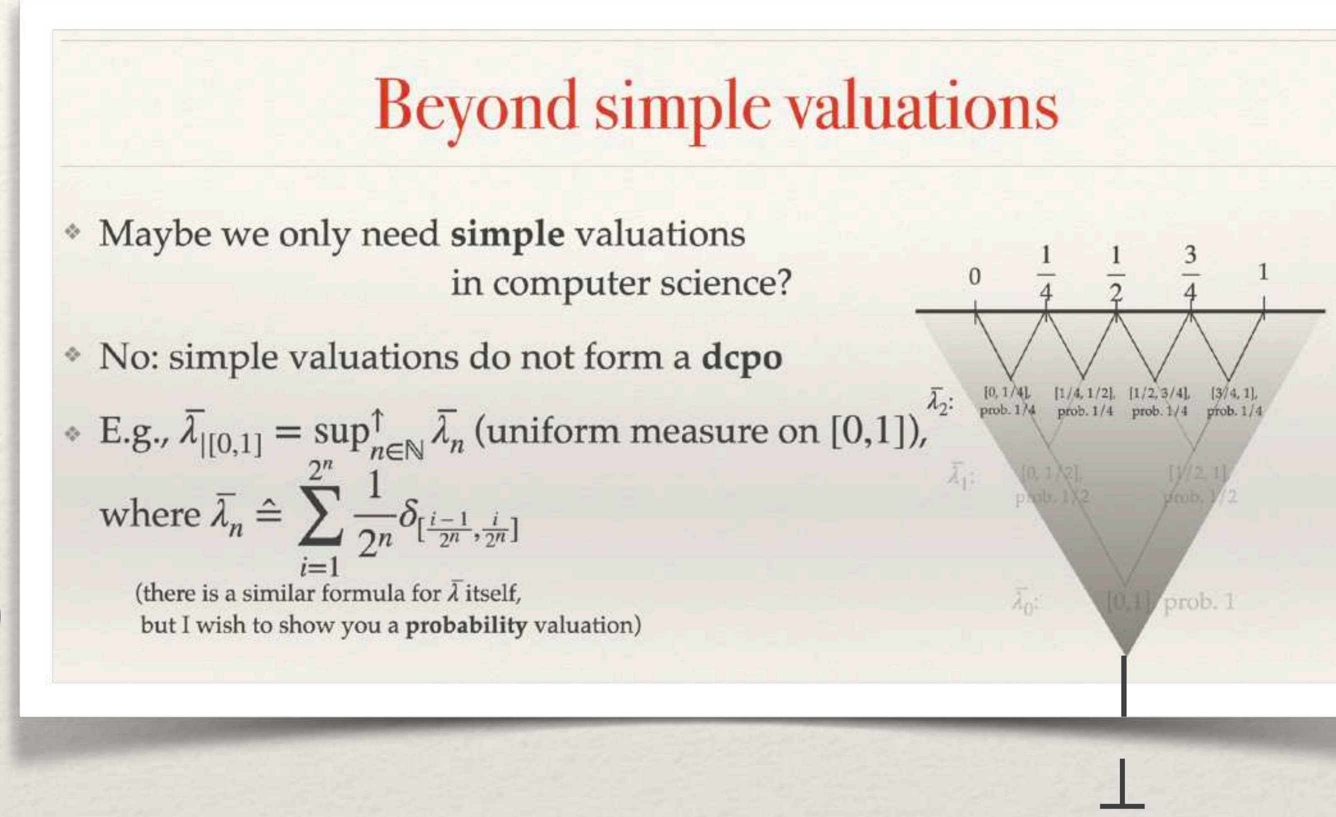

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- $[\![\mathbf{do}\ x = M; \mathbf{do}\ y = N; P]\!]$
  $= [\![\mathbf{do}\ y = N; \mathbf{do}\ x = M; P]\!]$
  (x not free in N, y not free in M)

M is a **commutative** monad

What does this compute?

```
[goubault@macbook-pro-de-jean Topics2023 % gimml
GimML comes with ABSOLUTELY NO WARRANTY (See file COPYRIGHT).
GimML for Darwin 20.5.0, by Jean Goubault-Larrecq (c) 2021.
[> use "ispcf1.ml"
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```

$$
\begin{aligned}
\mathit{longest\_decreasing\_run} \mathrel{\hat{=}} \mathbf{rec}(\lambda f_{\mathbf{real}\to\mathbf{int}\to\mathbf{T\ int}} \cdot \lambda x_{\mathbf{real}} \cdot \lambda n_{\mathbf{int}} \cdot\ & \mathbf{do}\ u = \mathbf{sample}[0,1]; \\
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printing slowed down for enhanced dramatic effect

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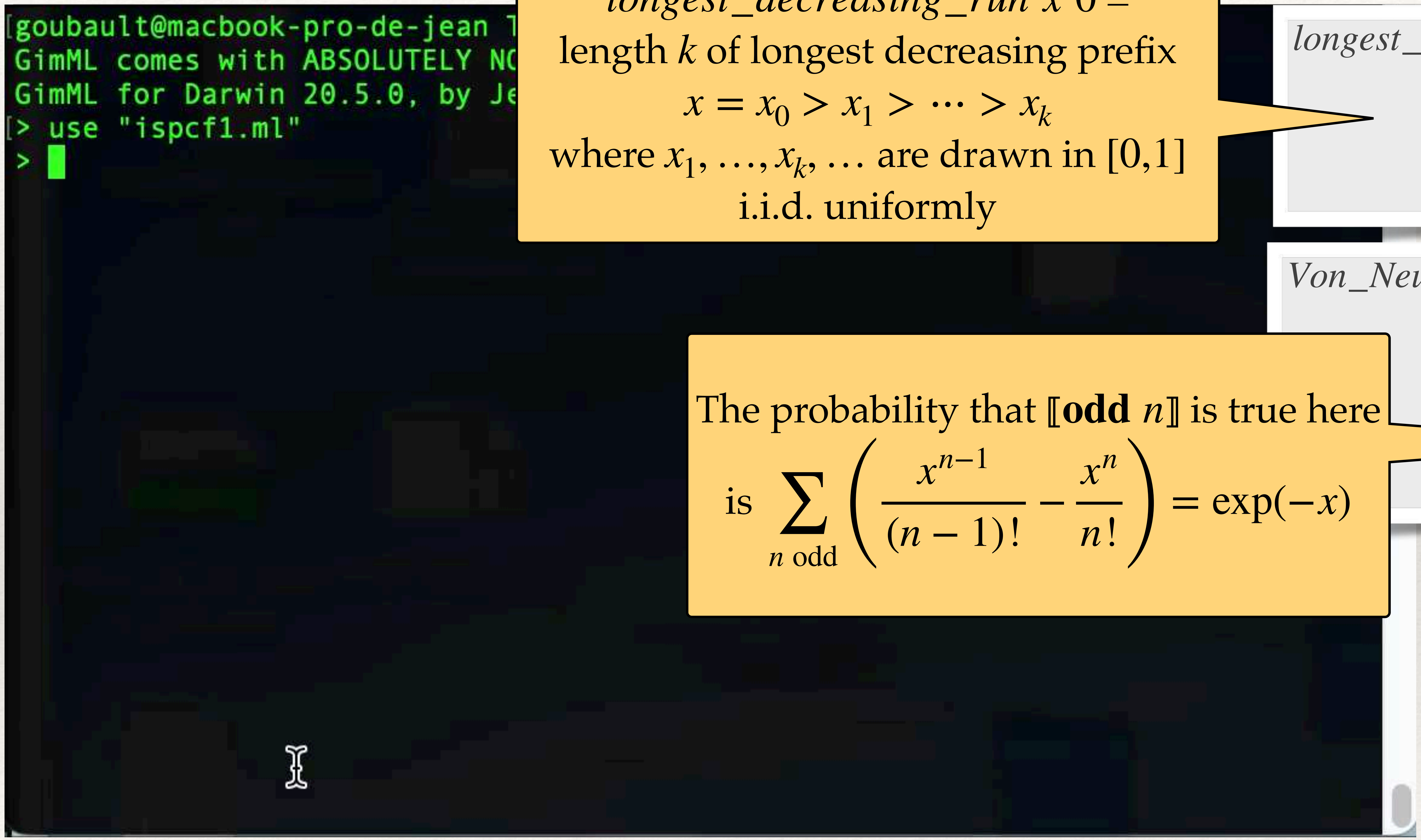

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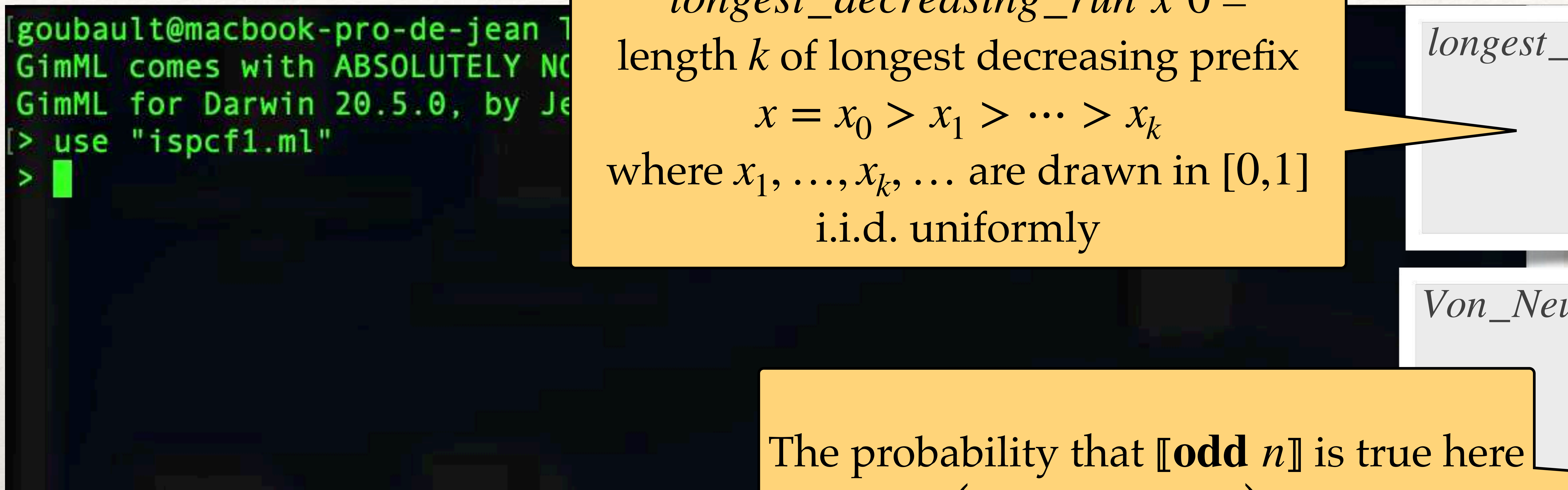

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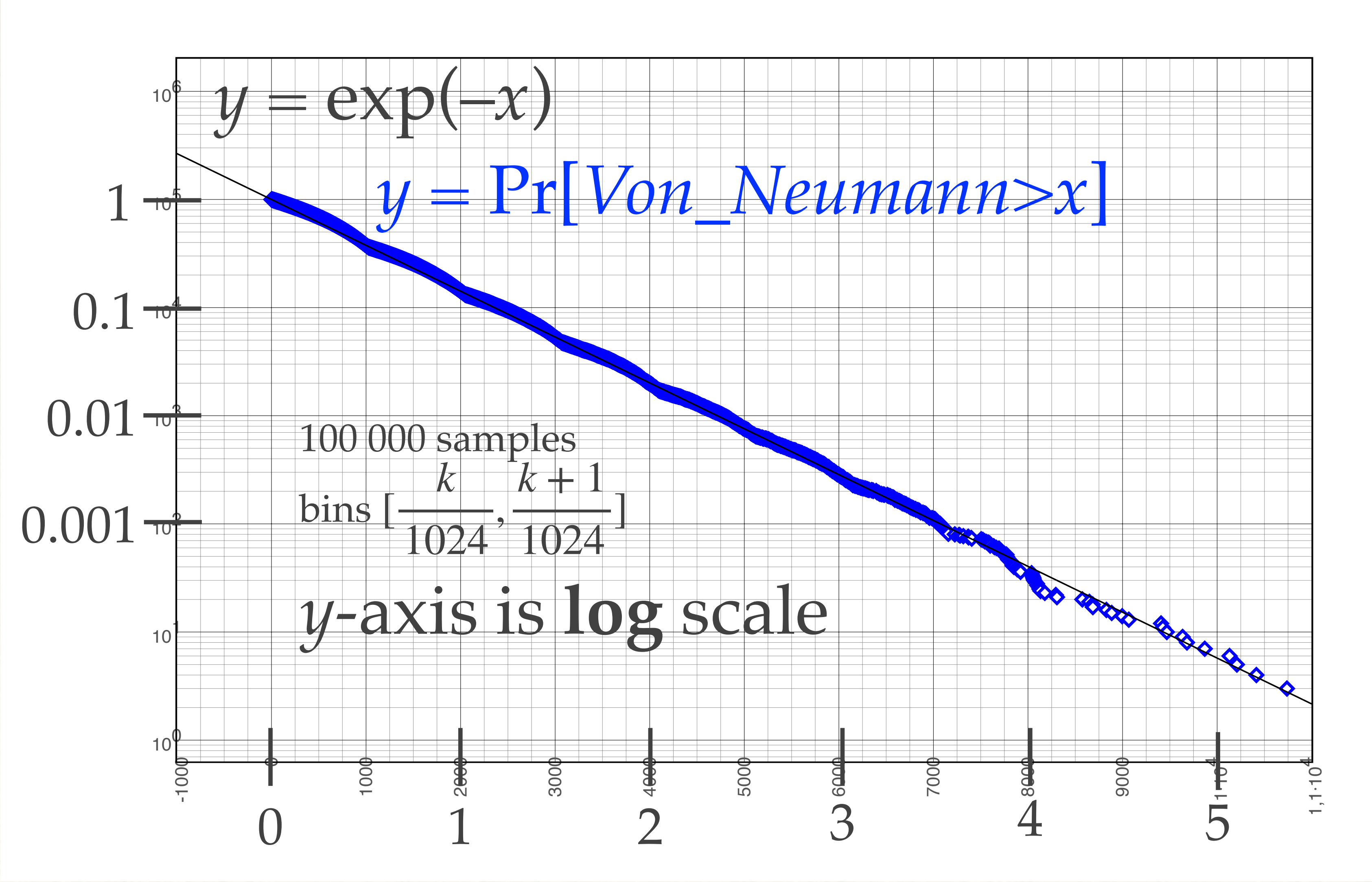

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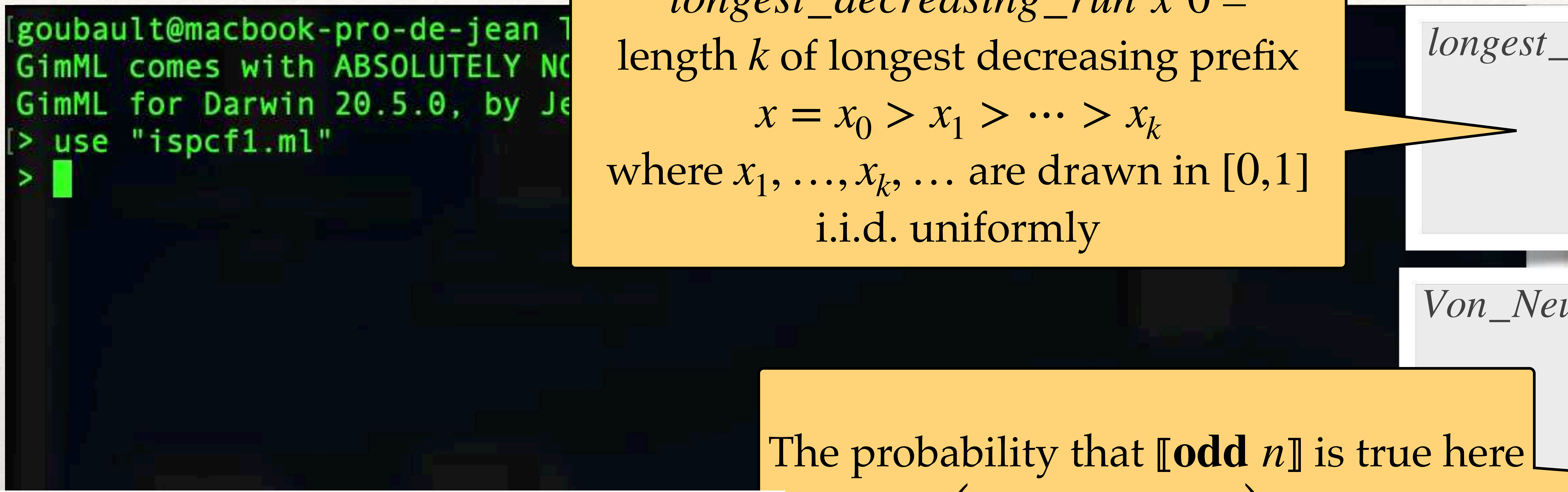

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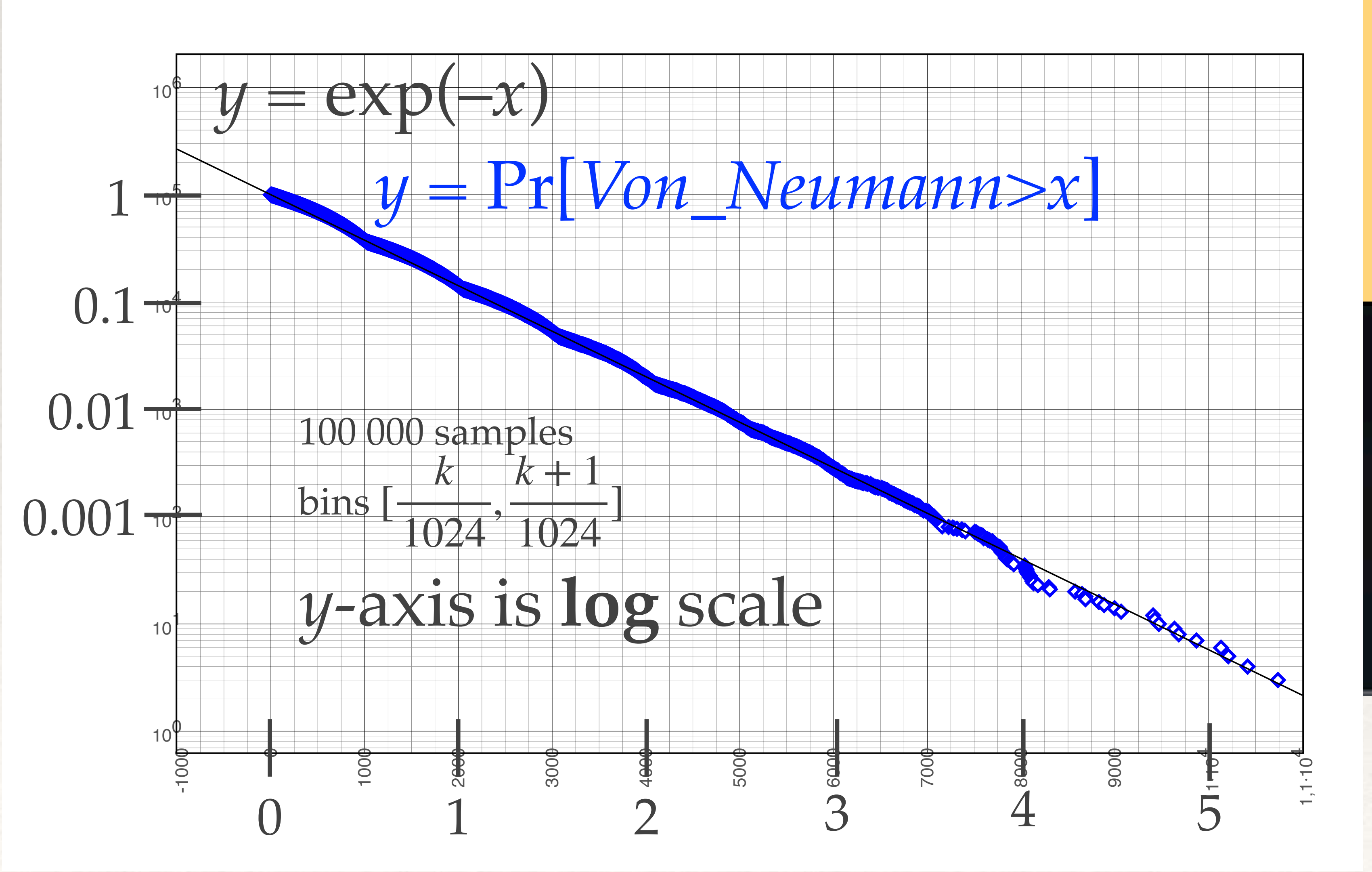

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Part II: separating minimal valuations from continuous valuations

Separation

- All this is well and good, but:
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- Are there any **non-minimal** subprobability valuations on a dcpo?

  Let me give an example [JGL Jia 21].

The Johnstone dcpo $\mathbf{J}$

- Johnstone's dcpo $\mathbf{J}$ (1981):
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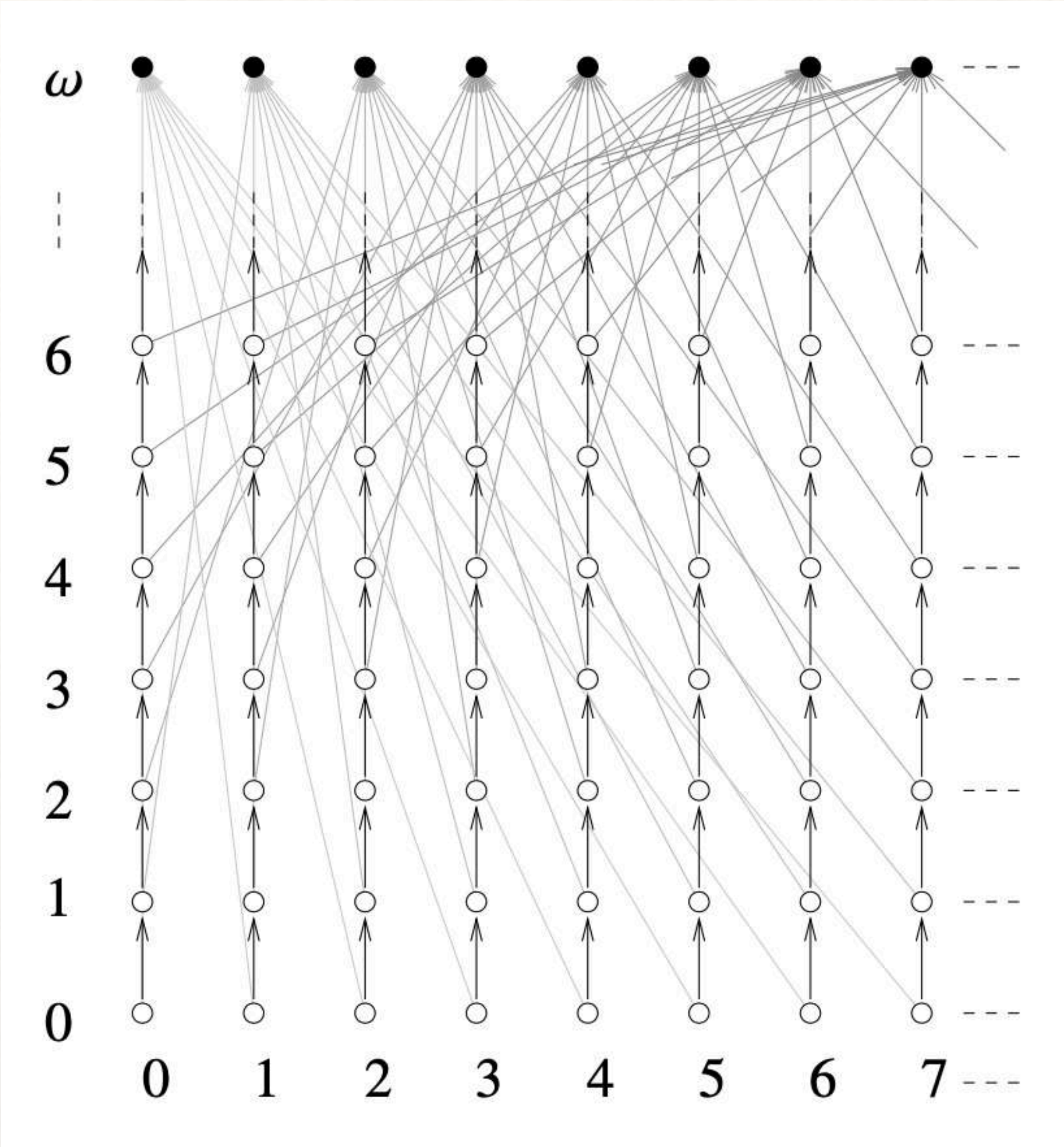

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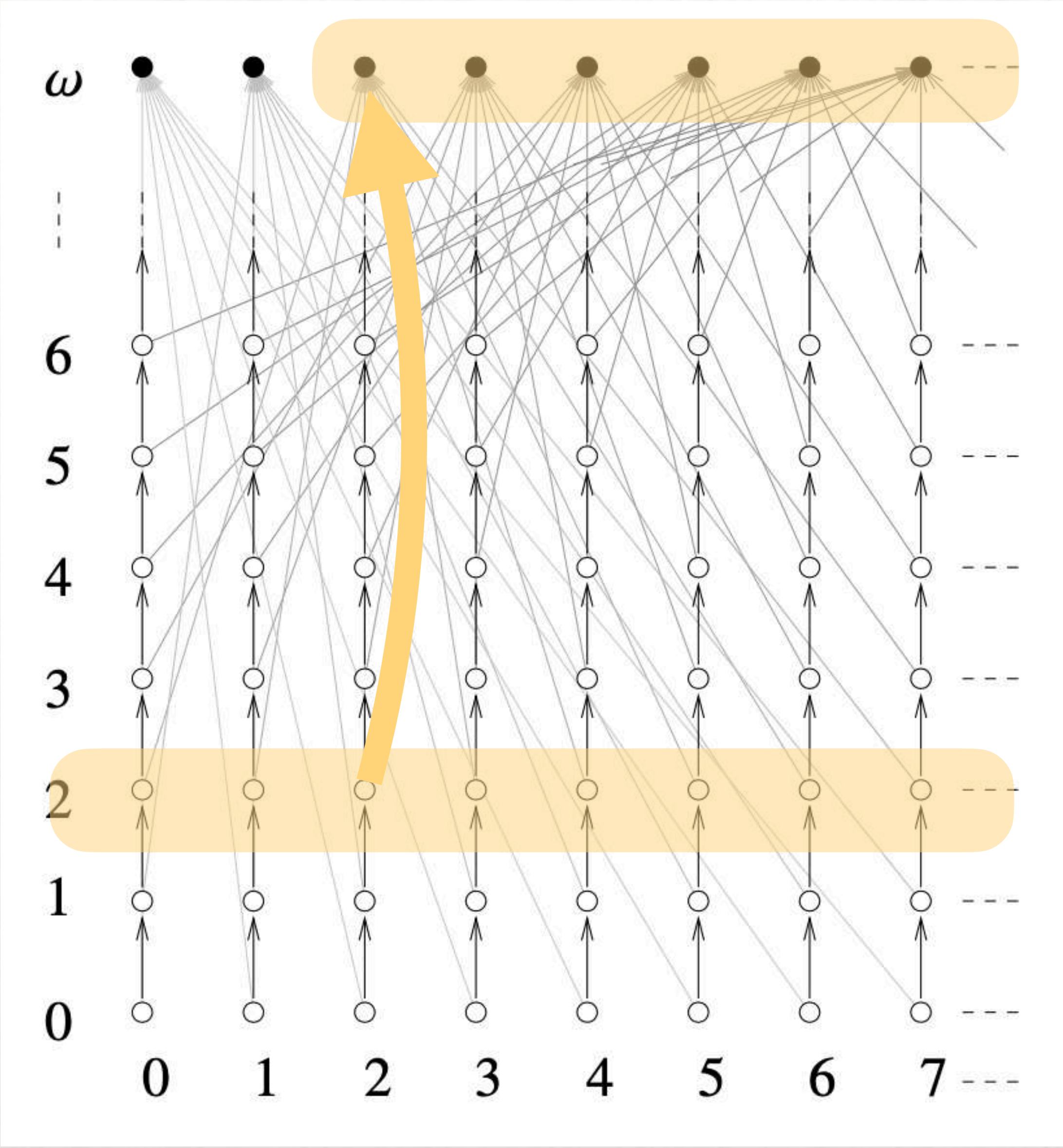

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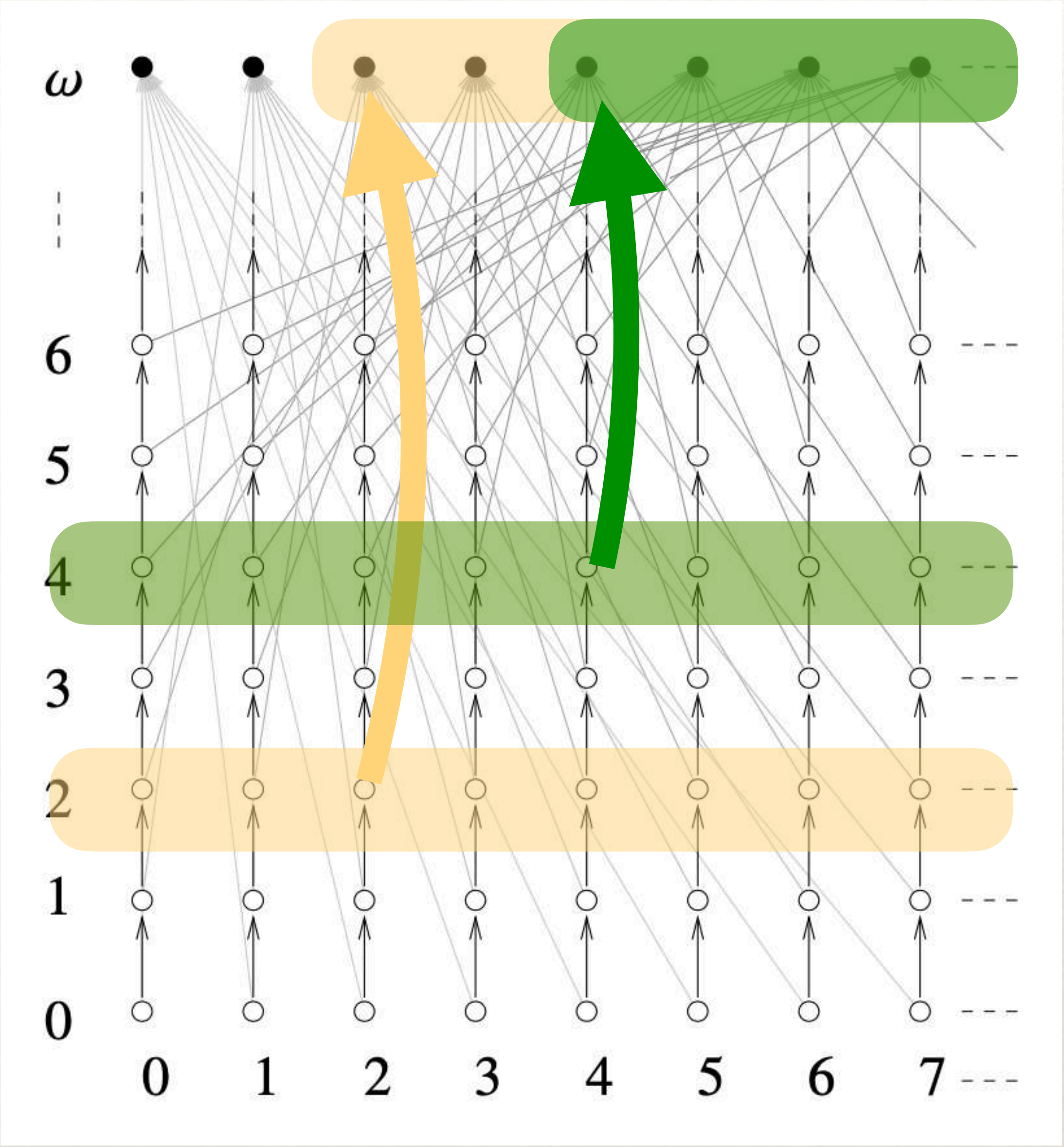

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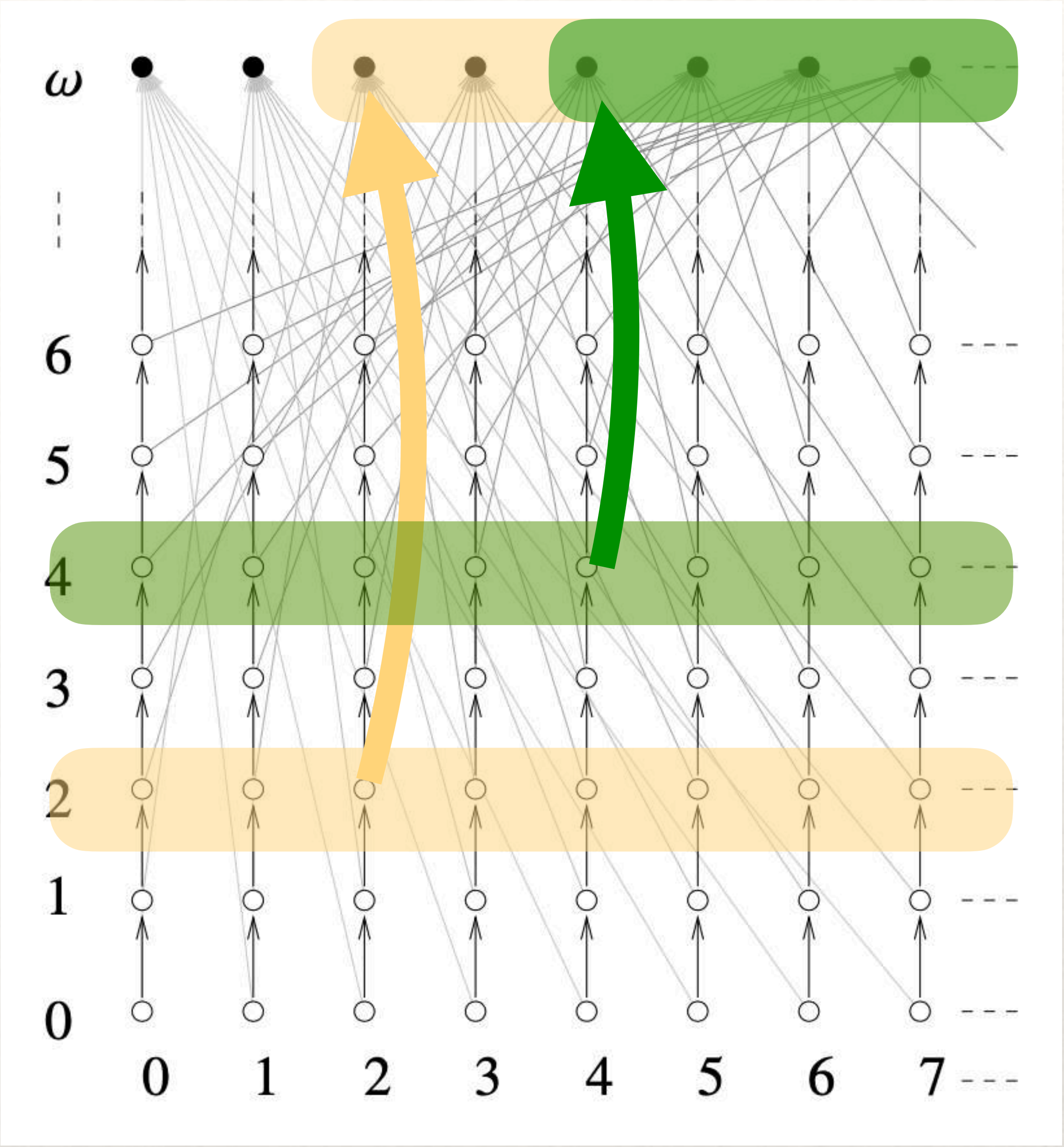

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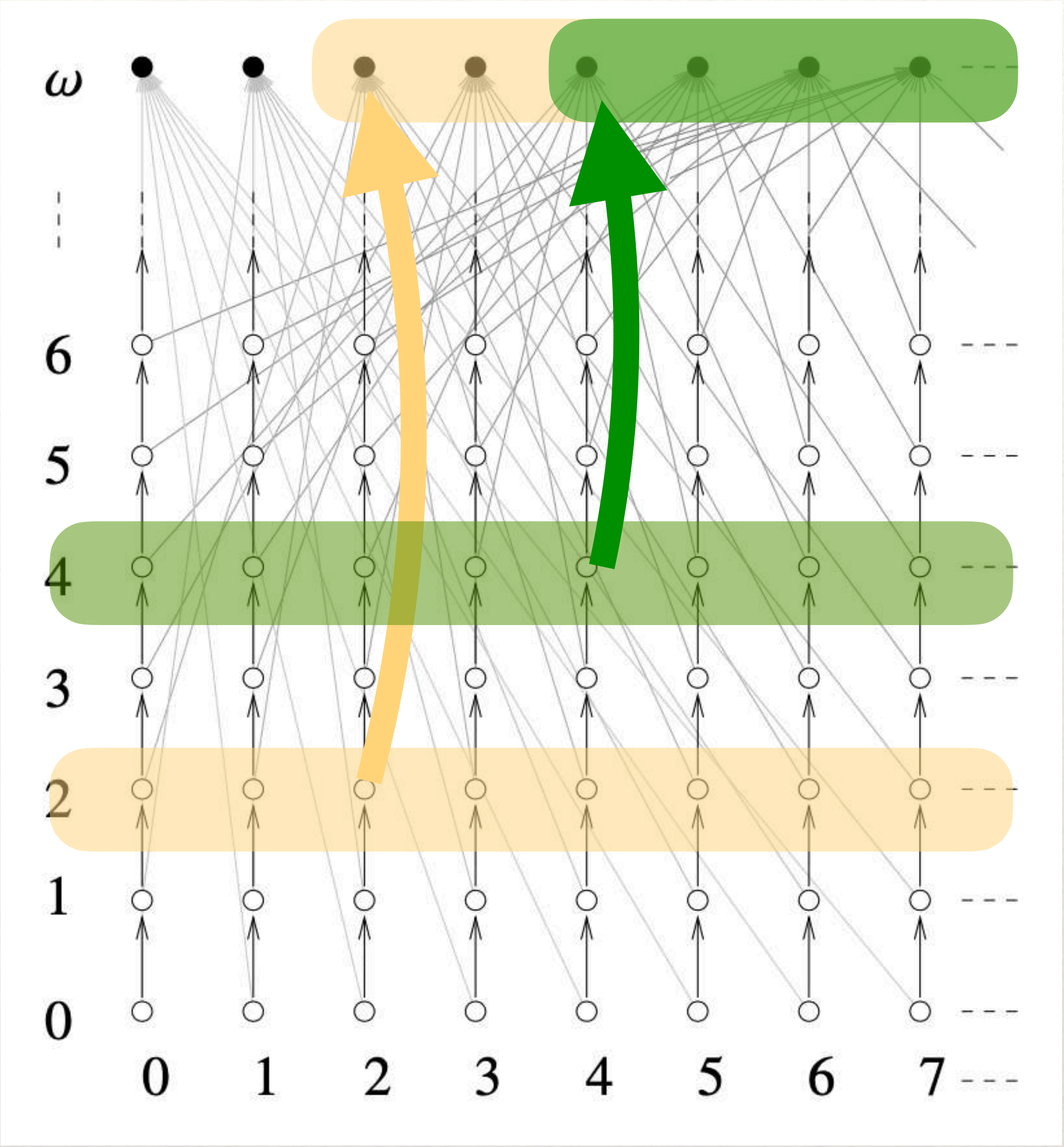

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- A famous counterexample in domain theory:
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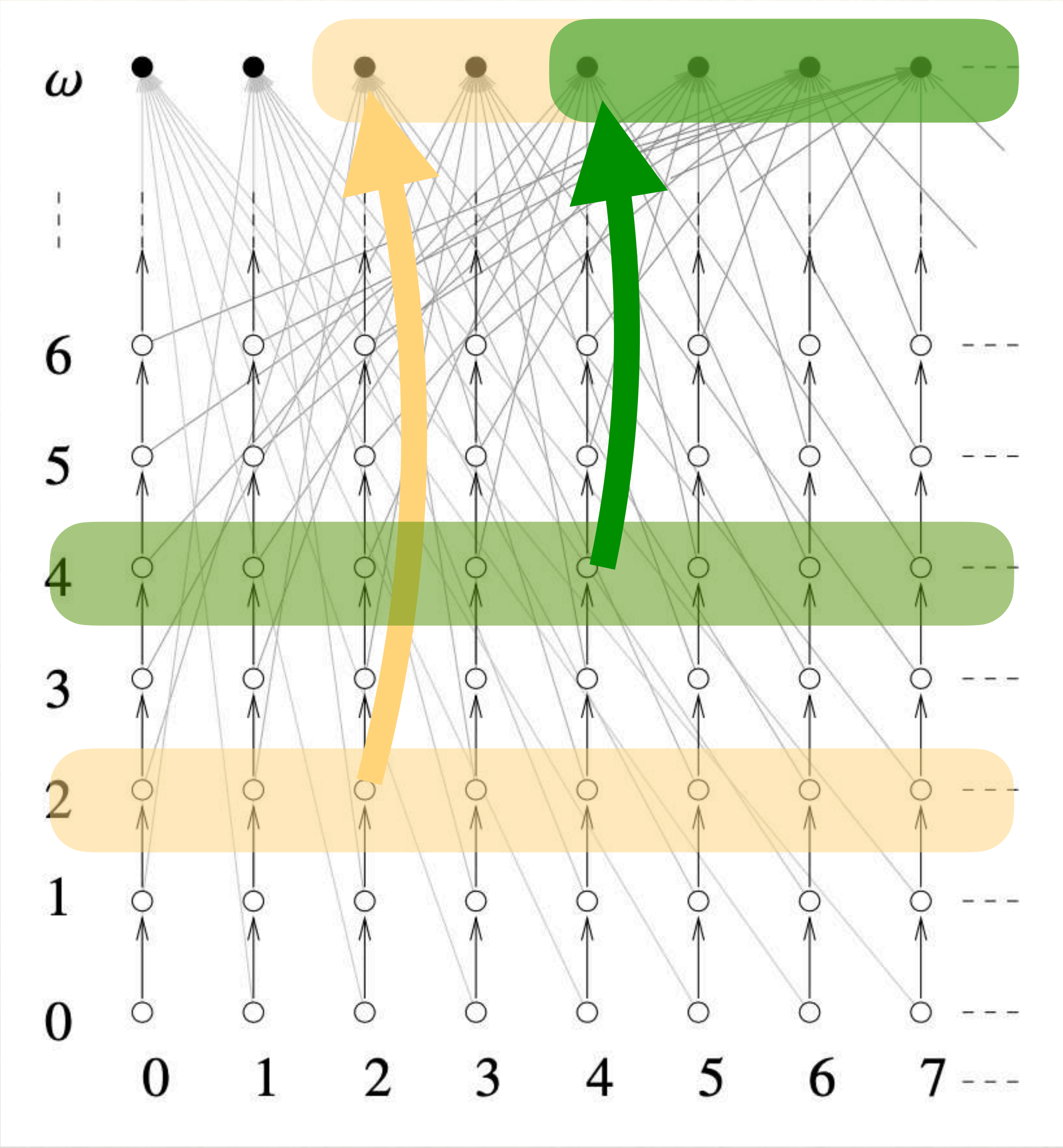
- I will write $\mathbf{J}_\sigma$ for $\mathbf{J}$ with the Scott topology

A funny valuation on $\mathbf{J}$



- On Johnstone's dcpo $\mathbf{J}$, there is a continuous valuation μ defined by:

  $\mu(U) = 1$ for every non-empty Scott-open set U
  $\mu(\emptyset) = 0$

- Modularity $\mu(U \cup V) + \mu(U \cap V) = \mu(U) + \mu(V)$ comes from the fact that $\mathbf{J}_\sigma$ is **hyperconnected**:

  any two non-empty open sets intersect.

  (Check it! Observe that every non-empty open set contains all points (m, ω) for m large enough.)

- We will show that μ is not minimal.

Discrete and good valuations

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Measures the **open** sets

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  is **good** [Alvarez-Manilla, Edalat, Saheb-Djahromi 00]
- Every subset of $\mathbf{J}_\sigma$ is Borel.
- One can show that every subprobability valuation ν on $\mathbf{J}_\sigma$

  is of the form $\theta + r\,.\,\mu$, where

  — θ is discrete (hence good)

  — $r \geq 0$

Namely, $\theta = \sum_{x \in \mathbf{J}} \nu(\{x\})\,.\,\delta_x$

$r = \nu(\mathbf{J}) - \sum_{x \in \mathbf{J}} \nu(\{x\})$

Good valuations on **J** are closed under directed suprema

- **Lemma.** If $\theta + r \,.\, \mu$ is a directed supremum of good valuations θ_i on $\mathbf{J}_{\sigma}$, then $r = 0$.
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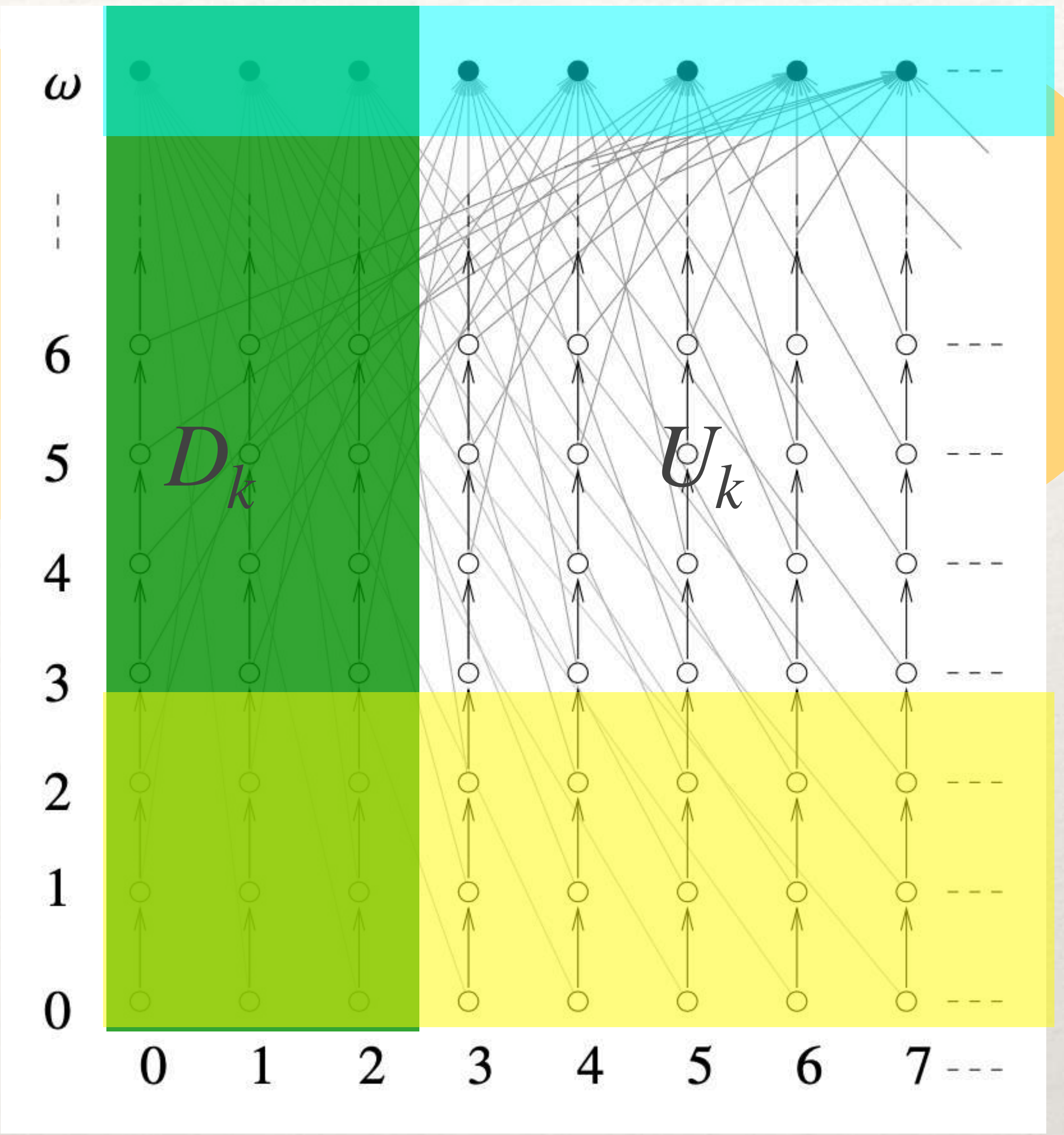

- Let D_k={k leftmost columns} (in green)
- $\downarrow D_k$ is closed (green+yellow)
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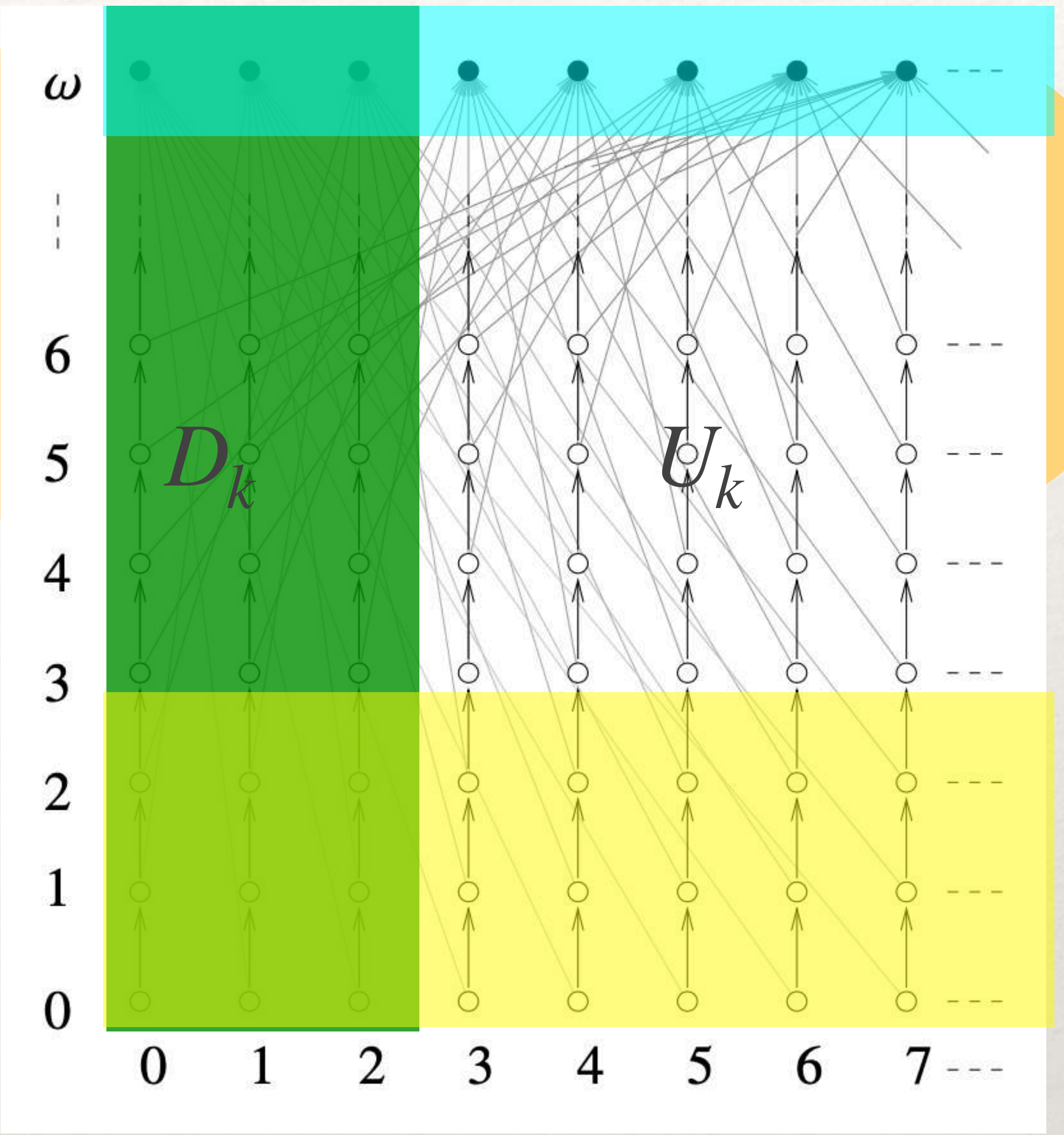

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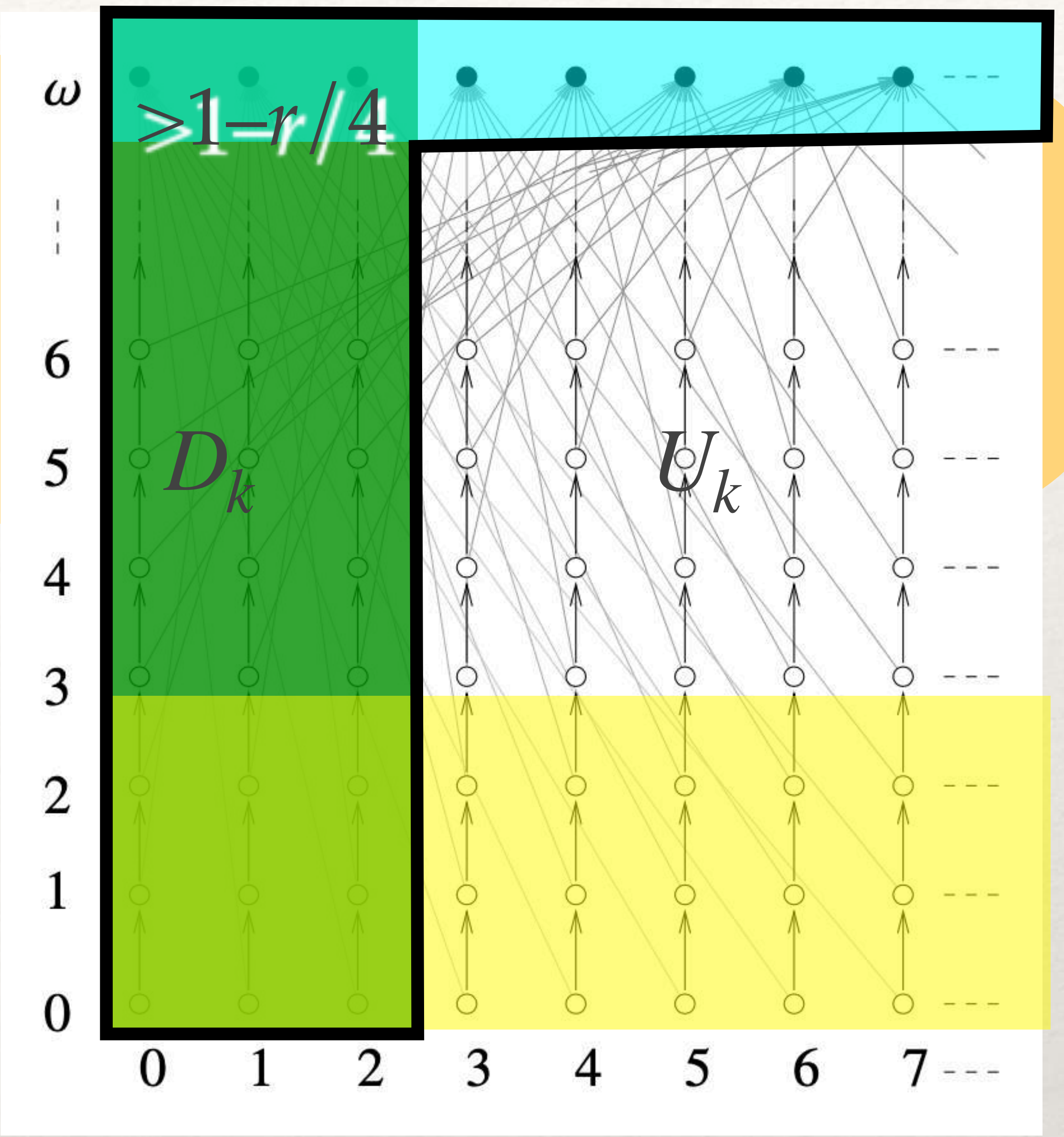
  - for each countable chain of **arbitrary subsets** E_k (necessarily Borel)
    $\theta_i(\bigcup_k^{\uparrow} E_k) = \sup_k^{\uparrow} \theta_i(E_k)$ because θ_i is **good**
    so $\theta_i(\uparrow D_k) > 1 - r/4$ for k large enough



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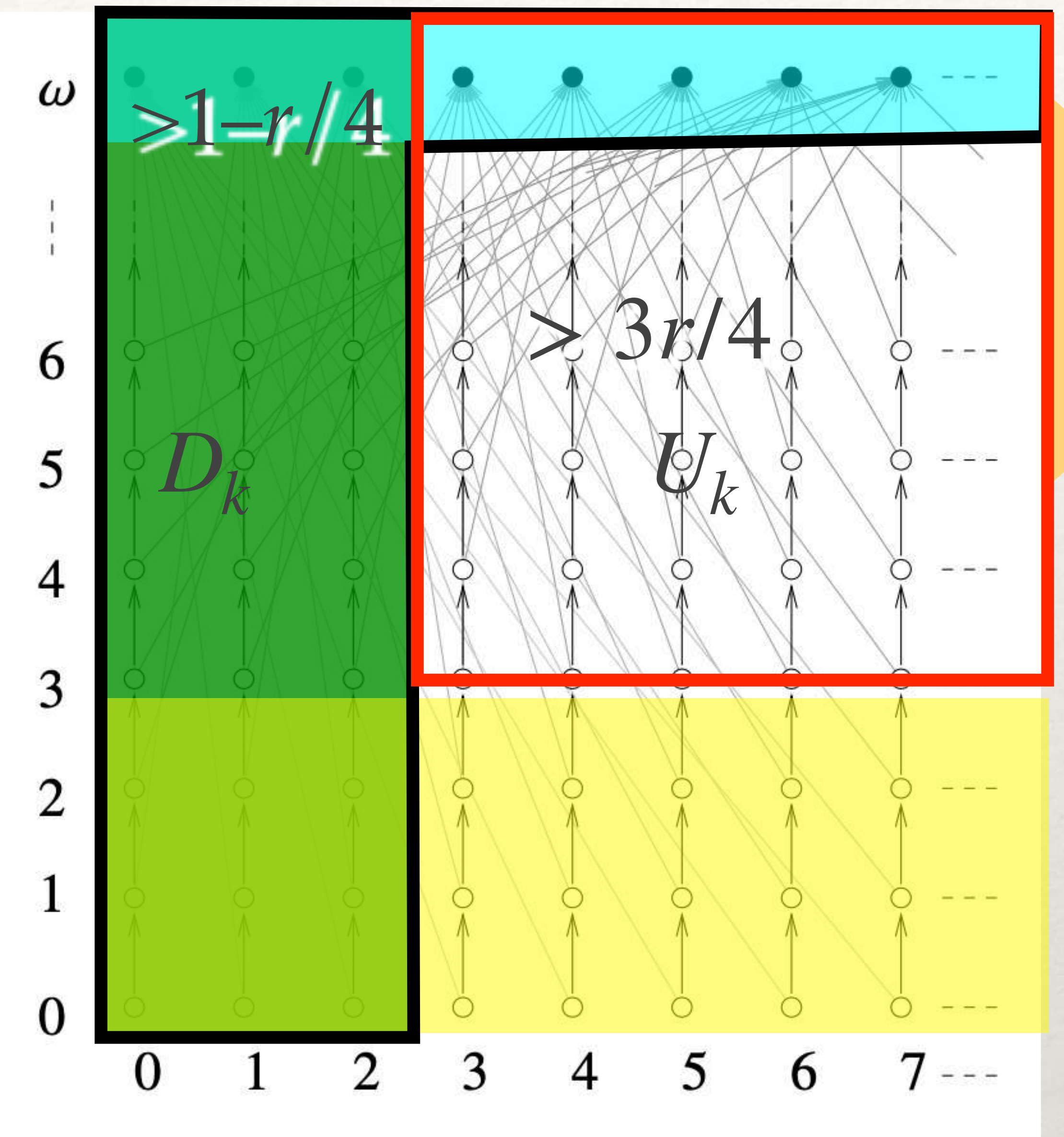
- **Lemma.** If $\theta + r\,.\,\mu$ is a directed supremum of good valuations θ_i on $\mathbf{J}_{\sigma'}$, then $r = 0$.



- *Proof sketch.* Imagine $r \neq 0$. Wlog., $(\theta + r\,.\,\mu)(\mathbf{J}) = 1$. Ingredients:
  - $\theta + r\,.\,\mu = \sup_i^{\uparrow} \theta_i$, so $\theta_i(\mathbf{J})$ tends to 1:

    $\theta_i(\mathbf{J}) > 1 - r/4$ for i large enough
  - for each countable chain of **arbitrary subsets** E_k (necessarily Borel)

    $\theta_i(\bigcup_k^{\uparrow} E_k) = \sup_k^{\uparrow} \theta_i(E_k)$ because θ_i is **good**

    so $\theta_i(\uparrow D_k) > 1 - r/4$ for k large enough
  - $(\theta + r\,.\,\mu)(U_k) \geq r\mu(U_k) = r$ (def. of μ)

    Since $(\theta + r\,.\,\mu)(U_k) = \sup_i^{\uparrow} \theta_i(U_k)$, $\theta_i(U_k) > 3r/4$ for i large enough

- Let D_k={k leftmost columns} (in green)
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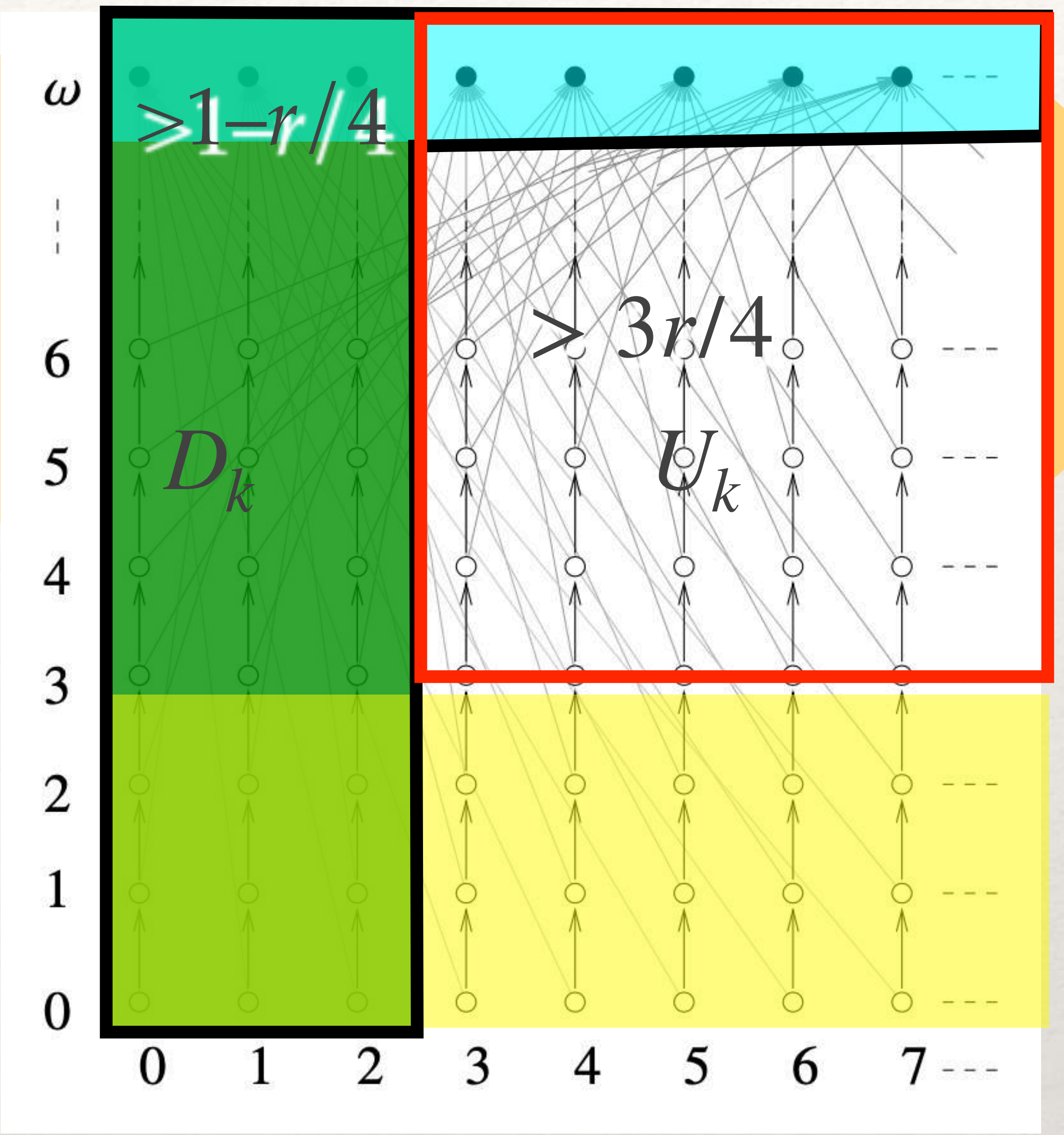

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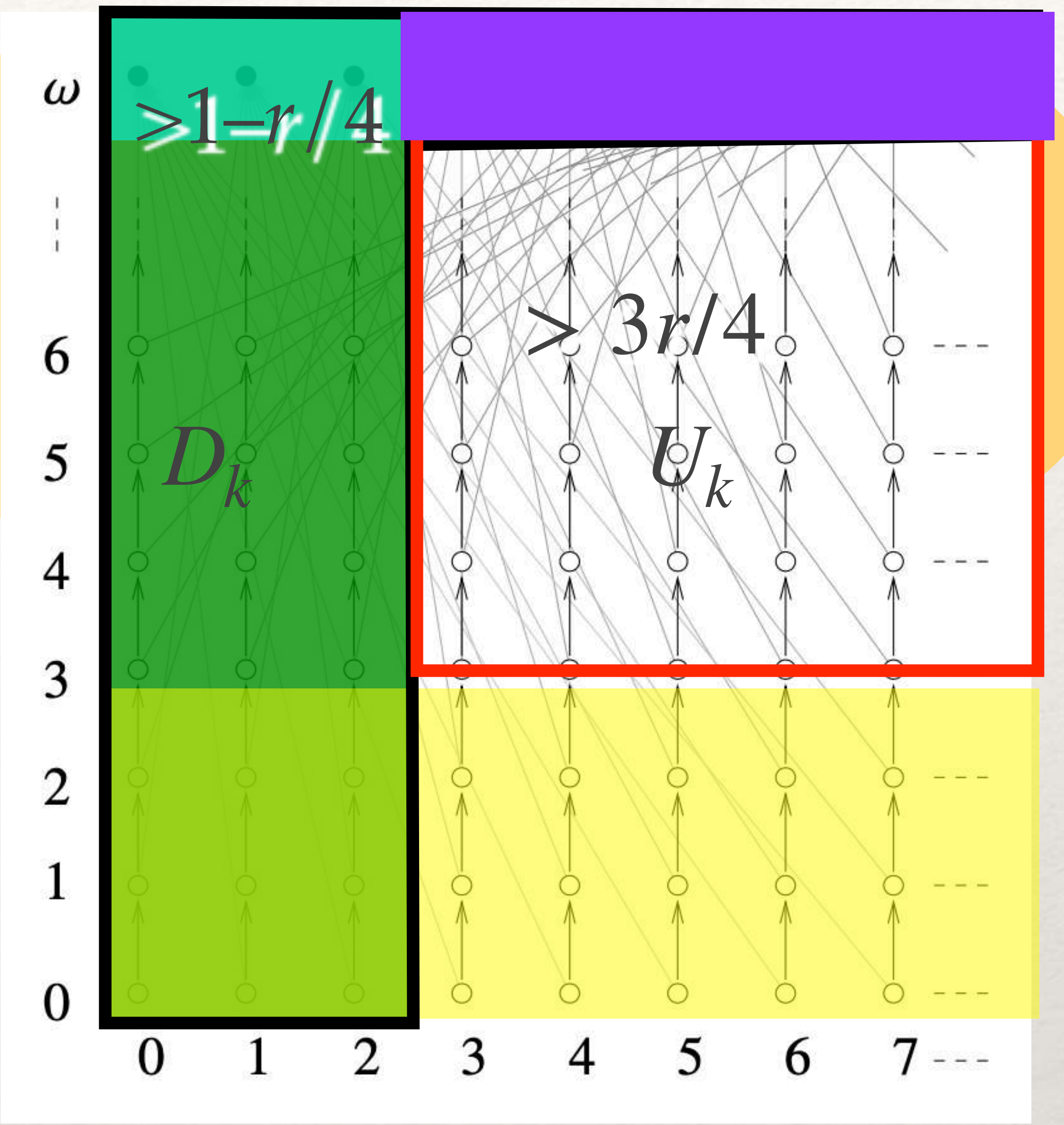

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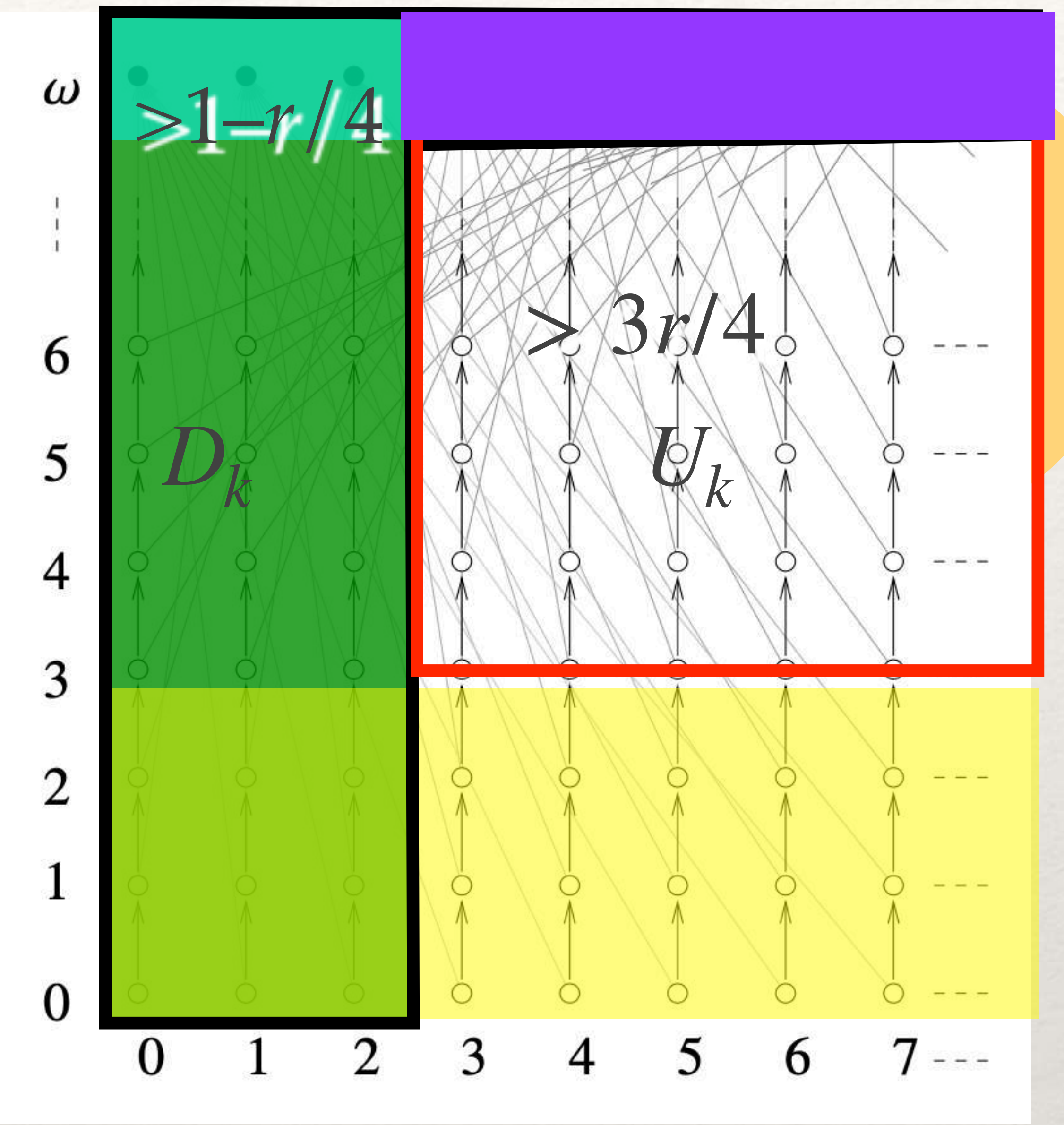

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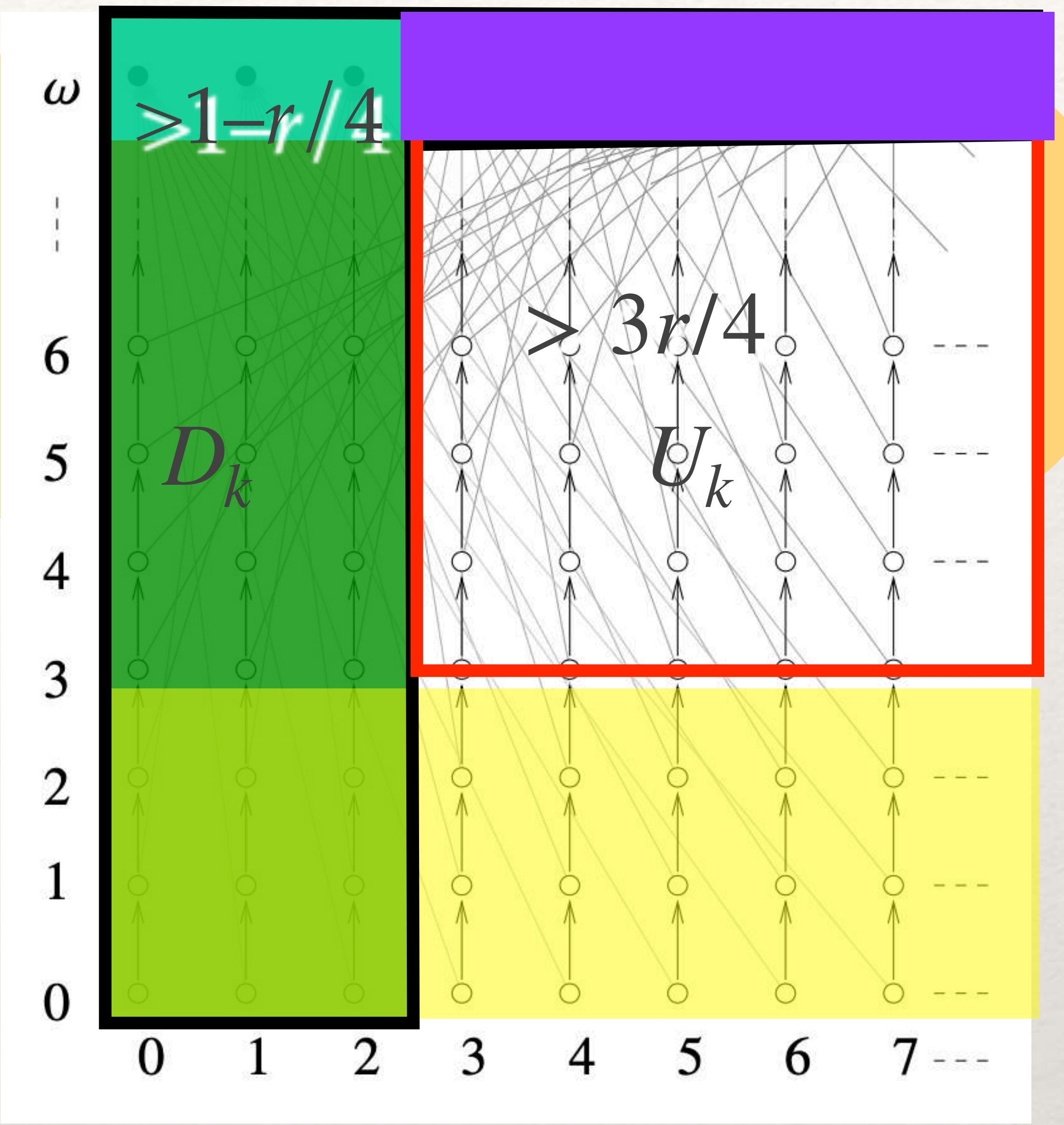

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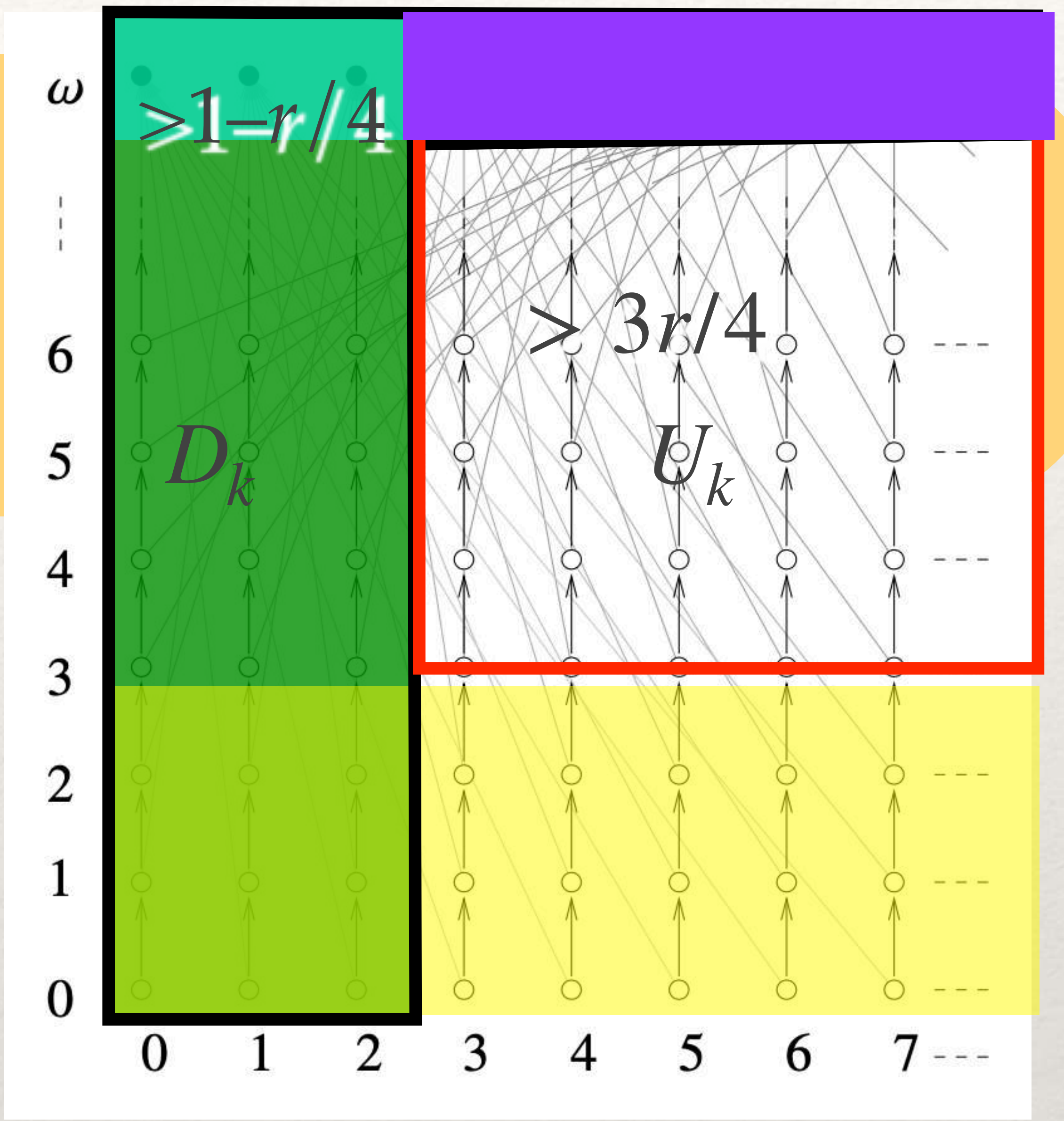

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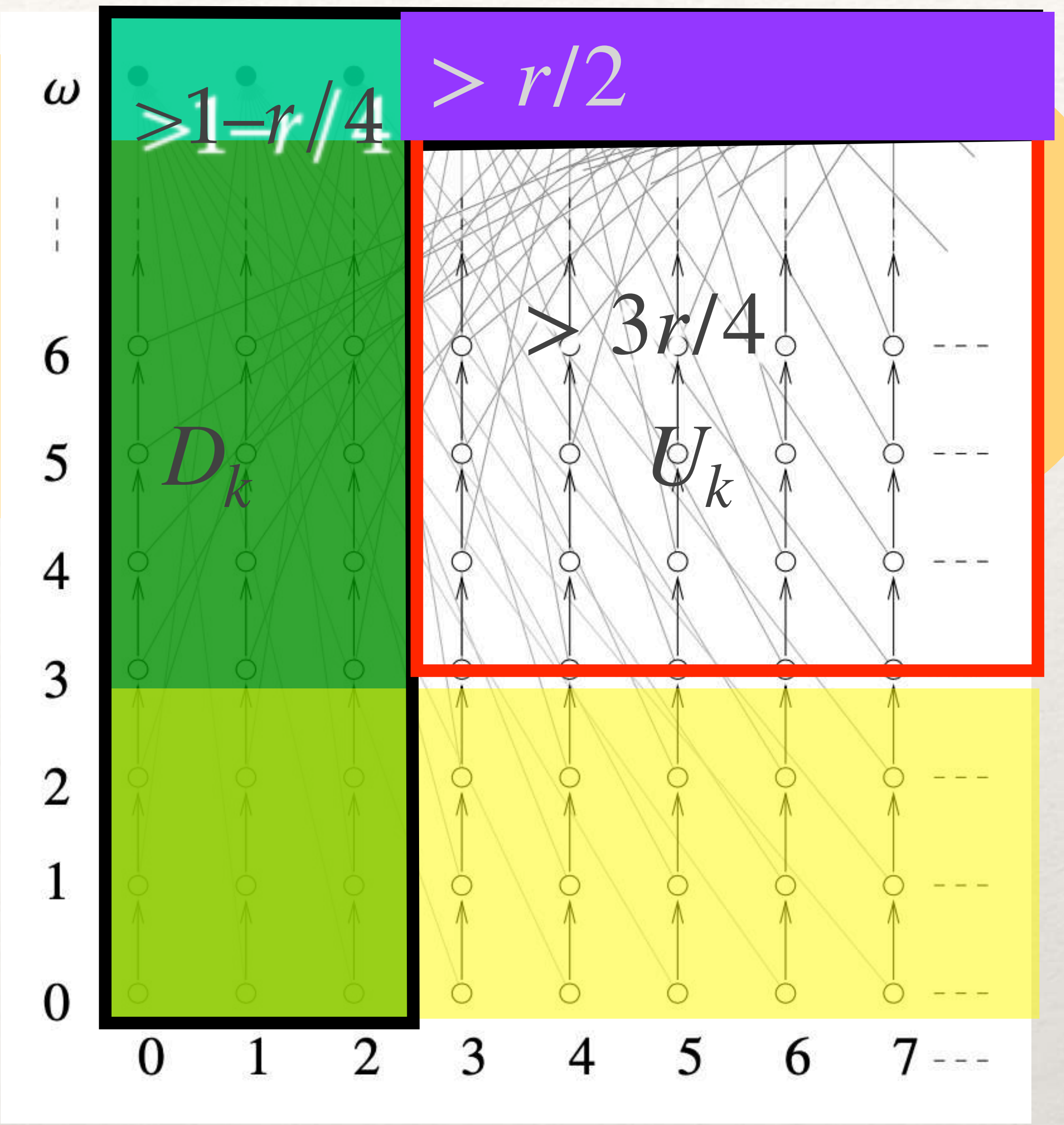

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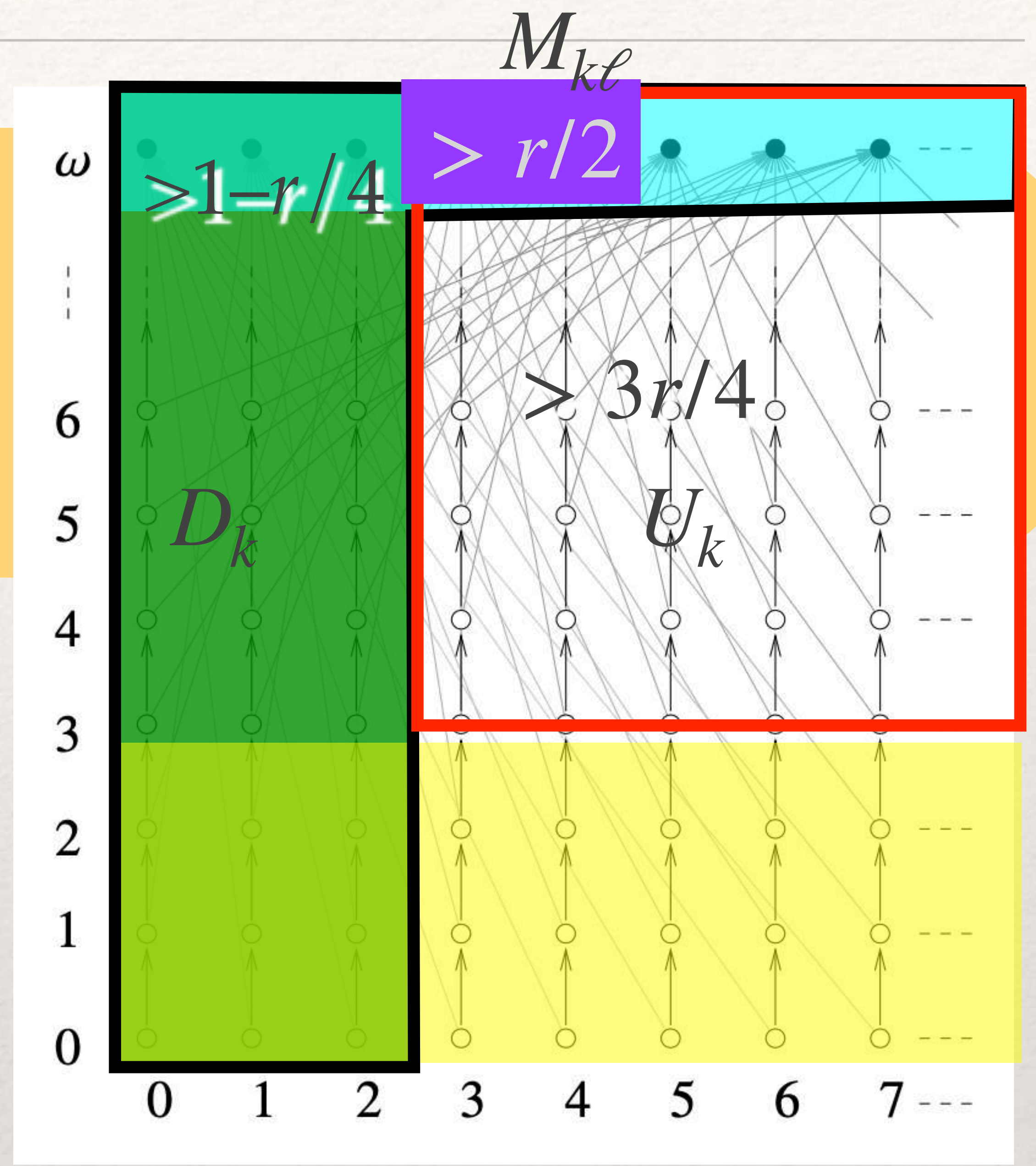

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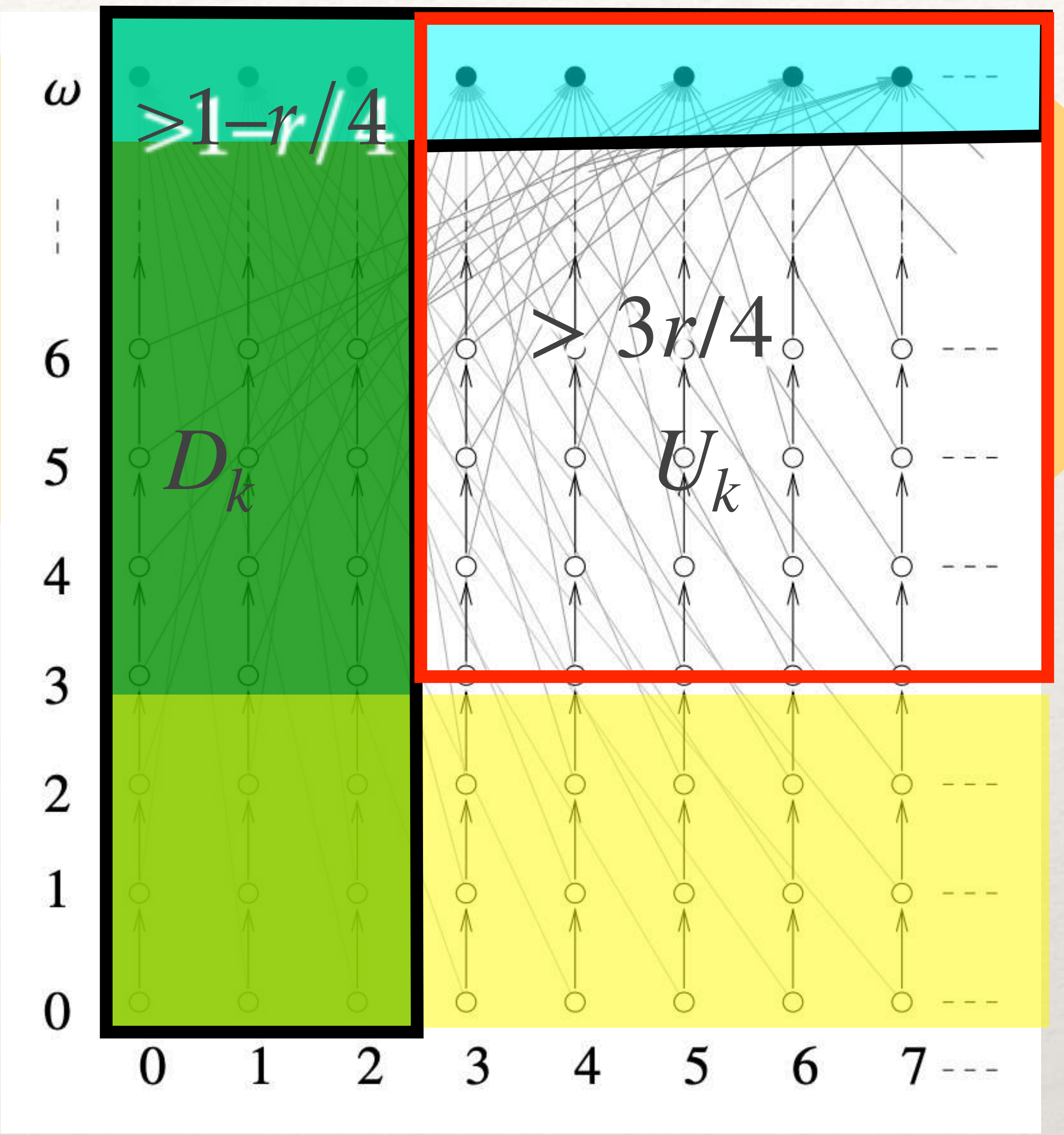

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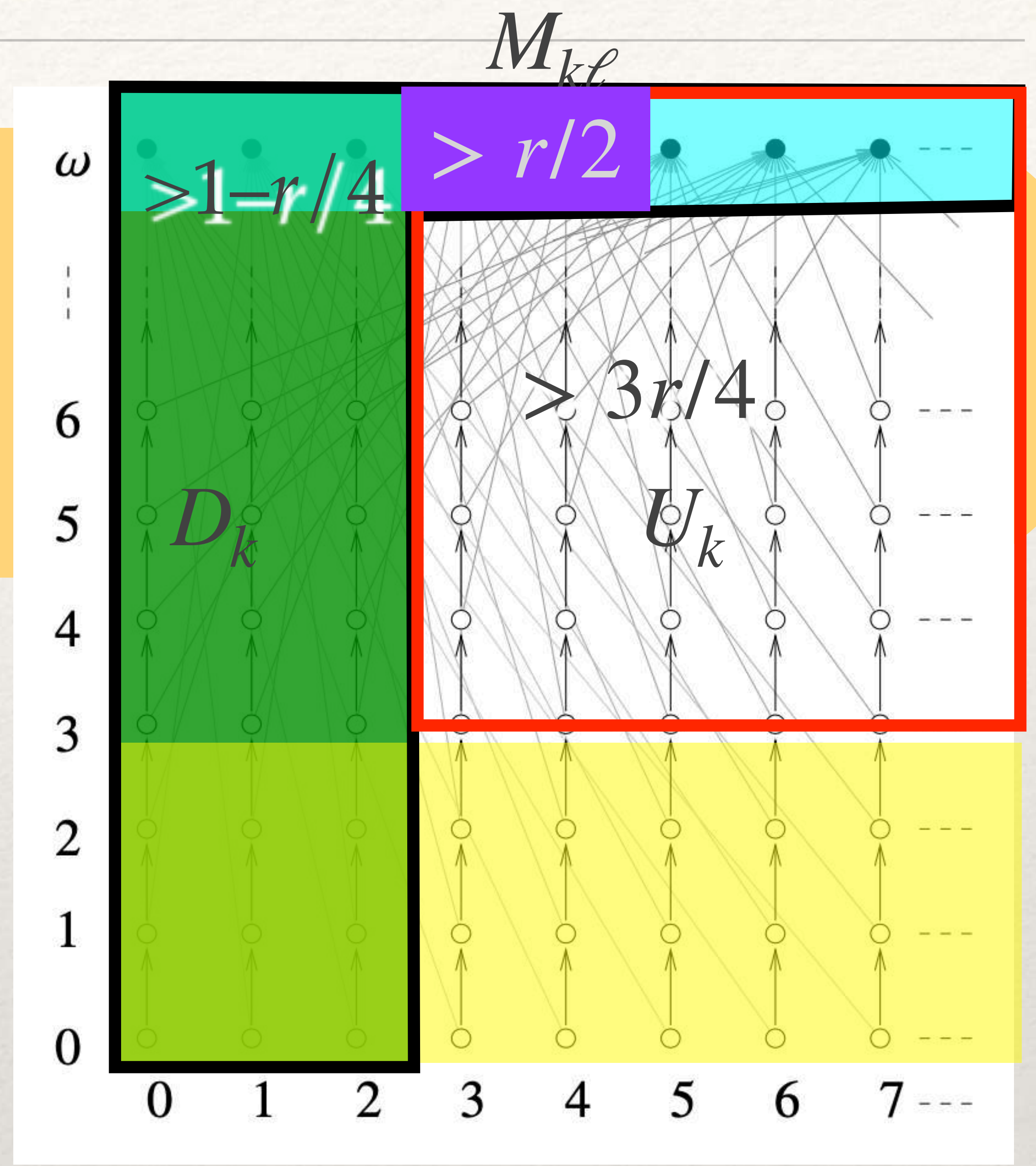

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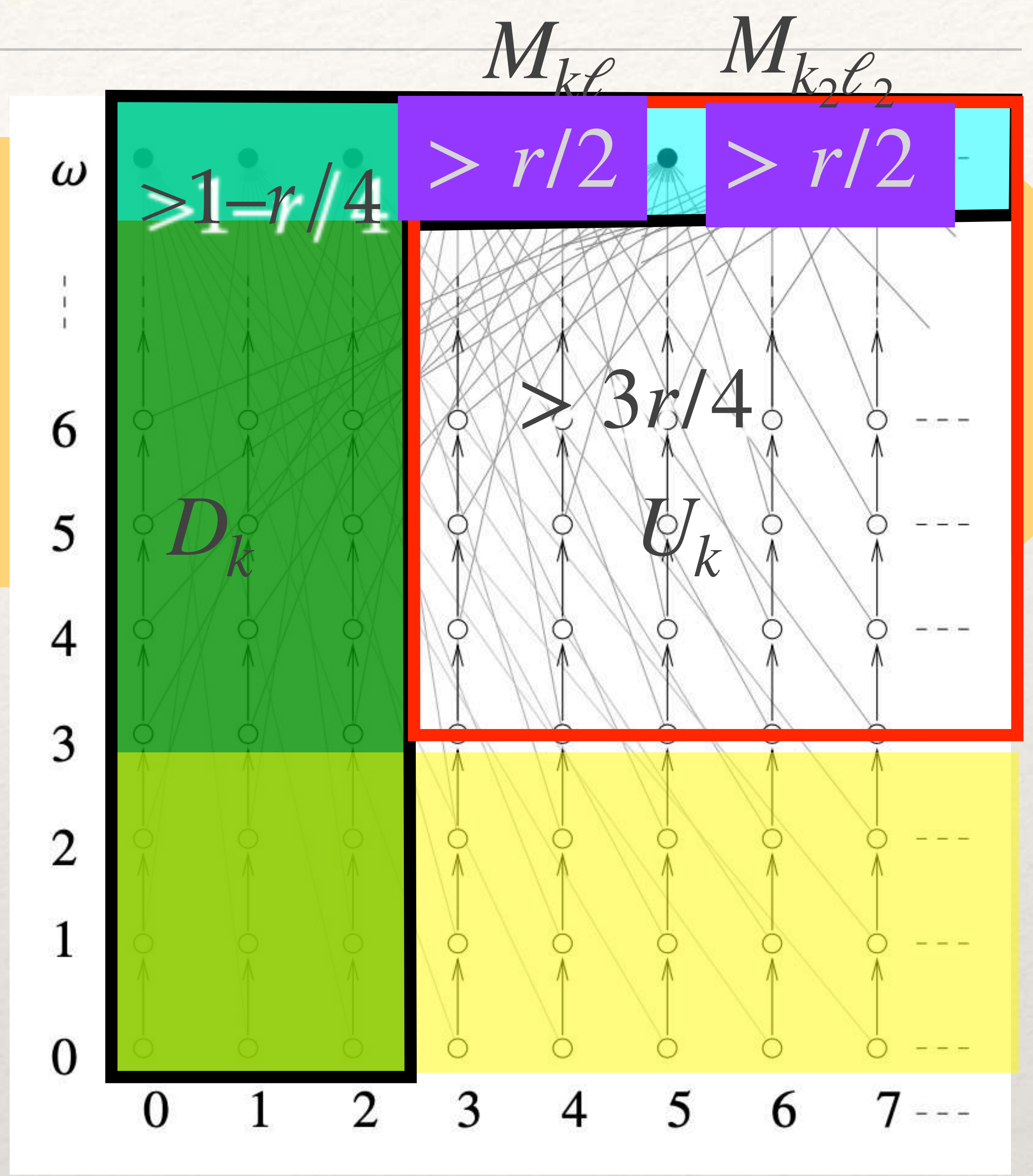

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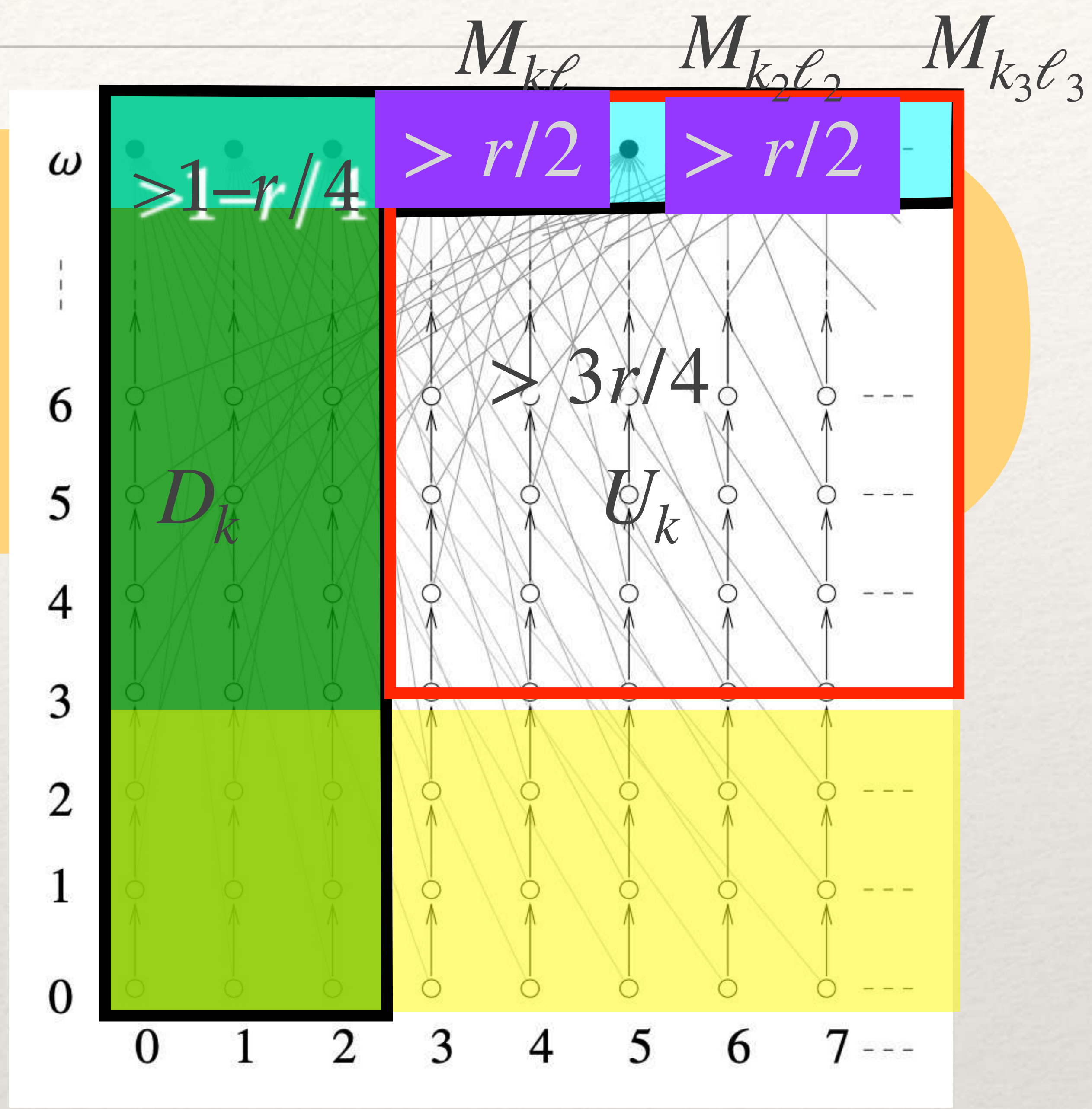

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- Then $\theta_{i_3}(M_{k_3\ell_3}) > r/2$, etc.
- Eventually, $\theta_{i_N}(M_{kl} \uplus \cdots \uplus M_{k_N\ell_N}) > Nr/2 > 1$: contradiction. $\square$

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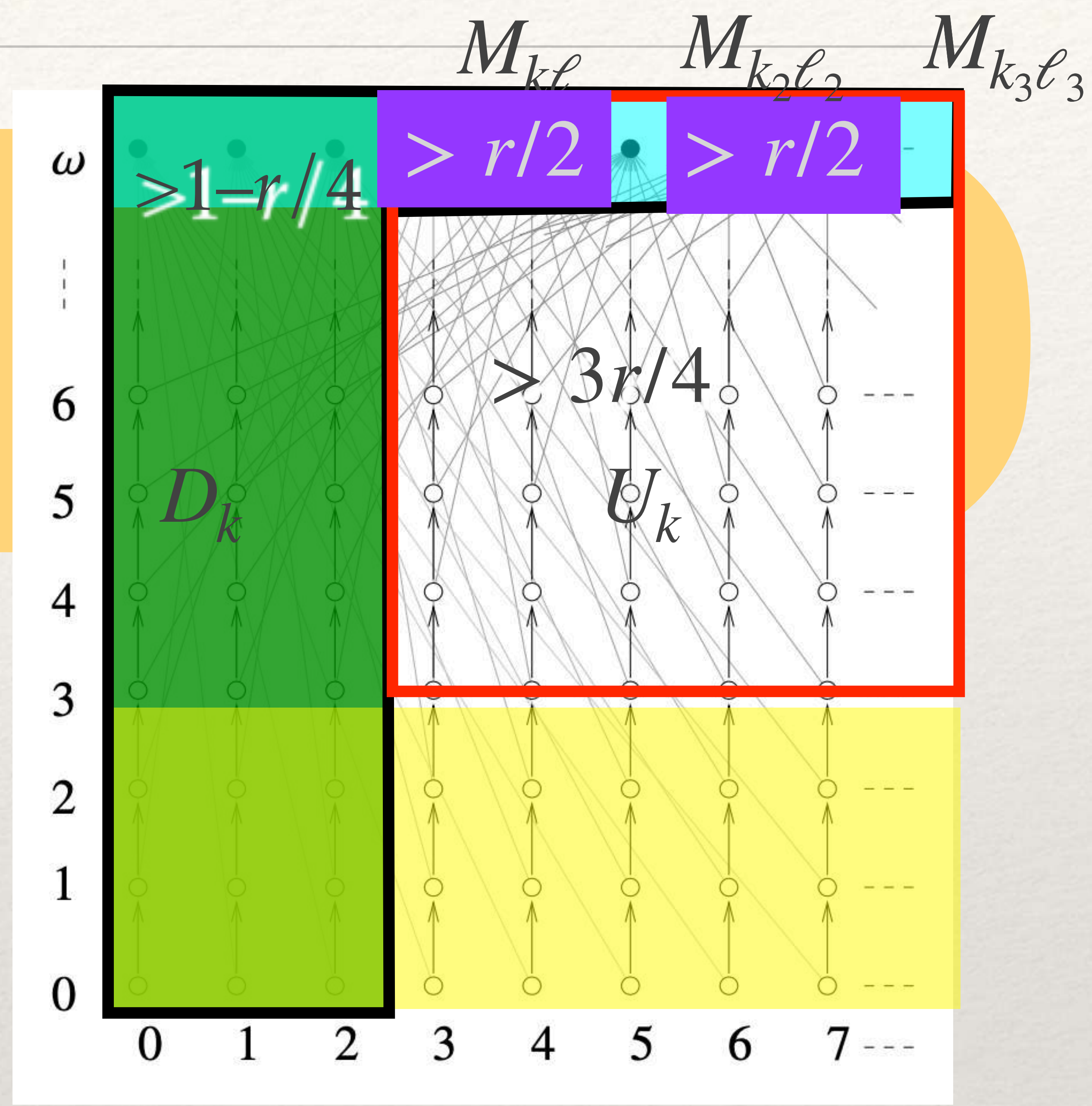

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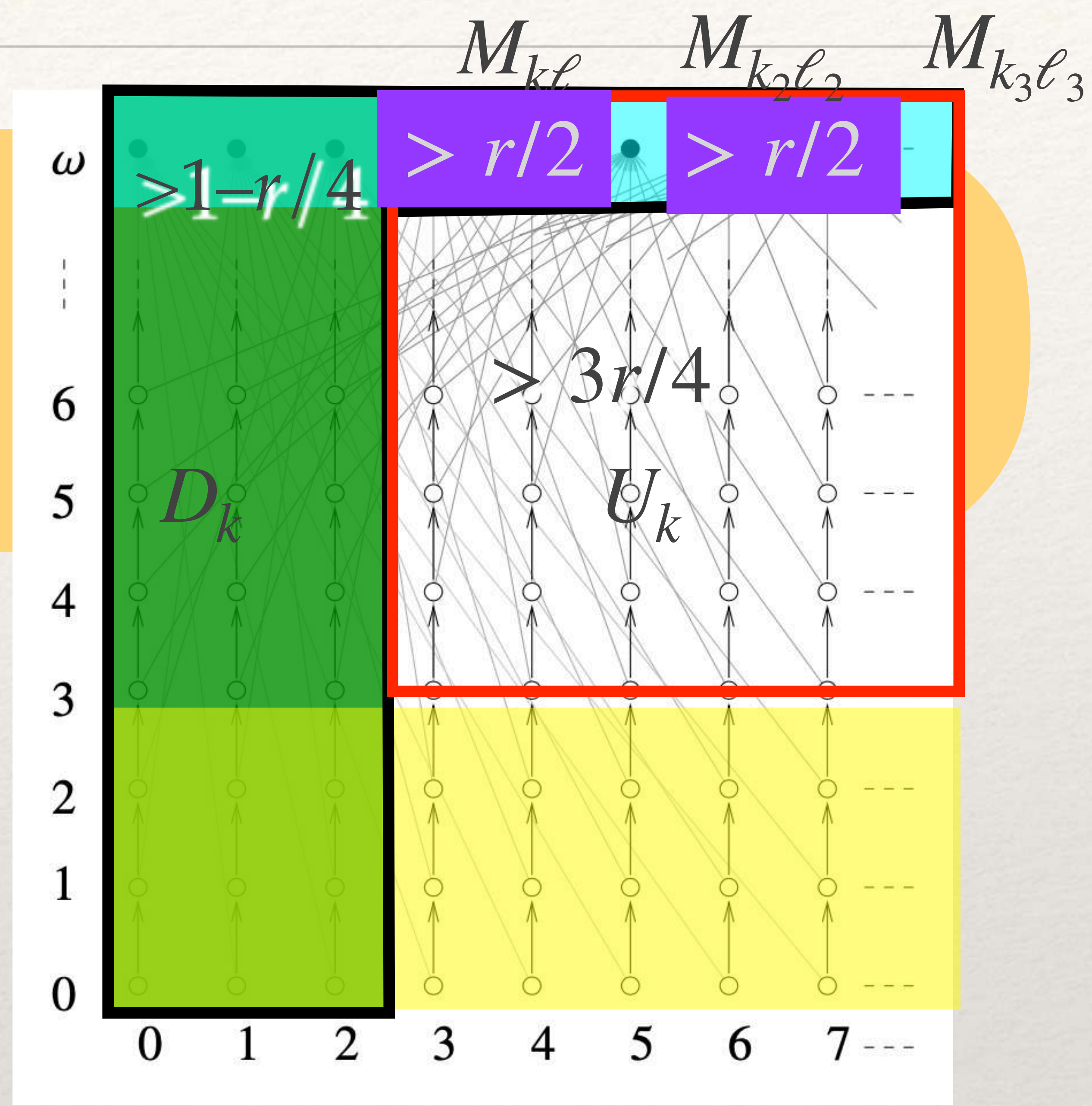

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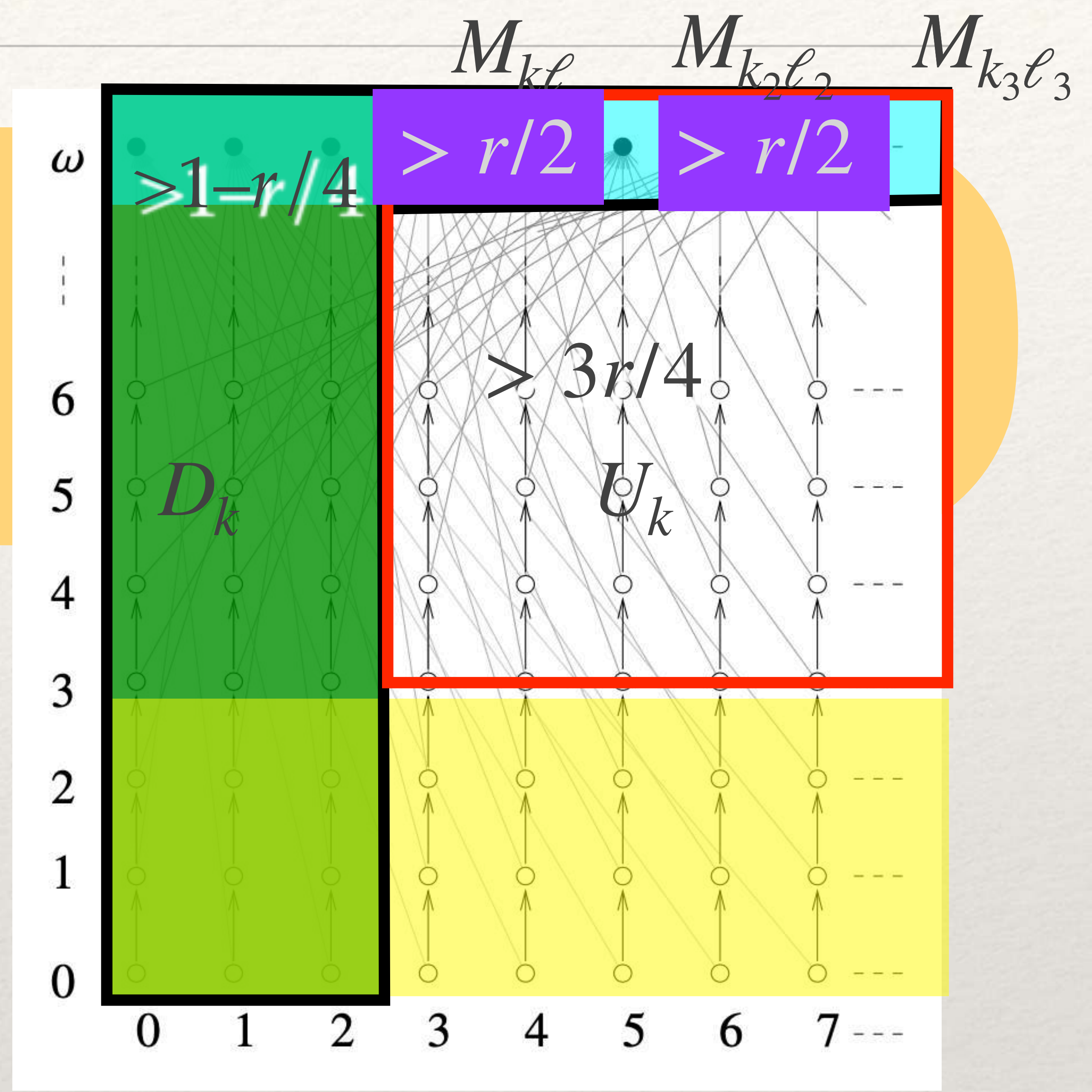

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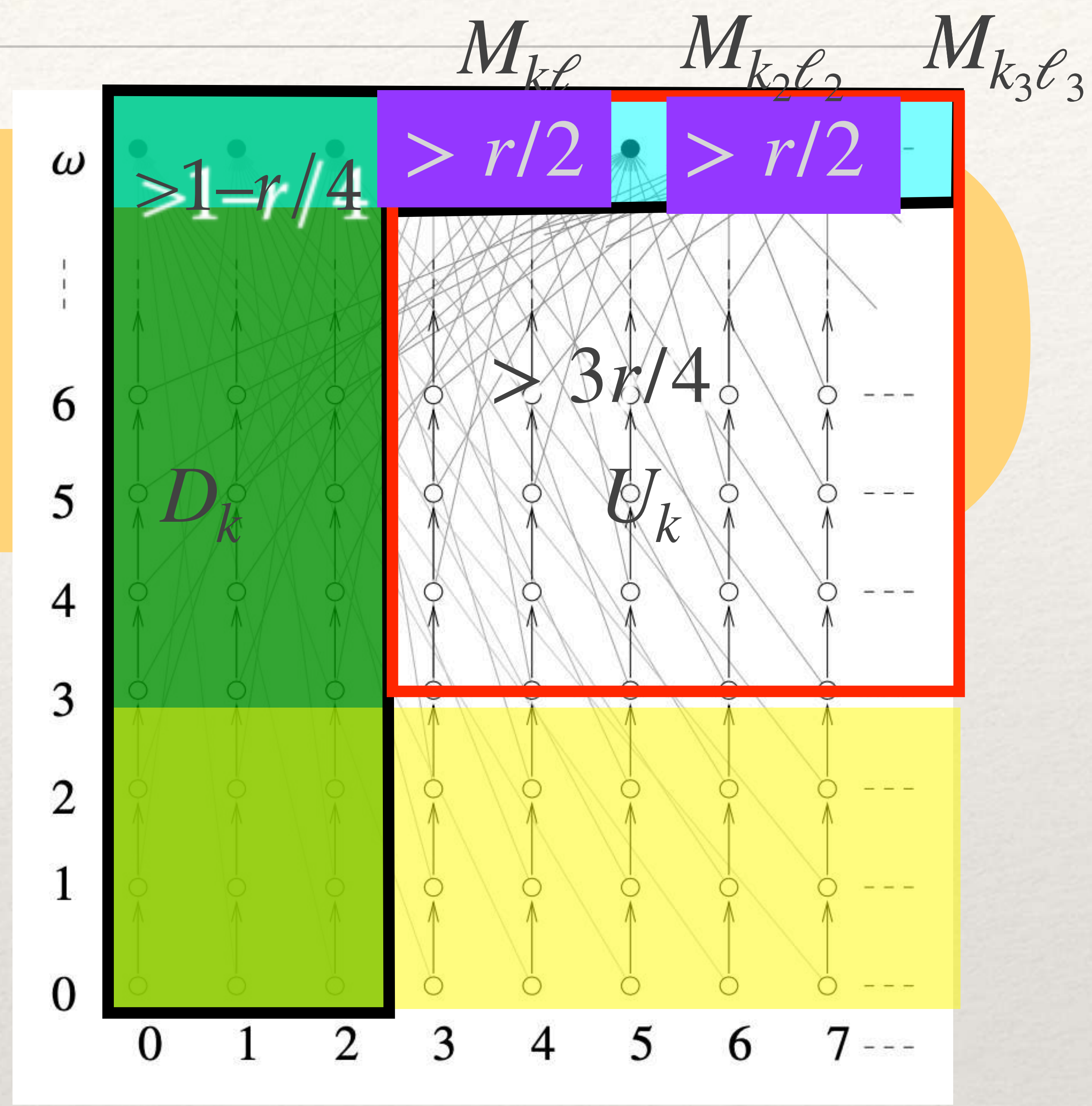

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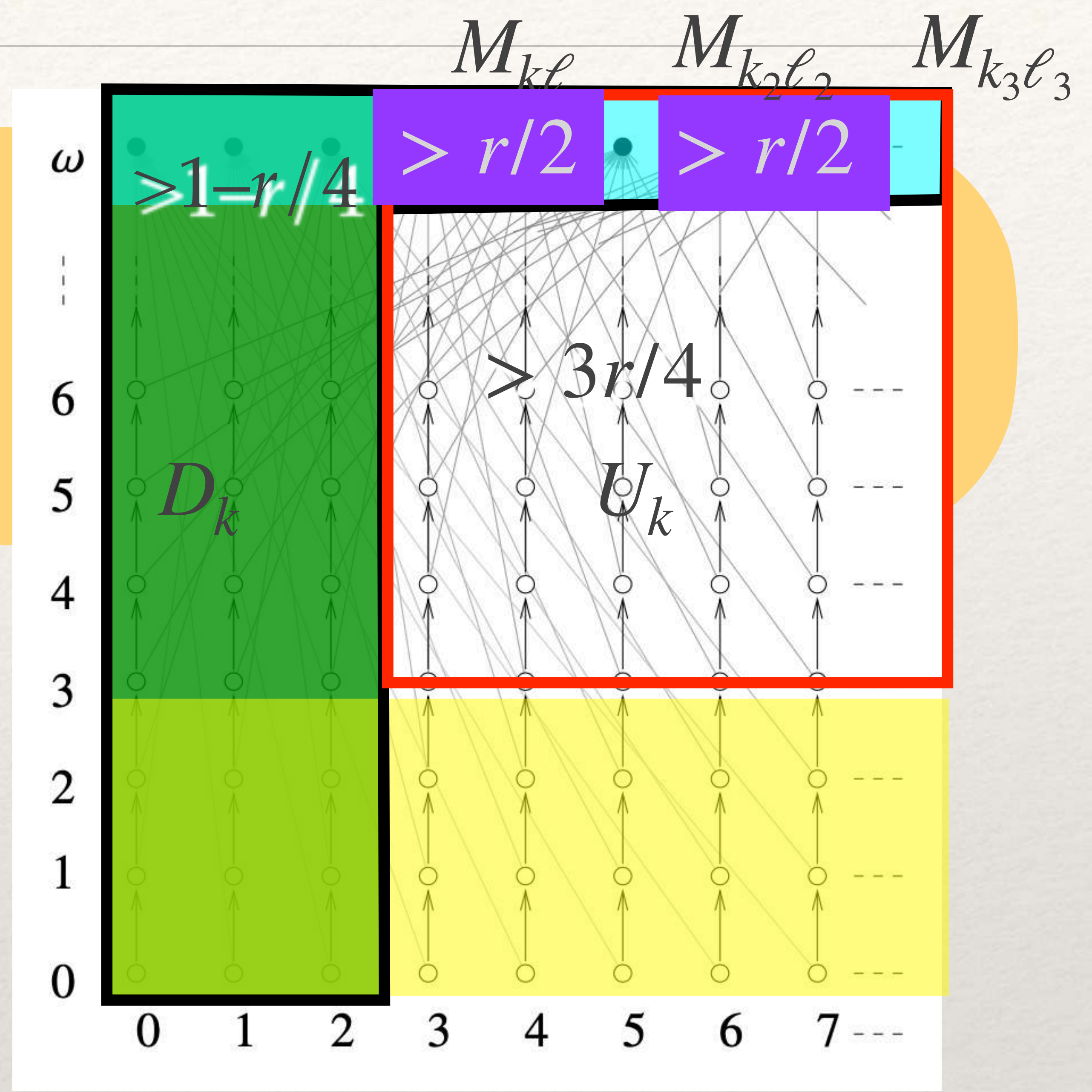

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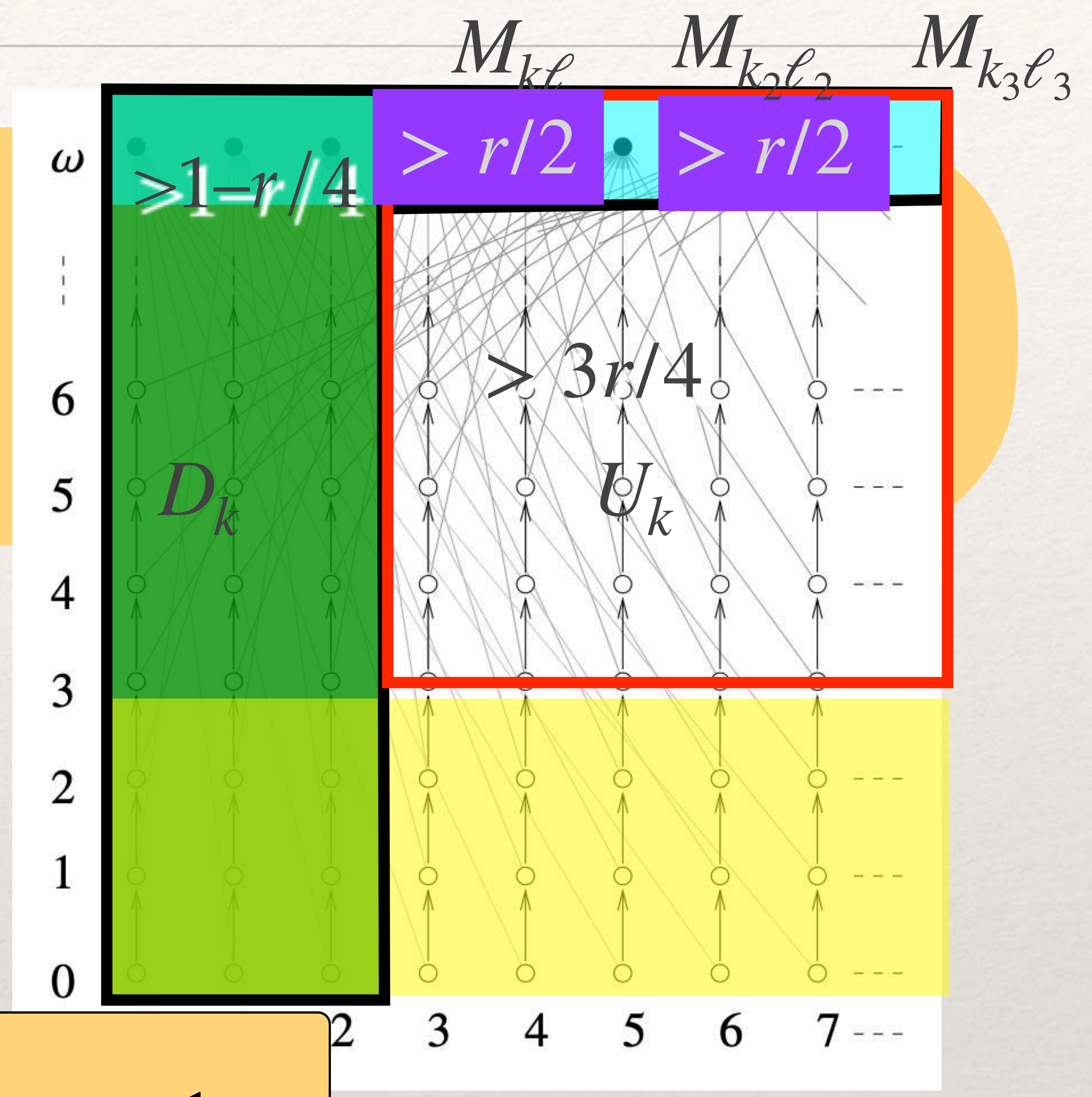

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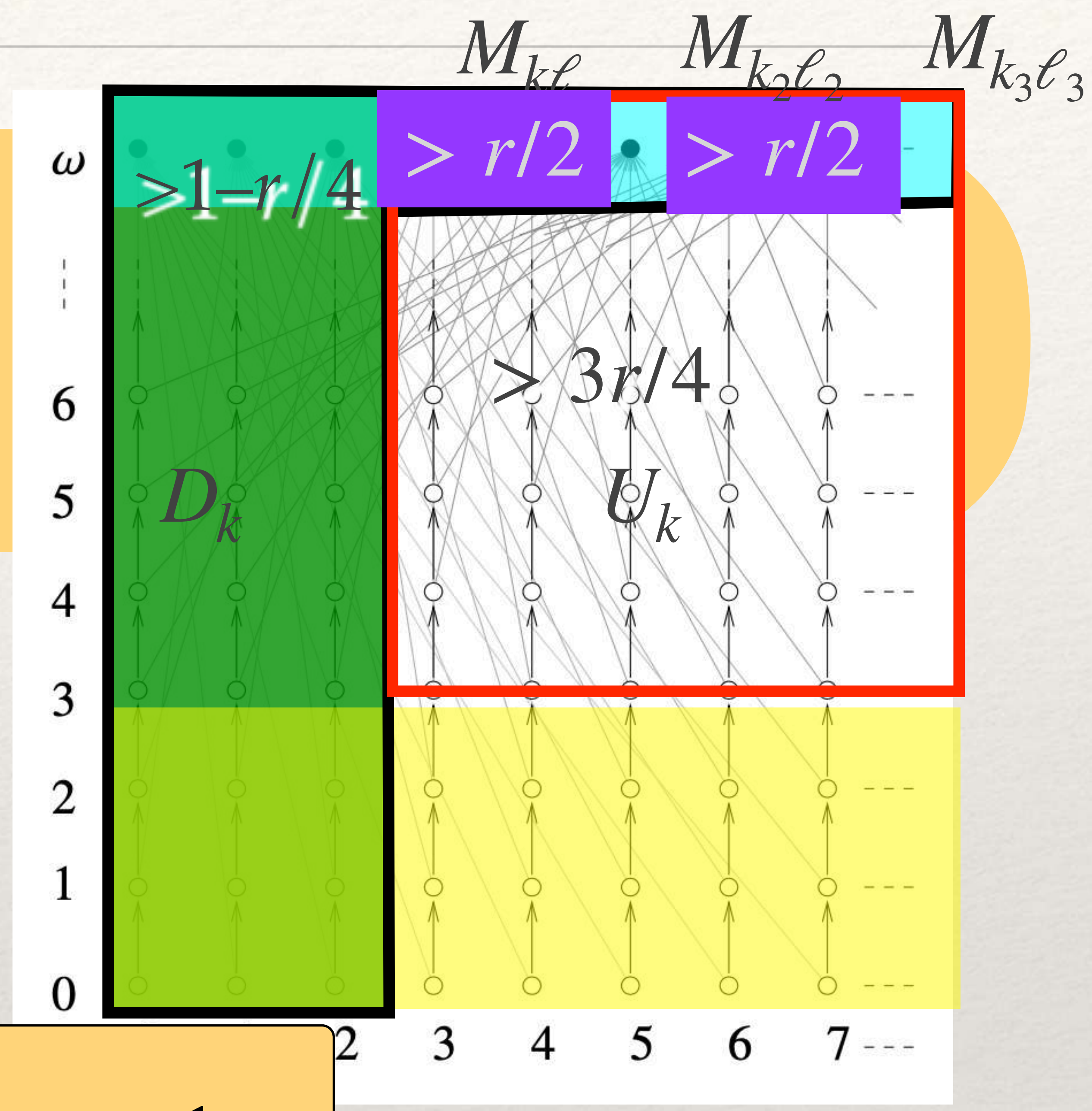

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- **Theorem.** μ is not minimal on $\mathbf{J}_\sigma$.

Conclusion and open problems

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- Open question: Does Fubini-Tonelli hold on **Dcpo**? [X. Jia]
  i.e., is $\mathbf{V}_{\leq 1}$ commutative on **Dcpo**?
  i.e., is every continuous valuation on a dcpo central?