
Graded differential linear logic: can we go higher-order?

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The smooth semantics

Formulas :

- Each MALL formula is a finite dimensional vector space :
 $\llbracket 1 \rrbracket := \mathbb{R} \quad \llbracket A \otimes B \rrbracket := \llbracket A \rrbracket \otimes \llbracket B \rrbracket \quad \llbracket A \oplus B \rrbracket := \llbracket A \rrbracket \uplus \llbracket B \rrbracket \quad \dots$
- Exponentials are interpreted by infinite dimensional vector spaces:
 - $\llbracket ?A \rrbracket := \mathcal{C}^\infty(\llbracket A \rrbracket', \mathbb{R})$ (*functions*)
 - $\llbracket !A \rrbracket := \mathcal{C}^\infty(\llbracket A \rrbracket, \mathbb{R})'$ (*distributions*)
- Negation is duality: $\llbracket A^\perp \rrbracket := \llbracket A \rrbracket' = \mathcal{L}(\llbracket A \rrbracket, \mathbb{R})$

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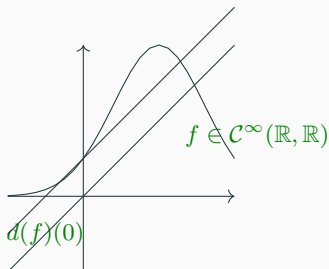
Proofs :

- Each proof is a **linear** map between the interpretation of the formulas.
- $A \Rightarrow B = !A \multimap B$ is $\mathcal{C}^\infty(A, B) \simeq \mathcal{L}(!A, B)$
- The **dereliction** states that $\mathcal{L}(A, B) \subseteq \mathcal{C}^\infty(A, B)$: it **forgets the linearity**.

Differential Linear Logic



Differential interaction nets. Ehrhard, Regnier (2004)



Differential linear logic is about linear extraction of a proof

$$\frac{\ell : A \vdash B}{\ell : !A \vdash B} \text{ d} \qquad \frac{f : !A \vdash B}{D_0(f) : A \vdash B} \bar{\text{d}}$$

- Other rules has to be added (cut-elimination)

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A} \text{ w} \quad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A} \text{ c} \quad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A} \text{ d} \quad \frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A} \text{ p}$$

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Differential Linear Logic

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 \\
 \frac{}{\vdash \delta_0 : !A} \bar{\text{w}} \quad \frac{\vdash \Gamma, \psi : !A \quad \vdash \Delta, \phi : !A}{\vdash \Gamma, \Delta, \psi * \phi : !A} \bar{\text{c}} \quad \frac{\vdash \Gamma, v : A}{\vdash \Gamma, D_0(_)(v) : !A} \bar{\text{d}}
 \end{array}$$

Differential Linear Logic

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 \end{array}$$

- They have nice **mathematical interpretation** (differential calculus)

$\bar{\text{d}}/\text{p}$ is the chain rule ...

From D_0 to differential equations

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Forgets linearity

$$\frac{f : !A \vdash B}{D_0(f) : A \vdash B} \bar{\text{d}}$$

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That is ℓ since $D_0(\ell) = \ell$

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Forgets linearity
Solves D

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Apply D_0
Applies D

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Solution? (LPDO)

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Solution? (LPDO)

Type?

Graded linear logic



A core quantitative coefficient calculus. Brunel et. al (2014)



Combining Effects and Coeffects via Grading. Gaboardi et. al (2016)

Exponential rules of B_SLL

$$\frac{\Gamma \vdash B}{\Gamma, !_0 A \vdash B} \text{ w}$$

$$\frac{\Gamma, !_x A, !_y A \vdash B}{\Gamma, !_{x+y} A \vdash B} \text{ c}$$

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$$\frac{!_{x_1} A_1, \dots, !_{x_n} A_n \vdash B}{!_{x_1 \times y} A_1, \dots, !_{x_n \times y} A_n \vdash !_y B} \text{ p}$$

$$\frac{\Gamma, !_x A \vdash B \quad x \leq y}{\Gamma, !_y A \vdash B} \text{ d}_I$$

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Additive law

Graded linear logic



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Multiplicative law

Additive law

Graded linear logic



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Multiplicative law

Additive law

Order

Graded linear logic



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Exponential rules of B_SLL

$$\begin{array}{c}
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Multiplicative law

Additive law

Order

$(S, +, 0, \times, 1, \leq)$ is an ordered semiring

Graded linear logic



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Exponential rules of B_{SLL}

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Multiplicative law

Additive law

Order

$(S, +, 0, \times, 1, \leq)$ is an ordered semiring

- Type system for resource consumption
- Coeffect analysis

The logic DB_SLL

- A syntactical differentiation of B_SLL

The exponential rules of DB_SLL

$$\begin{array}{cccc} \frac{\vdash \Gamma}{\vdash \Gamma, ?_0 A} \text{ w} & \frac{\vdash \Gamma, ?_x A, ?_y A}{\vdash \Gamma, ?_{x+y} A} \text{ c} & \frac{\vdash \Gamma, ?_x A \quad x \leq y}{\vdash \Gamma, ?_y A} \text{ d}_I & \frac{\vdash \Gamma, A}{\vdash \Gamma, ?_1 A} \text{ d} \\ \frac{}{\vdash !_0 A} \bar{\text{w}} & \frac{\vdash \Gamma, !_x A \quad \vdash \Delta, !_y A}{\vdash \Gamma, \Delta, !_{x+y} A} \bar{\text{c}} & \frac{\vdash \Gamma, !_x A \quad x \leq y}{\vdash \Gamma, !_y A} \bar{\text{d}}_I & \frac{\vdash \Gamma, A}{\vdash \Gamma, !_1 A} \bar{\text{d}} \end{array}$$

Theorem

When S is additive splitting, and the order is define through the sum, DB_SLL enjoys a cut elimination procedure.

The monoid of LPDOcc

Let $\mathcal{D}$ be the set of LPDOcc.

$$\mathcal{D} \simeq \mathbb{R}[X_1, \dots, X_n, \dots]$$

$$\left(D = \sum_{\alpha \in \mathbb{N}} k_{\alpha} \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} \right) \mapsto \left(P = \sum_{\alpha \in \mathbb{N}} k_{\alpha} X_1^{\alpha_1} \dots X_n^{\alpha_n} \right)$$

Proposition

The set of LPDOcc, endowed with composition, is an additive splitting commutative monoid.

An indexed differential linear logic

Exponential rules of IDiLL

$$\begin{array}{c} \frac{\vdash \Gamma}{\vdash \Gamma, ?_D A} w_I \quad \frac{\vdash \Gamma, ?_{D_1} A, ?_{D_2} A}{\vdash \Gamma, ?_{D_1 \circ D_2} A} c \quad \frac{\vdash \Gamma, ?_{D_1} A}{\vdash \Gamma, ?_{D_1 \circ D_2} A} d_I \\[10pt] \frac{}{\vdash !_D A} \bar{w}_I \quad \frac{\vdash \Gamma, !_D A \quad \vdash \Delta, !_D A}{\vdash \Gamma, \Delta, !_D A} \bar{c} \quad \frac{\vdash \Gamma, !_D A}{\vdash \Gamma, !_D A} \bar{d}_I \end{array}$$

From d_D to d_I : syntax has to change, and semantics as well

- $\llbracket !_D A \rrbracket = D(C^\infty(\llbracket A \rrbracket, \mathbb{R})')$ (~~solutions~~) (*parameters*)
- $\llbracket ?_D A \rrbracket = D^{-1}(C^\infty(\llbracket A \rrbracket', \mathbb{R}))$ (~~parameters~~) (*solutions*)

The duality transforms solutions into parameters

The smooth semantics for IDiLL

Definition

$$w : \begin{cases} \mathbb{R} & \rightarrow ?_{id} E \\ 1 & \mapsto cst_1 \end{cases} \quad \bar{w} : \begin{cases} \mathbb{R} & \rightarrow !_id E \\ 1 & \mapsto \delta_0 \end{cases}$$

$$c : \begin{cases} ?_{D_1} E \hat{\otimes} ?_{D_2} E & \rightarrow ?_{D_1 \circ D_2} E \\ f \otimes g & \mapsto \Phi_{D_1 \circ D_2} * (D_1(f).D_2(g)) \end{cases}$$

$$\bar{c} : \begin{cases} !_D E \hat{\otimes} !_D E & \rightarrow !_D E \\ \psi \otimes \phi & \mapsto \psi * \phi \end{cases}$$

$$d_I : \begin{cases} ?_{D_1} E & \rightarrow ?_{D_1 \circ D_2} E \\ f & \mapsto \Phi_{D_2} * f \end{cases} \quad \bar{d}_I : \begin{cases} !_D E & \rightarrow !_D E \\ \psi & \mapsto \psi \circ D_2 \end{cases}$$

Theorem

The smooth semantics is **compatible** with the cut-elimination procedure.

Higher order : the promotion rule

- In linear logic

$$\frac{!A_1, \dots, !A_n \vdash B}{!A_1, \dots, !A_n \vdash !B} \text{ p} \qquad \frac{!A \xrightarrow{f} B}{!A \xrightarrow{p_A} \textcolor{red}{!!}A \xrightarrow{!f} !B}$$

Higher order : the promotion rule

- In linear logic

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For $f \in \mathcal{C}(A, B), g \in \mathcal{C}(B, C)$, $\textcolor{red}{f}; g$ is:

$$!A \xrightarrow{p_A} \textcolor{red}{!!}A \xrightarrow{!f} !B \xrightarrow{g} C$$

Higher order : the promotion rule

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$$!A \xrightarrow{p_A} \textcolor{red}{!!}A \xrightarrow{!f} !B \xrightarrow{g} C$$

- In graded linear logic

$$\frac{!_{y_1} A_1, \dots, !_{y_n} A_n \vdash B}{!_{x \times y_1} A_1, \dots, !_{x \times y_n} A_n \vdash !_x B} \text{ p} \qquad \frac{!_y A \xrightarrow{f} B}{!\textcolor{red}{x \times y} A \xrightarrow{p_{A,x,y}} !_x !_y A \xrightarrow{!_x f} !_x B}$$

Three questions:

1. Can we compose solutions?
2. Can we multiply differential operators?
3. Can we interpret higher-order?

Promotion and LPDO

Three questions:

1. Can we compose solutions?
2. Can we multiply differential operators?
3. Can we interpret higher-order?
 - The function $x \mapsto e^x$ is a solution of $f' - f = 0$.
 - But e^{e^x} is not a solution of $D(f) = 0$ (even with polynomial coeffs!)
 - Not really a problem for us: each map can be chosen as a parameter.
 - If f solution of D_1 and g solution of D_2 , $g \circ f$ solution of ?

Three questions:

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Three questions:

1. Can we compose solutions?
2. Can we multiply differential operators?
3. Can we interpret higher-order?
 - Our *sum* is the composition of operators
 - Can we define $\odot$ such that $(\mathcal{D}, \circ, \odot)$ is a semiring?

Promotion and LPDO

Three questions:

1. Can we compose solutions?
2. Can we multiply differential operators?
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Definition

A *differential semiring* is a tuple $(\mathcal{S}, 0, 1, +, \times)$ s.t.

$0 \times x = 0$	(w)	$xy = 0 \Rightarrow x = 0 \text{ or } y = 0$	($\bar{w}$)
$(x + y)z = xz + yz$	(c)	mult. split. $(x_1x_2 = y_1 + y_2)$	($\bar{c}$)
$1 \times x = x$	(d)	$x + y = 1 \Rightarrow x = 0 \text{ or } y = 0$	($\bar{d}$)
$x \leq y \Rightarrow xz \leq yz$	(d _I)	$x(yz) = (xy)z$	(p)

The axiom (d_I) is implied by (c), thanks to the definition of the order.

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Promotion and LPDO

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An overview of models of DiLL

Model	Reflexivity	Smoothness	Higher-order
Kothe spaces	✓	✗	✓
Convenient spaces	✗	✓	✓
Nuclear Frechet spaces	✓	✓	✓

Promotion and LPDO

Three questions:

1. Can we compose solutions?
2. Can we multiply differential operators?
3. Can we interpret higher-order?

We need linearly independent families to interpret partial derivatives:

- For finitary formulas, we are isomorphic to $\mathbb{R}^n$.
- For (E, V) , we define $!_D(E, V)$ as $(!_D E, !_D V)$, where

$$V = (x_1, \dots, x_n, \dots) \rightarrow !_D V = (\delta_{x_1}, \dots, \delta_{x_n}, \dots)$$

- For MALL connectives over exponential formulas, usual constructions work

Syntactical issues

- The going up procedure:

$$\frac{\frac{\frac{\vdash \Gamma, ?_x A, ?_y A}{\vdash \Gamma, ?_{x+y} A} c}{\vdash \Gamma, ?_{x+y+z} A} d_I}{\vdash \Gamma, ?_{x+y+z} A} d_I \quad \rightsquigarrow \quad \frac{\frac{\frac{\vdash \Gamma, ?_x A, ?_y A}{\vdash \Gamma, ?_x A, ?_{y+z} A} d_I}{\vdash \Gamma, ?_{x+y+z} A} c}{\vdash \Gamma, ?_{x+y+z} A} c$$

Syntactical issues

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$$\frac{\frac{\frac{\vdash \Gamma, ?_x A, ?_y A}{\vdash \Gamma, ?_{x+y} A} c}{\vdash \Gamma, ?_{x+y+z} A} d_I}{\vdash \Gamma, ?_{x+y+z} A} d_I \quad \rightsquigarrow \quad \frac{\frac{\vdash \Gamma, ?_x A, ?_y A}{\vdash \Gamma, ?_x A, ?_{y+z} A} d_I}{\vdash \Gamma, ?_{x+y+z} A} c$$

- Does not work with promotion:

$$\frac{\frac{\frac{\vdash ?_y A, B}{\vdash ?_{xy} A, !_x B} p}{\vdash ?_{xy+z} A, !_x B} d_I}{\vdash ?_{xy+z} A, !_x B} d_I \quad \rightsquigarrow \quad \frac{\frac{\vdash ?_y A, B}{\vdash ?_{y+\textcolor{red}{t}} A, B} d_I}{\vdash ?_{xy+x\textcolor{red}{t}} A, !_x B} p$$

with $\textcolor{red}{t} = \frac{z}{x}$. But we can't divide!

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$$\frac{\frac{\frac{\vdash ?_y A, B}{\vdash ?_{xy} A, !_x B} p}{\vdash ?_{xy+z} A, !_x B} d_I}{\vdash ?_{xy+z} A, !_x B} d_I \rightsquigarrow \frac{\frac{\frac{\vdash ?_y A, B}{\vdash ?_{y+t} A, B} d_I}{\vdash ?_{xy+x t} A, !_x B} p}{\vdash ?_{xy+x t} A, !_x B} p$$

with $t = \frac{z}{x}$. But we can't divide!

- Even worse with $\bar{d}_I$:

$$\frac{\frac{\frac{\vdash ?_y A, B}{\vdash ?_{xy} A, !_x B} p}{\vdash ?_{xy} A, !_x+z B} \bar{d}_I}{\vdash ?_{xy} A, !_x+z B} \bar{d}_I \rightsquigarrow \frac{\frac{\vdash ?_y A, B}{\vdash \dots} \bar{d}_I}{\vdash \dots} \bar{d}_I$$

Cut elimination from DiLL

- Solution through equivalent rules

$$\frac{\vdash \Gamma, !_x A}{\vdash \Gamma, !_{x+y} A} \bar{d}_I := \frac{\vdash \Gamma, !_x A \quad \overline{\vdash !_y A}^{\bar{w}_I}}{\vdash \Gamma, !_{x+y} A} \bar{c}$$

- For most of the rules, we decorate

$$\frac{\frac{\vdash \Gamma}{\vdash \Gamma, ?_0 A} w \quad \frac{\frac{\vdash \Delta, !_x A^\perp \quad \vdash \Xi, !_y A^\perp}{\vdash \Delta, \Xi, !_{x+y=0} A^\perp} \bar{c}}{\vdash \Gamma, \Delta, \Xi} cut \quad \rightsquigarrow$$

$$\frac{\frac{\vdash \Gamma}{\vdash \Gamma, ?_0 A} w \quad \vdash \Delta, !_0 A^\perp}{\vdash \Gamma, \Delta} cut$$

$$\frac{\frac{\vdash \Gamma, \Delta}{\vdash \Gamma, \Delta, ?_0 A} w \quad \vdash \Xi, !_0 A^\perp}{\vdash \Gamma, \Delta, \Xi} cut$$

- Some cases can't be decorated

No more sums in proofs

- In DiLL, cut elimination introduces sums of proofs
- Semantically, $D(f.g) = D(f)g + fD(g)$
- In the syntax: proofs get very technical

$$\frac{\frac{\vdash ?A, ?A}{\vdash ?A} c \quad \frac{\vdash A^\perp}{\vdash !A^\perp} \bar{d}}{\vdash} cut \quad \rightsquigarrow \quad \frac{\frac{\vdash ?A, ?A}{\vdash ?A} \quad \frac{\frac{\vdash A^\perp}{\vdash !A^\perp} \bar{d}}{cut}}{\vdash} cut \quad \frac{}{\vdash !A^\perp} \bar{w} + \frac{\frac{\vdash ?A, ?A}{\vdash ?A} \quad \frac{}{\vdash !A^\perp} \bar{w}}{cut} \quad \frac{\vdash A^\perp}{\vdash !A^\perp} \bar{d} cut$$

- No more sums with the grading

$$\frac{\frac{\vdash ?_x A, ?_y A}{\vdash ?_{x+y} A} c \quad \frac{\vdash A^\perp}{\vdash !_1 A^\perp} d}{\vdash} cut \quad x + y = 1 \Rightarrow x = 0, y = 1 \text{ or } x = 1, y = 0$$

A polarized setting

$$\begin{aligned} P, Q &:= 1 \mid 0 \mid P \otimes Q \mid P \oplus Q \mid ?_x^p P \mid !_x^p P \\ N, M &:= \top \mid \perp \mid N \wp M \mid N \& M \mid ?_x^n N \mid !_x^n N \\ (\textcolor{red}{?}_x^p P)^\perp &= \textcolor{red}{!}_x^n P^\perp \quad (\textcolor{red}{?}_x^n N)^\perp = \textcolor{red}{!}_x^p N^\perp \end{aligned}$$

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 \end{aligned}$$

$$\begin{array}{c}
 \frac{\vdash \Gamma}{\vdash \Gamma, ?_0^n N} \text{ w} \quad \frac{\vdash \Gamma, ?_x^n N, ?_y^n N}{\vdash \Gamma, ?_{x+y}^n N} \text{ c} \quad \frac{\vdash \Gamma, N}{\vdash \Gamma, ?_1^n N} \text{ d} \\
 \\
 \frac{}{\vdash !_0^n N} \bar{\text{w}} \quad \frac{\vdash \Gamma, !_x^n N \quad \vdash \Delta, !_y^n N}{\vdash \Gamma, \Delta, !_{x+y}^n N} \bar{\text{c}} \quad \frac{\vdash \Gamma, N}{\vdash \Gamma, !_1^n N} \bar{\text{d}} \\
 \\
 \frac{\vdash \Gamma, ?_x^n N \quad x \leq y}{\vdash \Gamma, ?_y^n N} \text{ d}_I \quad \frac{\vdash \Gamma, !_x^n N \quad x \leq y}{\vdash \Gamma, !_y^n N} \bar{\text{d}}_I \\
 \\
 \frac{\vdash ?_x^n N, Q}{\vdash ?_{y \times x}^n N, !_y^p Q} \text{ p}
 \end{array}$$

The Laplace transform for distributions:

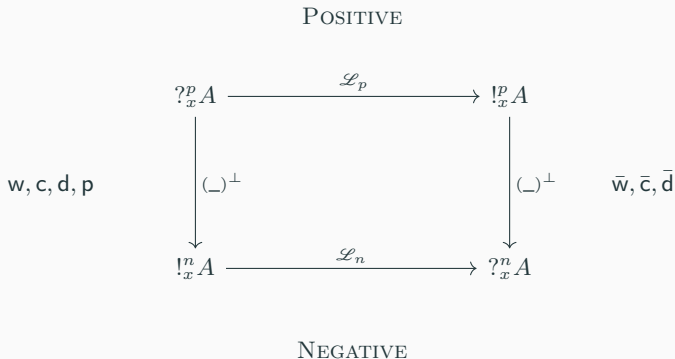
$$\mathcal{L} : \begin{cases} !A & \rightarrow ?A' \\ \psi & \mapsto (\ell \in A' \mapsto \psi(x \in A \mapsto e^{\ell(x)})) \end{cases}$$

$$\mathcal{L}(\delta_0) = cst_1 \quad \mathcal{L}(\psi * \phi) = \mathcal{L}(\psi) . \mathcal{L}(\phi) \quad \mathcal{L}(D_0(_)(v)) = eval_v$$

Laplace transform turns costructural rules into structural ones.

The Laplace transform

- Comes from a model of functions with exponential growth
- This model is graded, higher order (and gives codigging)



What about differential operators?

For $P = \sum_{\alpha} a_{\alpha} X^{\alpha}$, we note $P(\partial) = \sum_{\alpha} a_{\alpha} \partial^{\alpha}$.

$$\mathcal{L}(P(\partial)(\psi)) = P.\mathcal{L}(\psi) \qquad \mathcal{L}(P.\psi) = P(\partial)(\mathcal{L}(\psi))$$

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PARAMETERS

$$\begin{array}{ccc}
 ?_P A & \xrightarrow{\mathcal{L}^{-1}} & !_{P(\partial)} A \\
 \downarrow (_)^{\perp} & & \downarrow (_)^{\perp} \\
 !^P A & \xrightarrow{\mathcal{L}} & ?^{P(\partial)} A
 \end{array}
 \quad \begin{array}{l} w, c, d, p \\ \bar{w}, \bar{c}, \bar{d} \end{array}$$

SOLUTIONS

What do we have?

- A non higher-order logic, where one can interpret solutions and parameters of LPDO
- A way to eliminate cuts with the promotion
- A polarized version, with higher-order, coming from concrete models

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- A semiring of LPDO

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Thank you!