A game characterization of the parity index of regular tree languages

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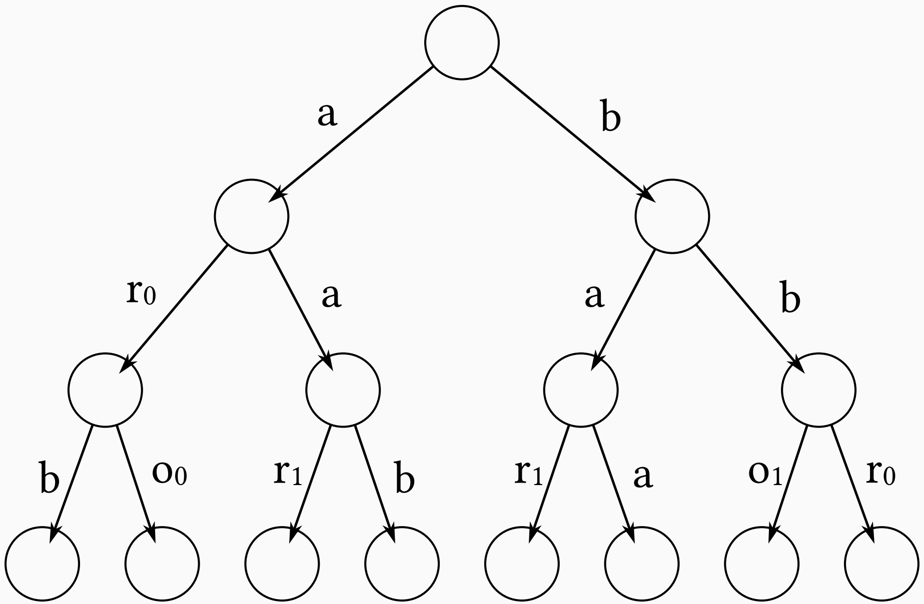
ω-regular tree languages

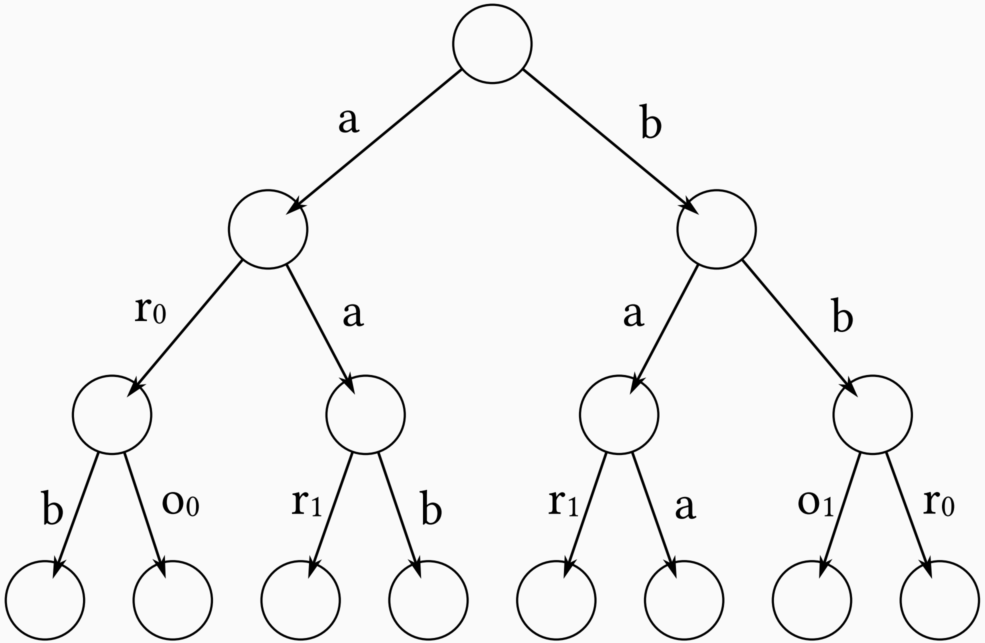
Infinite executions

We look at programs with branching infinite executions. More specifically, we consider their execution traces, labelled with a finite alphabet Σ.
These execution can be represented by infinite trees, with labelled transitions.



μ-calculus transition systems

Properties on trees can be expressed in your favourite branching logic (μ-calculus, MSO...)



For instance, a formula can describe the following specification:

- There is no c in the tree
- Each branch has an infinity of b
- All a are followed by a branch seeing a's until seeing a b

A ω-regular tree language consists in the set of infinite trees satisfying a given formula φ.

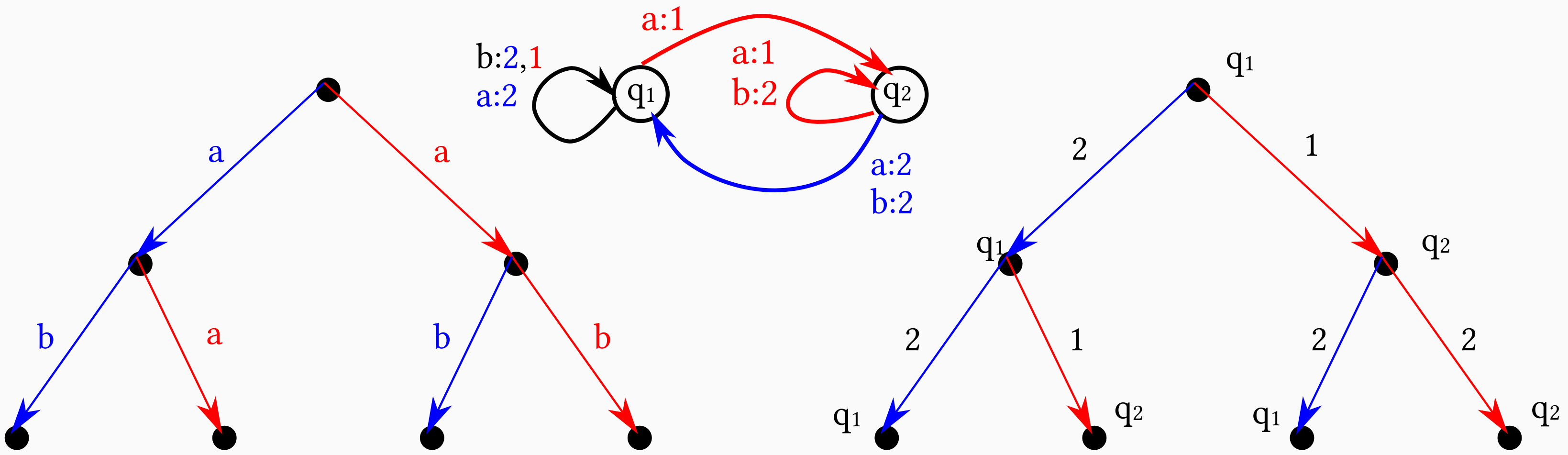


Figure 3: A run of a parity automaton on a tree t outputs a re-labelling of t with integers.

A tree t is accepted by an automaton A if there exists a run of A on t where all the branches are eventually dominated by an even value.

A ω-regular language is said I-feasible if it is recognized by an automaton with parity index I.

The alternation depth of a formula φ recognizing L corresponds to the index interval I necessary to recognize it. It is denoted the Mostowski index of this language.

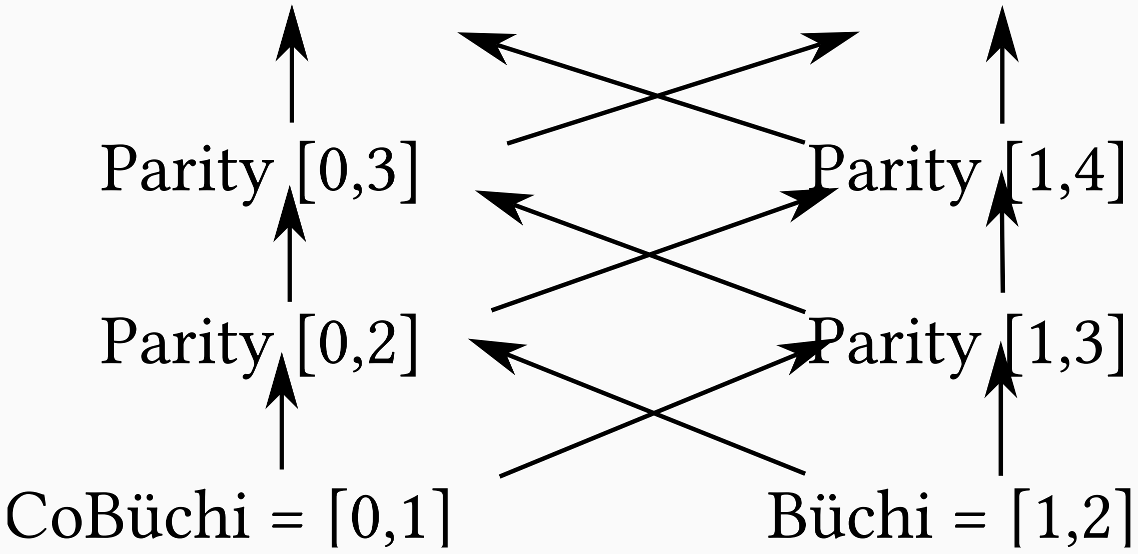


Figure 4: A part of the Mostowski hierarchy.

Existing Results

For the following classes, we can decide where is a language (described by an automaton) located in this class

- if the input automaton is deterministic [Niwinski & Walukiewicz 05]
- If the language is Büchi [Urbański 00]

Some other classes, out of the Mostowski hierarchy, can also be decided (weak when the input is a Büchi automaton, game automata, branch languages...)

In general, deciding the Mostowski index of a given language is an open problem.

Characterization of Mostowski feasibility

Theorem (Colcombet, Löding 08): The problem of deciding the J-feasibility of a regular tree language is reducible to the existence of a uniform $n \in \mathbb{N}$ such that some automaton $A(n)$ with counters bounded by n is universal

Our work starts from this result, trying to extend it and to use more convenient objects.

Equivalences reached

The following statements are equivalent, given a language$L$ described by a (guidable) automaton A:

- L is J-feasible
- There exists a uniform $n \in \mathbb{N}$ such that Eve wins some game $G(A, J, n, t)$ exactly on the trees $t \in L$
- We can exhibit a common structure on the acceptation games of the trees in L by A

Parity transposition games

Parity Games

Parity games consist in graphs with $\mathbb{N}$-labelled edges, vertices partitioned between two players, Adam and Eve.

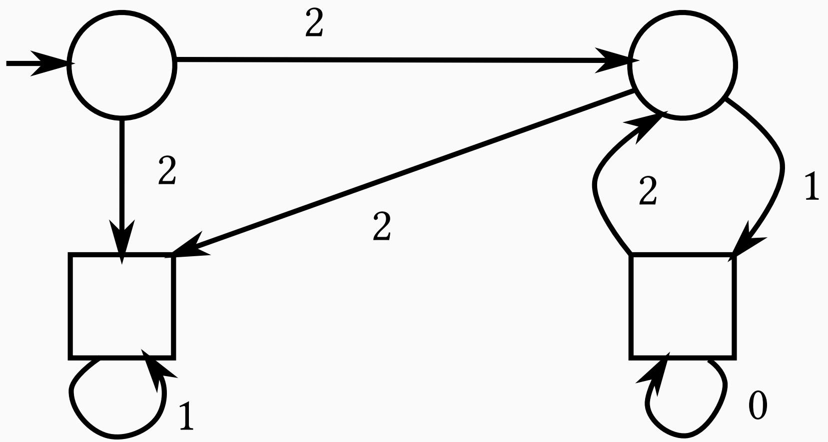


Figure 5: Eve, controlling the circle vertices, has a winning strategy, through always going right.

The acceptation of a tree t by an automaton A can be seen as a game A_t.

Register games

Let J an index, G a parity game on some index I. The parity transposition game $\mathcal{T}_J(G)$ consists in a game played over G, where the output is restricted to J.
It acts as a parity transducer where Eve chooses the current mapping from I to J, with some constraints.

Results on parity transposition games

Two main results are to note [Lehtinen 18]:

Proposition: Whenever Adam has a winning strategy in G, $\forall J$, Adam has a winning strategy in $\mathcal{T}_J(G)$.

Proposition: Whenever Eve has a winning strategy in G of parity index I, $\exists$ a minimal $J \subseteq I$ such that Eve has a winning strategy in $\mathcal{T}_J(G)$.

Now with counters: Parity transposition games

We extend parity transition games with counters bounded by N. They act as an extra buffer for Eve's memory.
We denote the corresponding game $\mathcal{T}_J^N(G)$.

The victory of Eve is non-decreasing along N.

Characterization of the J-feasibility

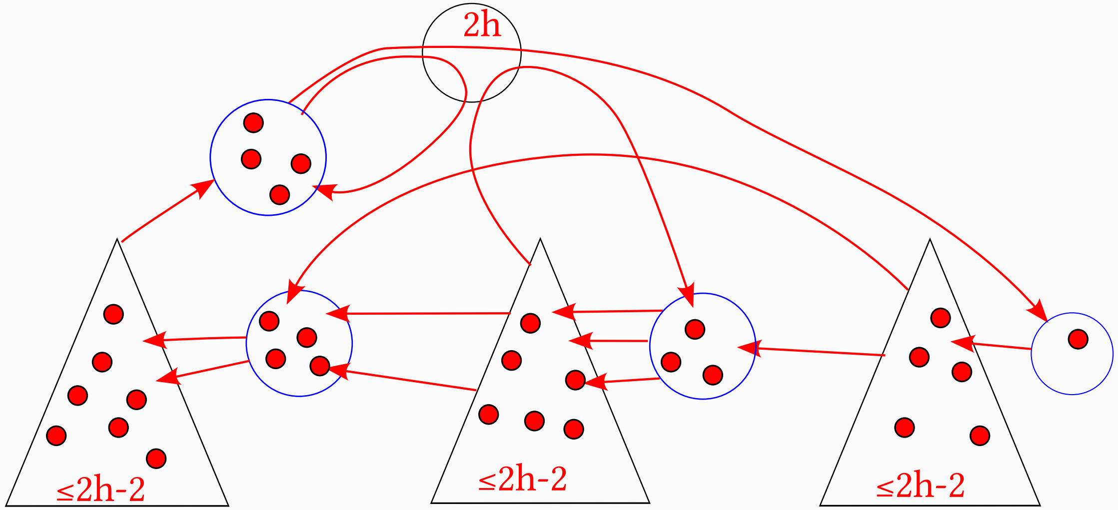
Theorem: For L an ω-regular tree language recognized by some (guidable) automaton A, the following are equivalent:

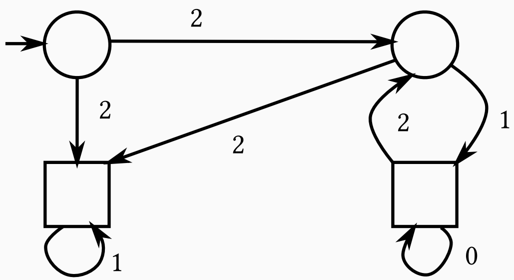
- L is J-feasible
- $\exists N \in \mathbb{N}$ such that $\forall t \in L$, Eve wins $\mathcal{T}_J^N(A_t)$.

Strahler number of a game

Attractor decomposition

Given a game G won by Eve, an attractor decomposition consists in a tree-like structure, composed of even edges and their attractors. A play can only remain in an attractor during a finite number of steps. Rightwards movement in the tree is necessarily even.





(The two figures do not correspond to the same game)

Given a finite tree, its n-Strahler number consists in the tallest $(n+1)$-ary complete minor that can be extracted from the tree with children deletion and edge-contraction along a single child.

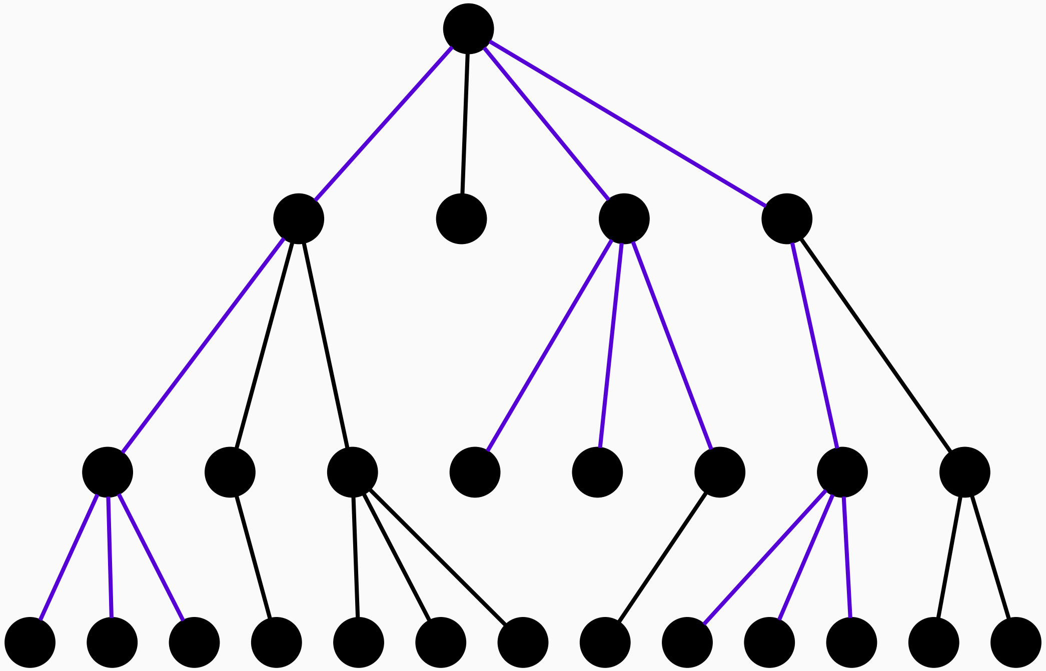


Figure 6: A tree of height 4 and of 2-Strahler 3, with a corresponding minor in purple.

n-Strahler number of attractor decompositions

Using the previous theorem, and extending a result by Daviaud, Jurdziński and Thejaswini, we are able to explicit a link between this n-Strahler number and the J-feasibility of a language.

Theorem: For L an ω-regular tree language recognized by some (guidable) automaton A, the following are equivalent:

- L is $[1, 2h]$-feasible
- $\exists N \in \mathbb{N}$ such that $\forall t \in L$, there exists an attractor decomposition of A_t of N-Strahler h.

Conclusion

We provide alternative characterizations for the Mostowski index, using games and their structure

Future ideas

- Digging deeper with universal n-Strahler trees
- Characterizing the attractor decompositions that can be obtained from a given automaton

Thank you for your attention.