Recognizability in functional programs

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University of Lille

Journées annuelles GT Vérif et GT SCALP du GDR IFM 2024

λ-calculus

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Simply typed λ-calculus

Types: $\mathcal{A}$ is a finite set of atomic types and $(A \to B)$ is a type when A and B are types.

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Higher-order signature $\Sigma = \{a^A, b^B, \dots\}$ is a set of typed constant.

λ-calculus

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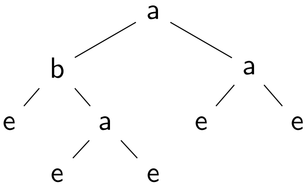
Syntax basics

Trees



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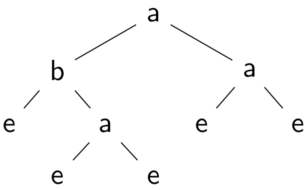

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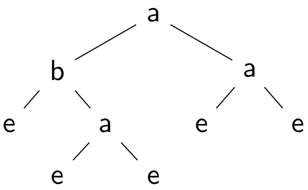

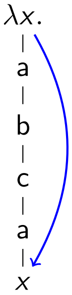

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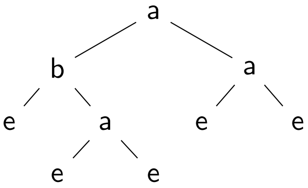

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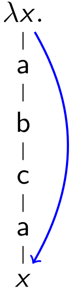
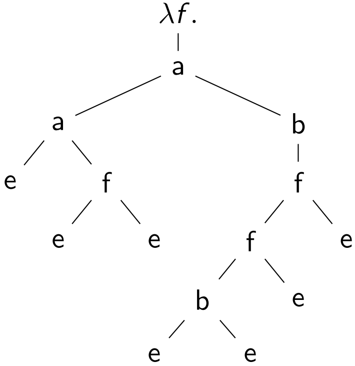
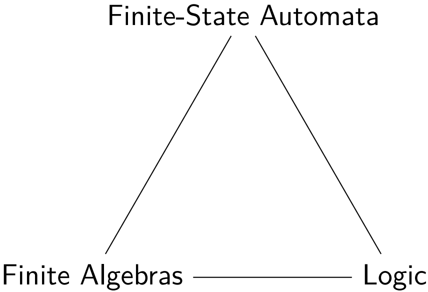


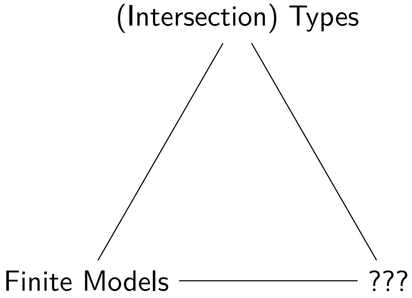
Complex operations



Recognizability

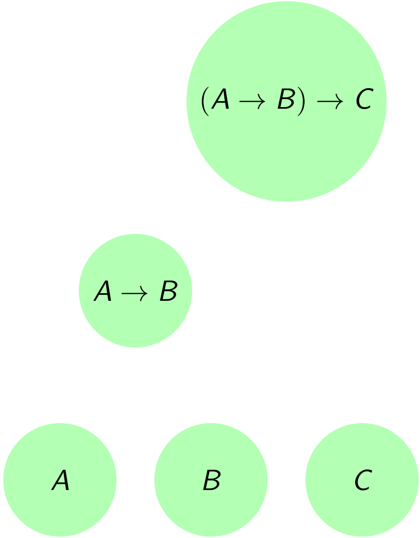


Recognizability in the simply typed λ-calculus [S.09]



Finite models

$[\![M, \nu]\!]$



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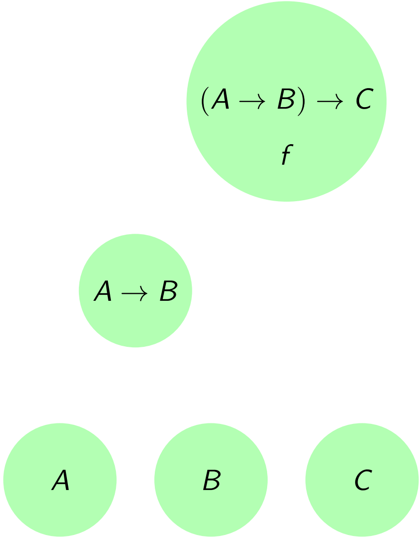

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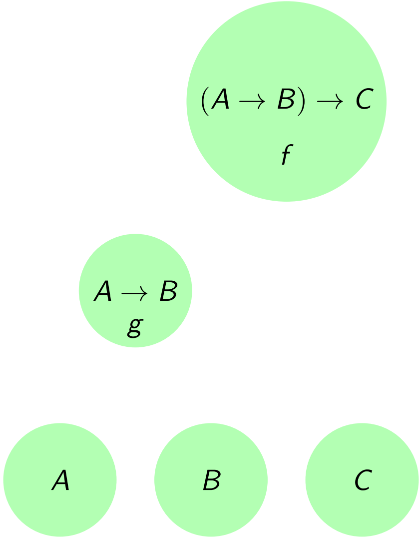

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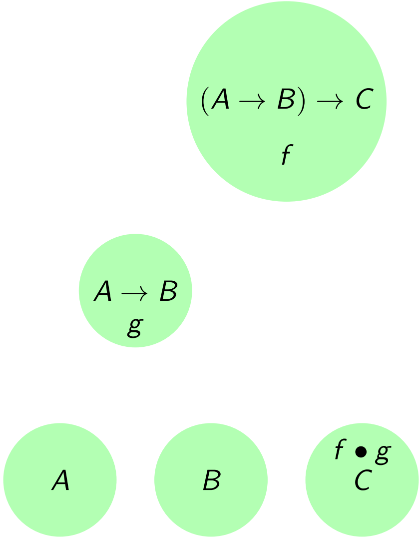

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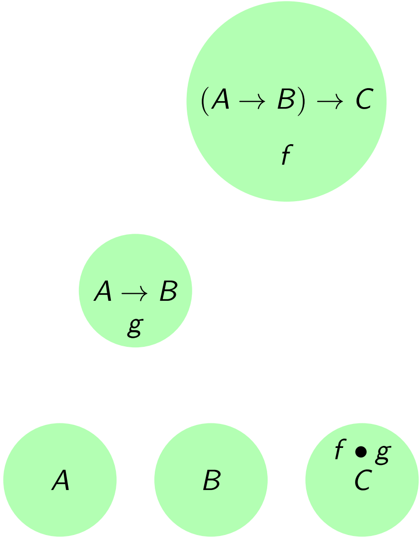

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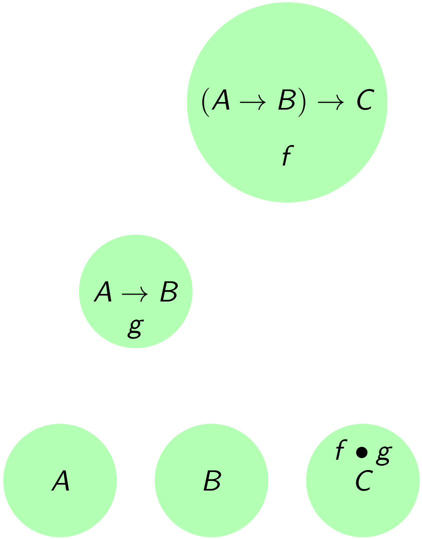

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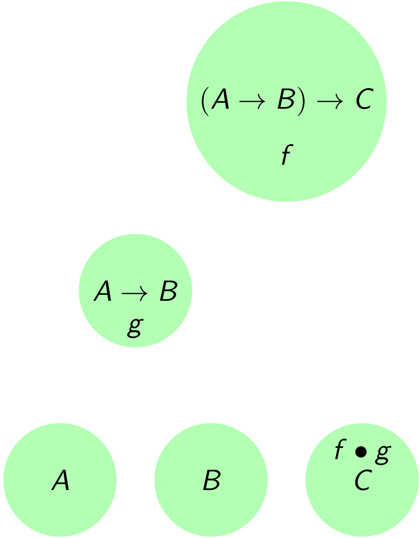
Lemma (Correctness)

If $M =_\beta N$, then for every ν, $[\![M, \nu]\!] = [\![N, \nu]\!]$.

A wealth of possibilities

standard, monotonous, stable, strongly stable ...models, bi-domains etc.



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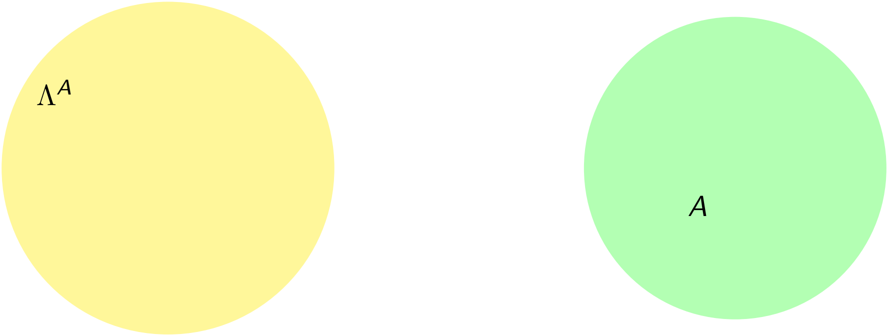

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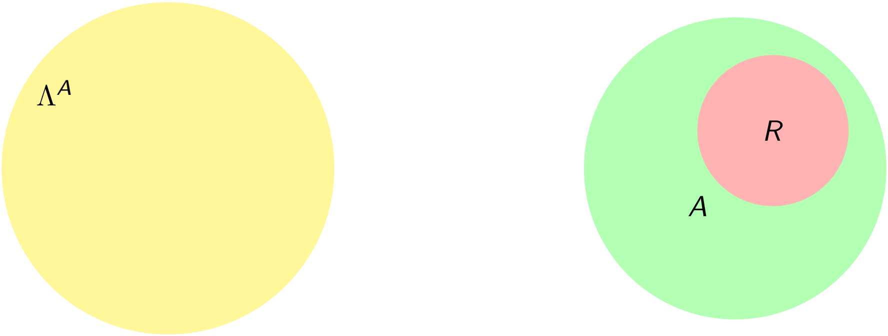
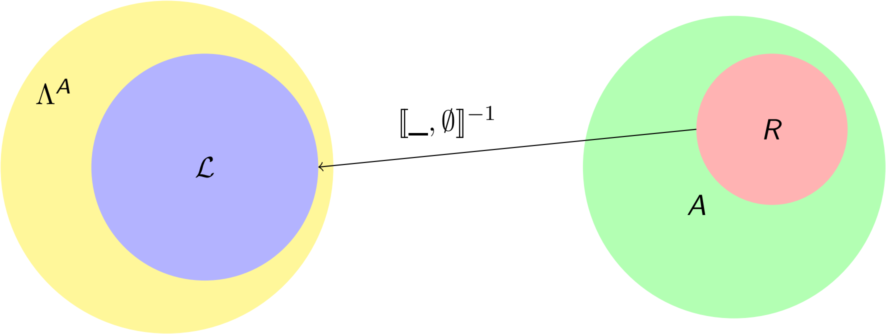


Recognizability in the simply typed λ-calculus

$\mathcal{L}$ is recognizable iff:

$$\mathcal{L} = \{M \mid [\![M, \emptyset]\!] \in R\}$$



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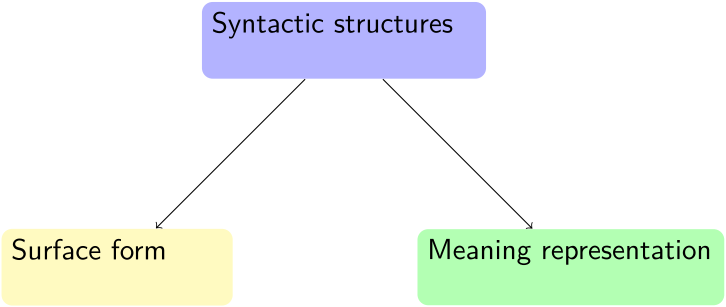
First application

Simple proof of decidability of 4^{th} order matching.

Finiteness: parsing algorithms

Abstract Categorial Grammars [de Groote 01, Muskens 01]



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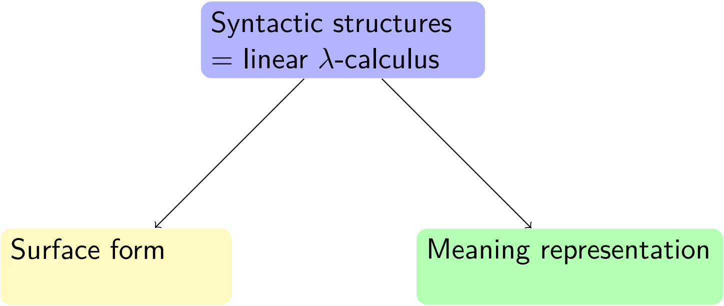

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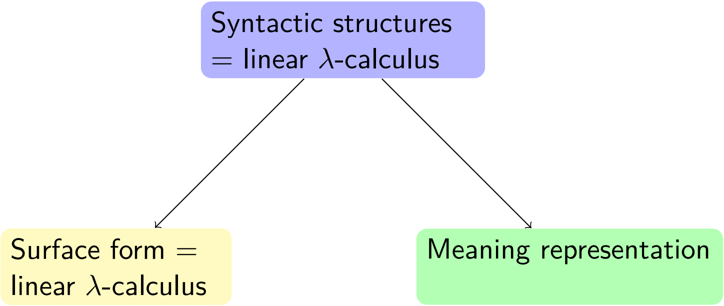

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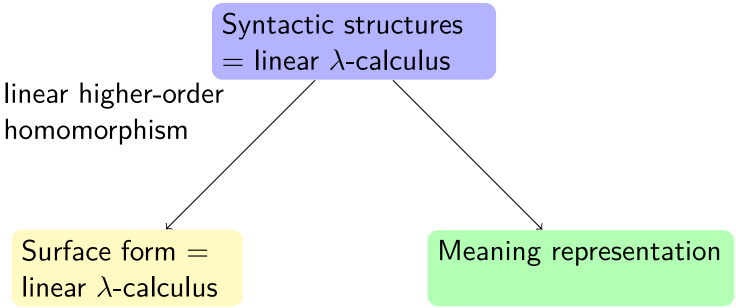

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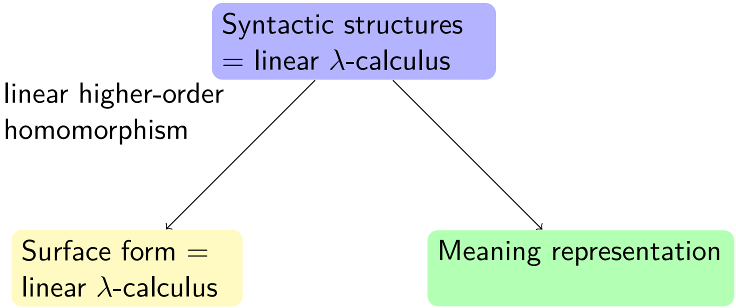

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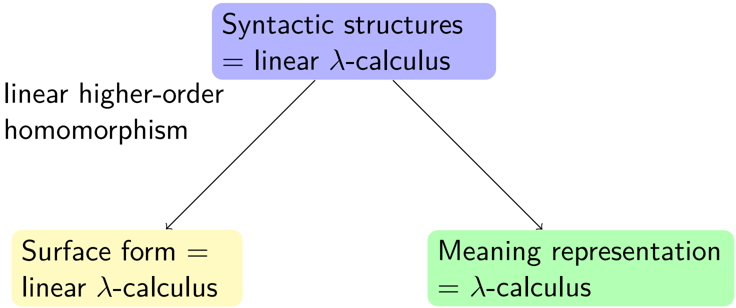

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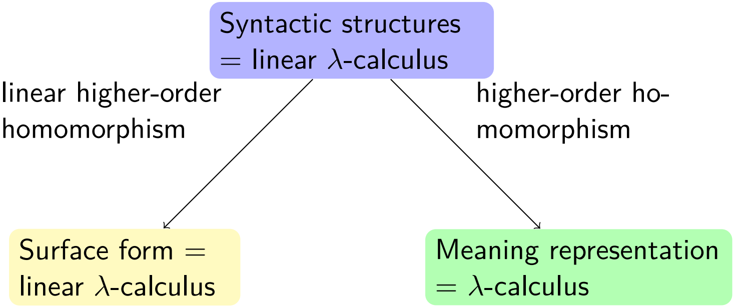

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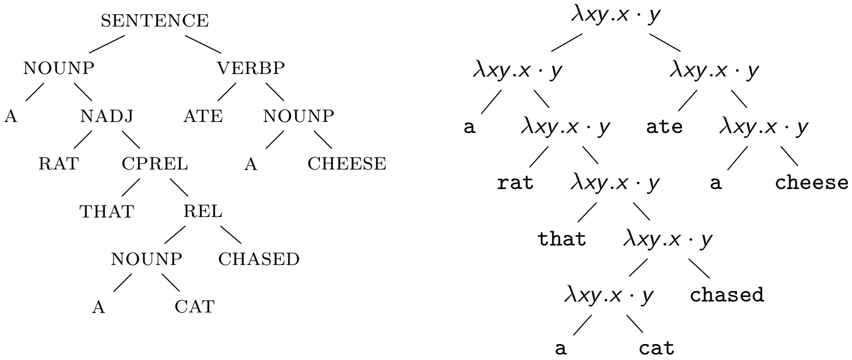
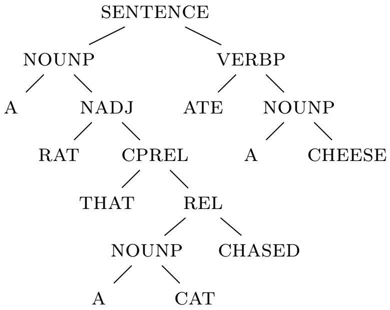
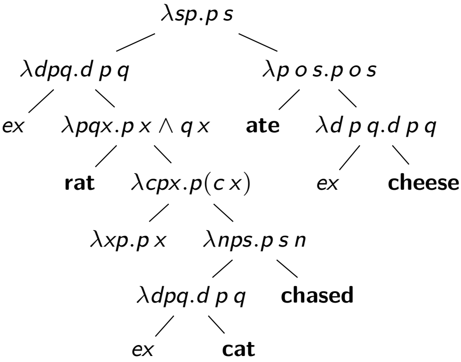
Example: Surface Realization



`a rat that a cat saw ate a cheese`

Example: Montague Semantics

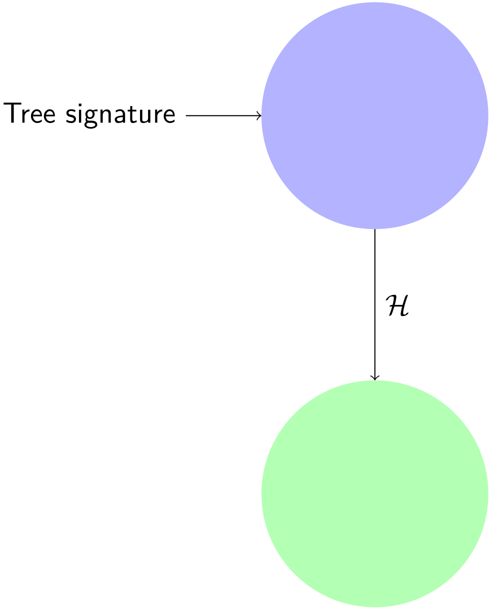




where $ex = \lambda pq.\exists x.p\,x \wedge q\,x$.

$$\exists x.\mathbf{rat}\,x \wedge (\exists y.\mathbf{cat}\,y \wedge \mathbf{chased}\,y\,x \wedge (\exists u.\mathbf{cheese}\,u \wedge \mathbf{ate}\,x\,u))$$

Decidability of parsing and text generation



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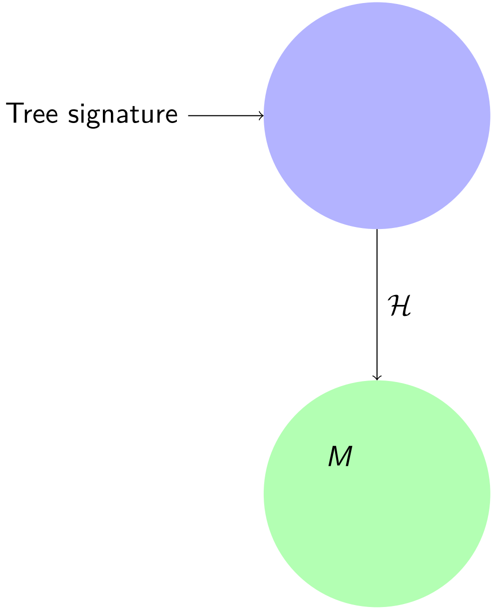

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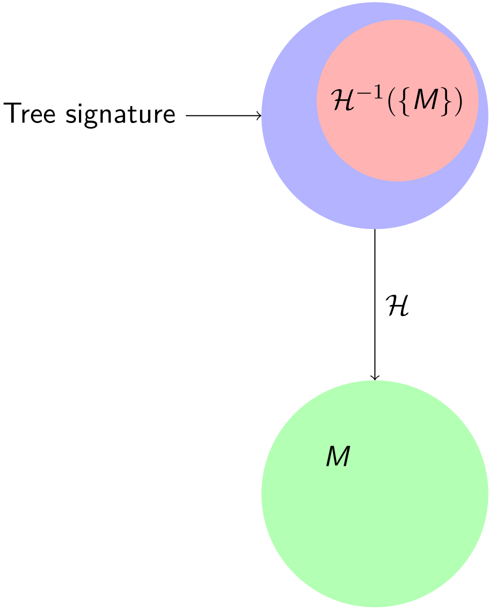

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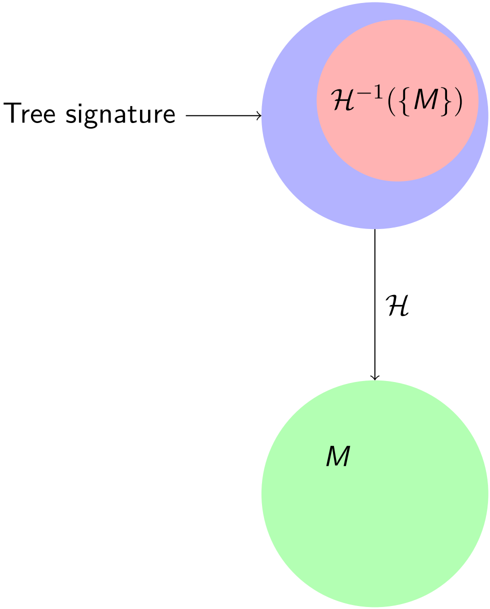

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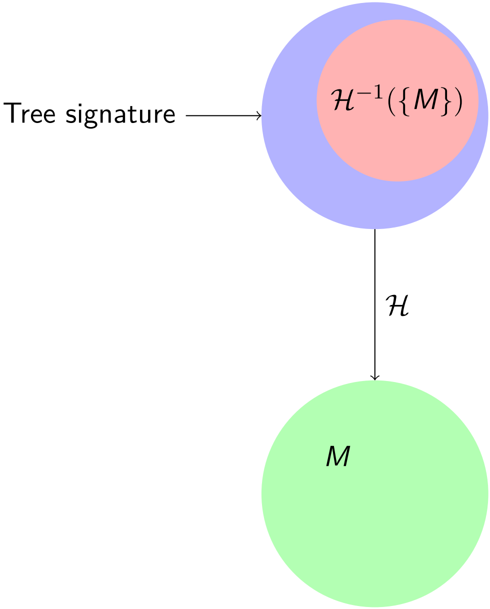

- The complexity is non-elementary (but polynomial in the linear case)
- A semantic argument extends the decidability result to higher-order OI grammars *[Kobele, S. 15]*

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With intentional models of λY-calculus, part of the properties can be used in parsing can be internalized in the semantics.

Infiniteness: Program Verification

Schematology

Programs

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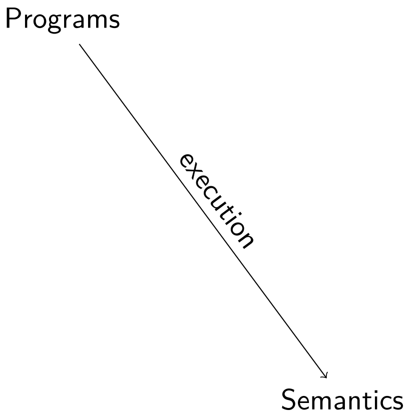

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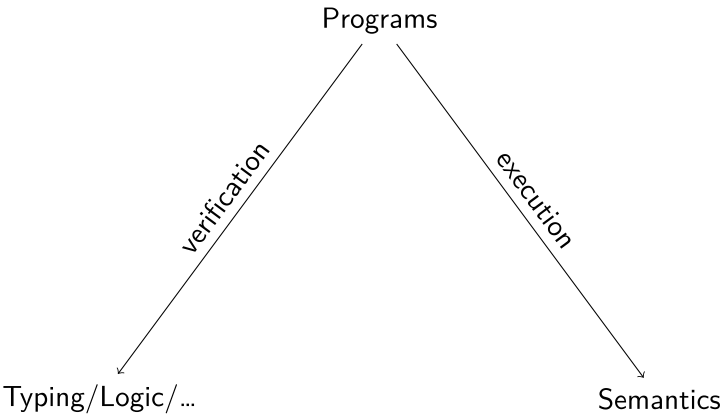

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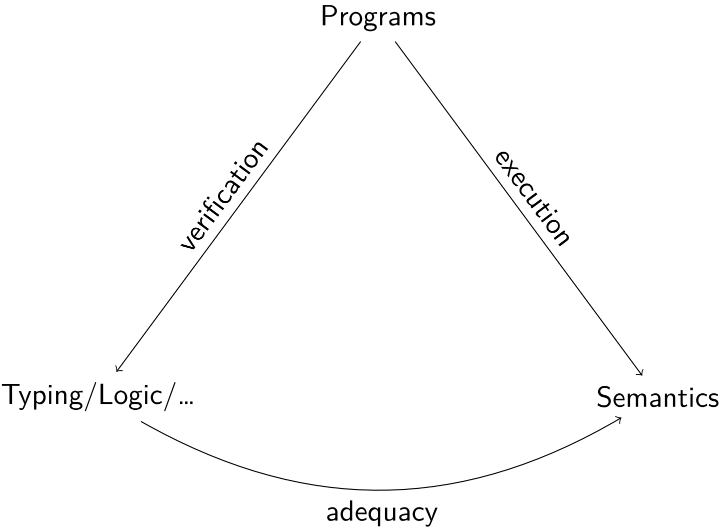

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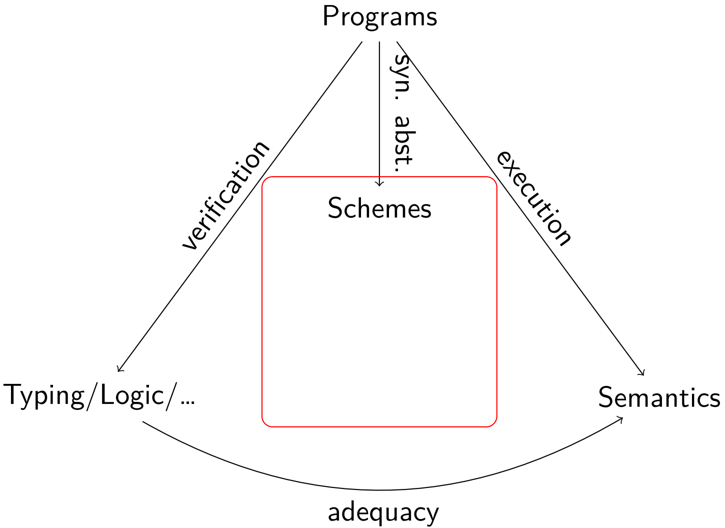

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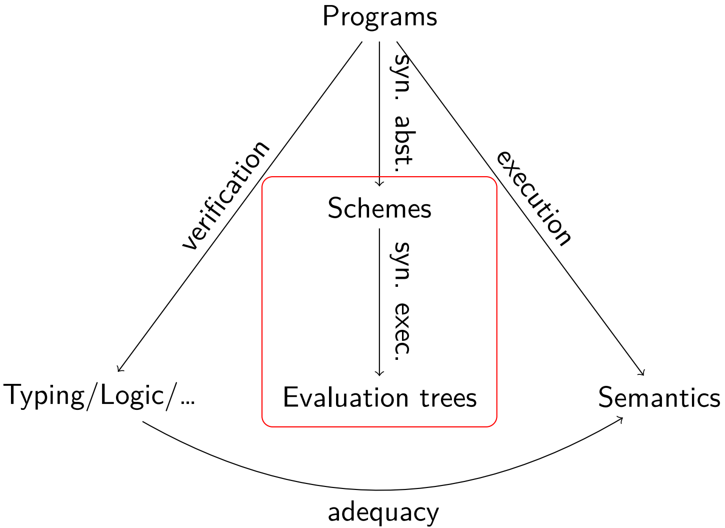

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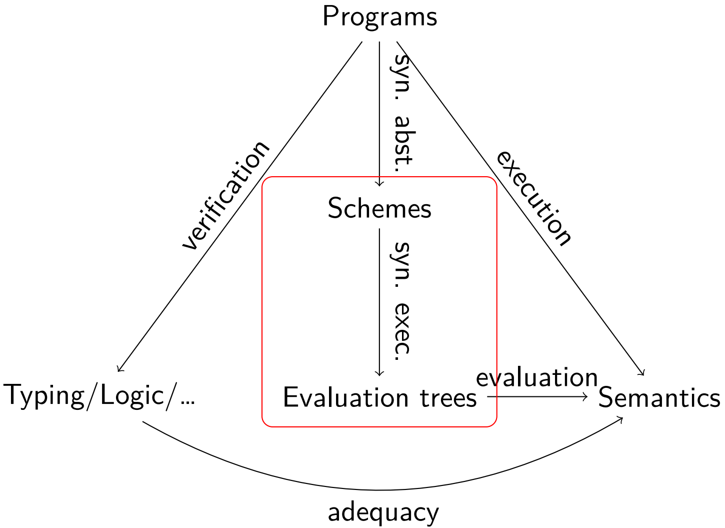

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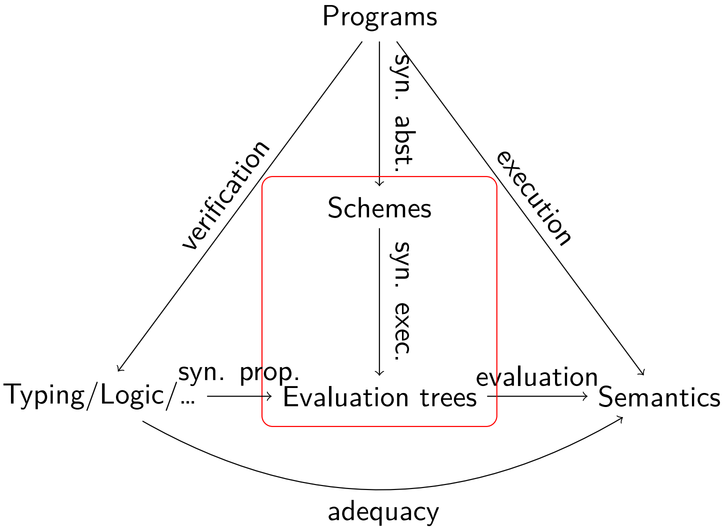

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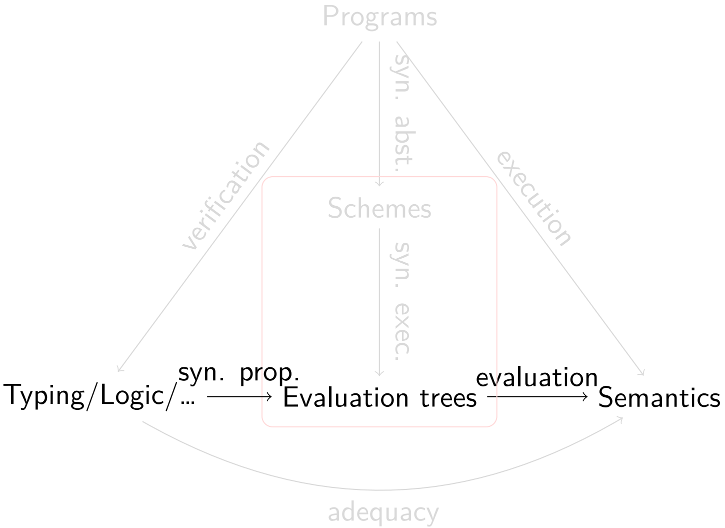

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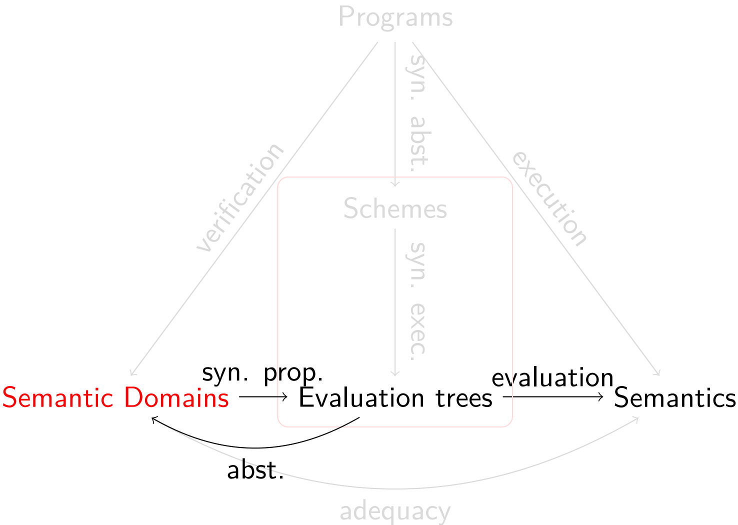
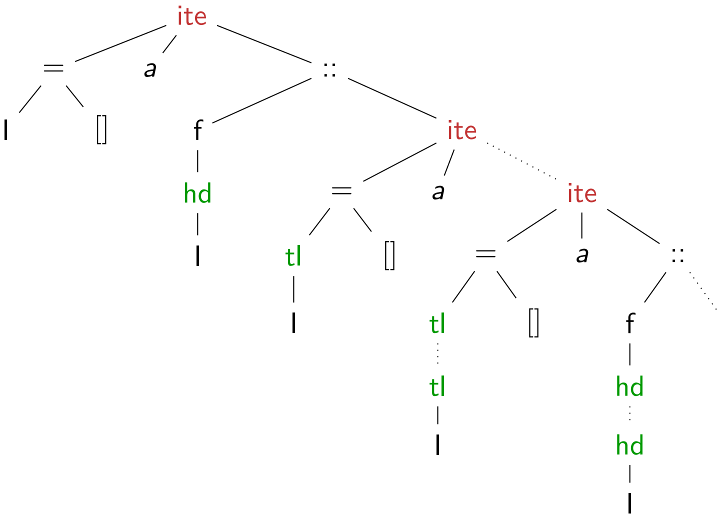


Higher-order control flow

$$\text{fold f a l} = \text{if l=[] then a else f (hd l) (fold f a (tl l))}$$

$$M = Y\lambda\text{fold f a l.ite } (=\text{l } []) \text{ a } (\text{f } (\text{hd l}) (\text{fold f a } (\text{tl l})))$$



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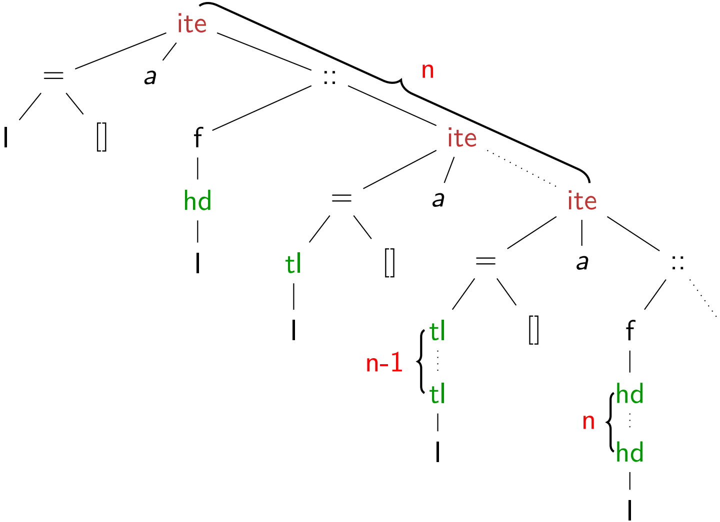

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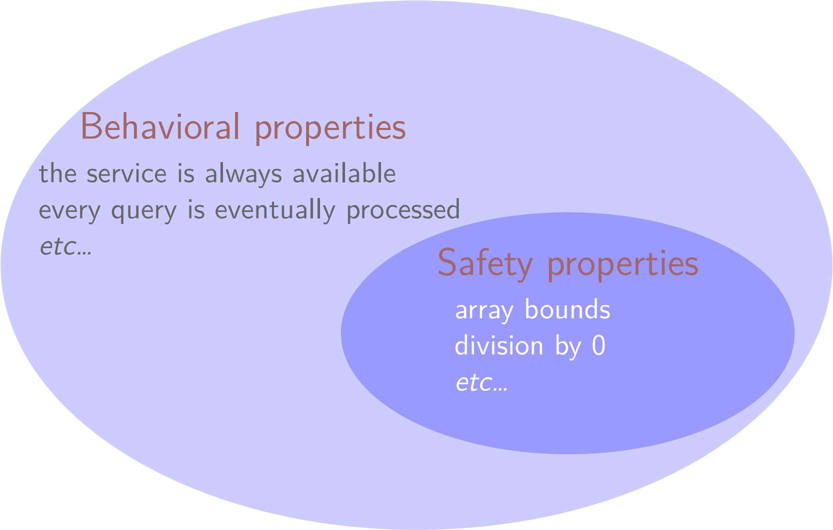

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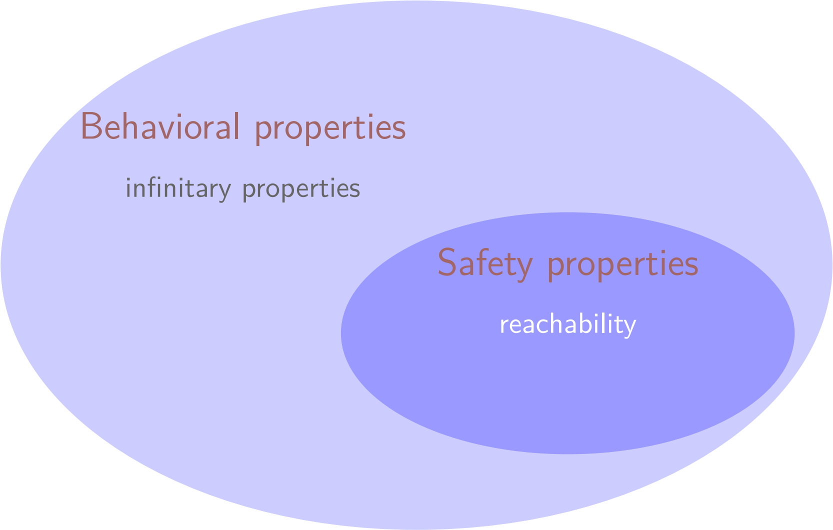
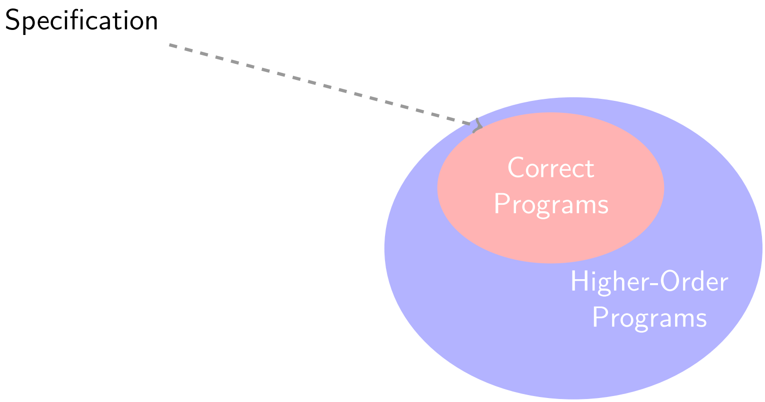


Finite abstractions

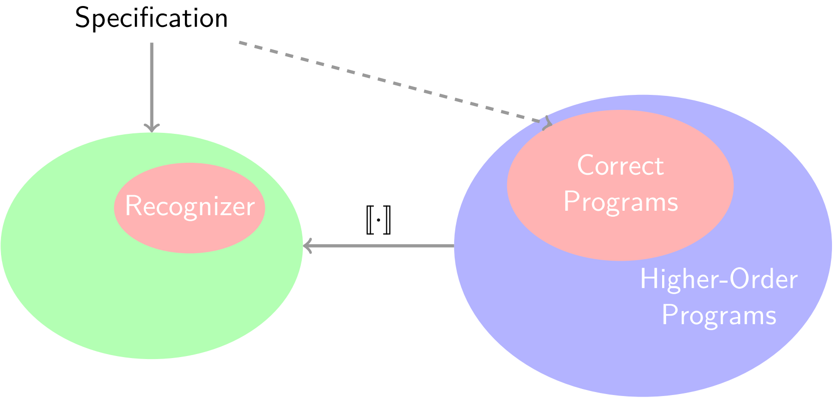
Reachability ⟶ Finite state automata

Infinitary properties ⟶ Parity automata
Monadic Second
Order Logic

Programs and recognizability



Programs and recognizability



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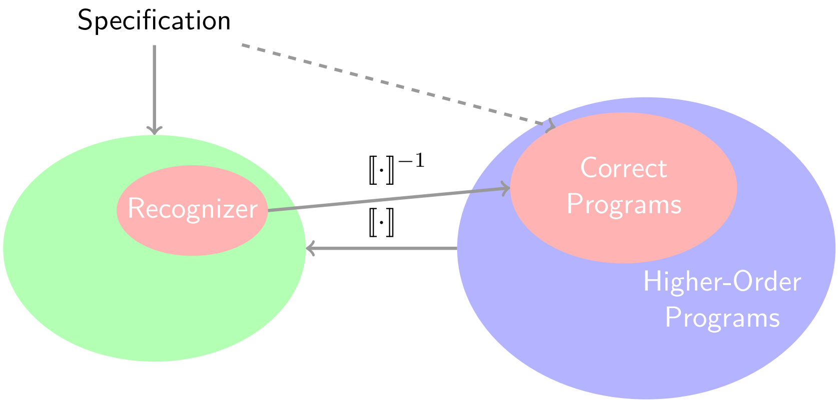

Motivations and results

- Relating finite state methods with denotational methods
- Reveal the invariants behind behavioral properties
- Obtain decidability results by finiteness properties

Some results:

- New proof of Ong's Theorem with Krivine Machine and semantics [Walukiewicz S. 11]
- Limitation of Scott models [Walukiewicz S. 13]
- Transfer Theorem for term evaluation [Walukiewicz S. 13]
- Finite models for weak MSOL based on wreath products of models [Walukiewicz S. 15]
- Finite models for MSOL [Walukiewicz S. 15]

Example: unfolding

$$\text{Graph} \xrightarrow{\textit{unfold}} \text{Tree}$$

MSOL-compatibility of unfolding

For all Σ.
For all φ there is $\hat{\varphi}$ s.t. for all $G \in Graph(\Sigma)$:

$$G \models \hat{\varphi} \text{ iff } \mathit{Unf}(G) \models \varphi$$

Remark: this theorem implies Rabin's Theorem.

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Other example: Muchnik iteration.

$$M \xrightarrow{eval} BT(M)$$

Transfer Theorem

For all Σ, $\mathcal{T}$, $\mathcal{X}$.
For all φ there is $\hat{\varphi}$ s.t. for all $M \in Terms(\Sigma, \mathcal{T}, \mathcal{X})$:

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Note: no limitation on Y-variables.

Transducers

Deforestation

```
q = sum(filter p (map f l))
```

```
def query(l):
    res = 0
    for e in l:
        if p(f e):
            res += f e
    return res

q = query(l)
```

Higher-order transducers

```
sum :: [a] -> Int
sum [] = 0
sum (n:l) = n + sum l

filter :: (a -> Bool) -> [a] -> [a]
filter p [] = []
filter p (a:as) | p a       = a : filter p as
                | otherwise = filter p as

map :: (a -> b) -> [a] -> [b]
map _ [] = []
map f (a:as) = f a : map f as
```

The composed transducer

```
q = sum(filter p (map f l))
query [] = 0
query (a:as) | p(f a)     = f a + query as
             | otherwise = query as
```

Closure of higher-order transducers under composition

- Higher-order transducers are closed under composition, provided they can inspect the input with regular properties: this implements *deforestation* in a very general setting.
- Finite models and recognizability give direct constructions of the compositions of transducers (Gallot, Lemay, S.).
- Several properties need to be investigated:
  - How complex are the regular properties to be checked on input?
  - When is the size of the composition reasonable?

We have Encouraging results for the transducers equivalent to MSOT (Gallot, Lemay, S. 20)

Perspectives

Evaluation of terms in finite models

Parsing HO Grammars
and HO verification

evaluation of higher-order
programs in finite models

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Parsing in the almost affine case

fixpoint computation strategy via datalog program transformation

Evaluation of terms in finite models

Parsing HO Grammars and HO verification	Parsing in the almost affine case
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Abstract Interpretation - abstraction refinements - fixpoint acceleration techniques	

Evaluation of terms in finite models

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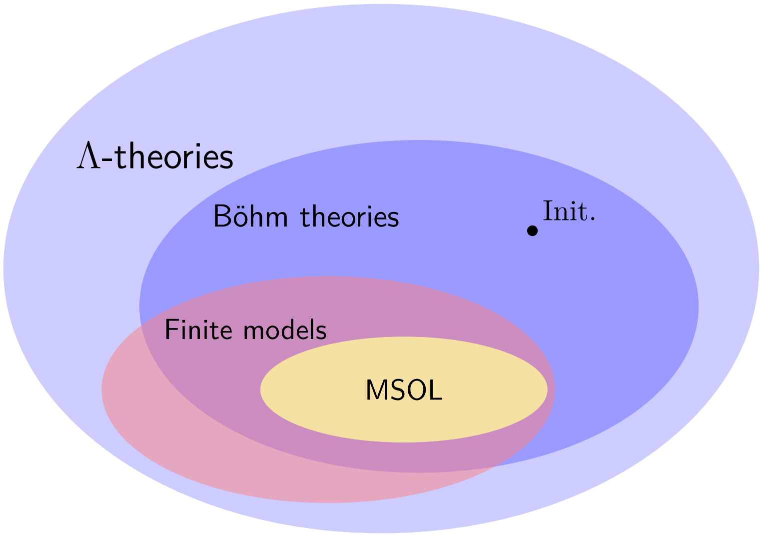
Abstract Interpretation

- abstraction refinements
- fixpoint acceleration techniques

Structure of models

- sequential algorithms, etc...
- linear logic

Theory of Böhm trees



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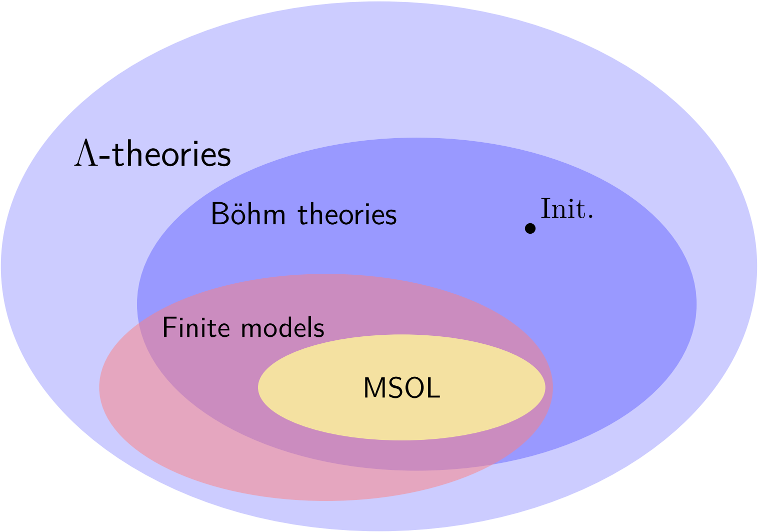

- expressiveness of finite Böhm models?

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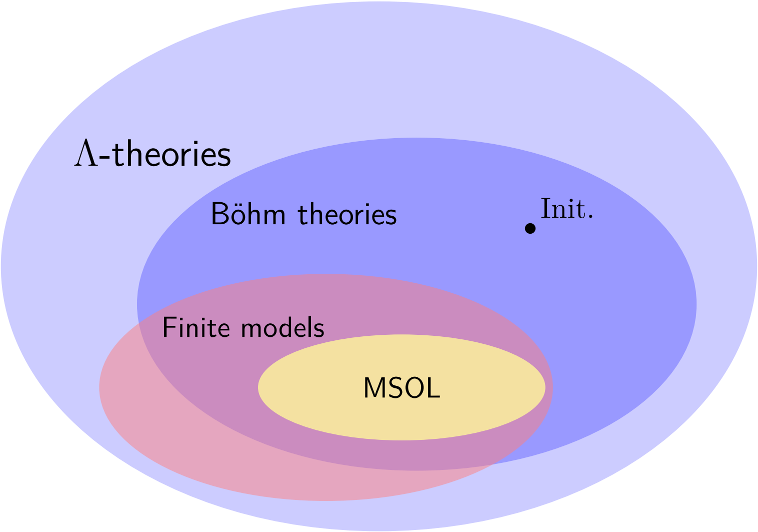

- expressiveness of finite Böhm models?
- axiomatization of finite Böhm models?