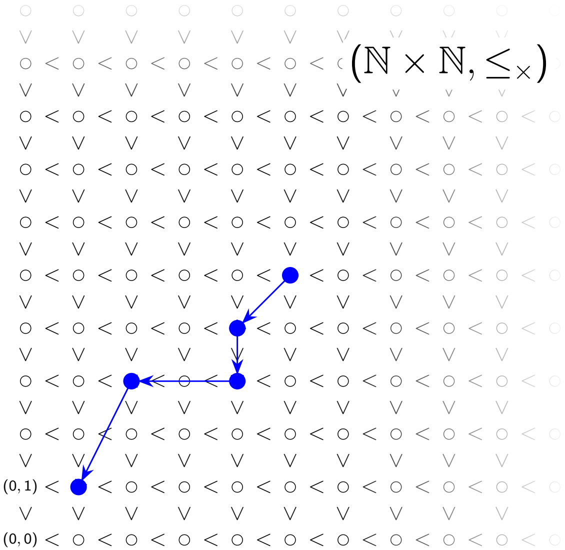
Measuring well quasi-orders and complexity of verification

(Slides from my PhD defense)

Isa Vialard

November 20, 2024

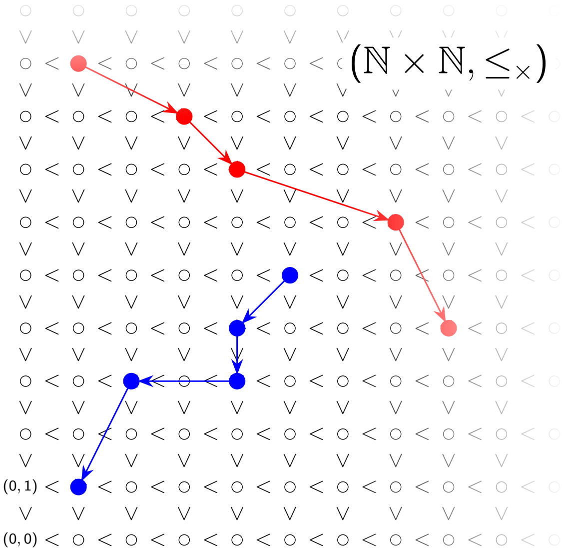
Some interesting sequences



— decreasing sequence

$(5,5) > (4,4) > (4,3)$
$> (2,3) > (1,1)$

Some interesting sequences



— decreasing sequence

— incomparable sequence
(or antichain)
i.e. pairwise incomparable

$(1,9) \perp (3,8), (4,7), (7,5), \ldots$

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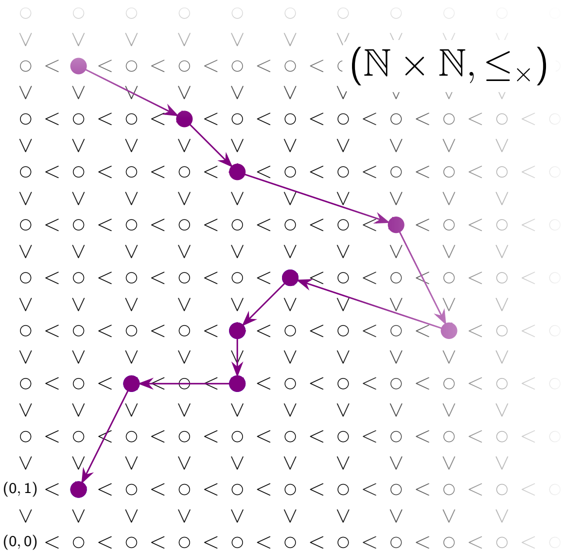


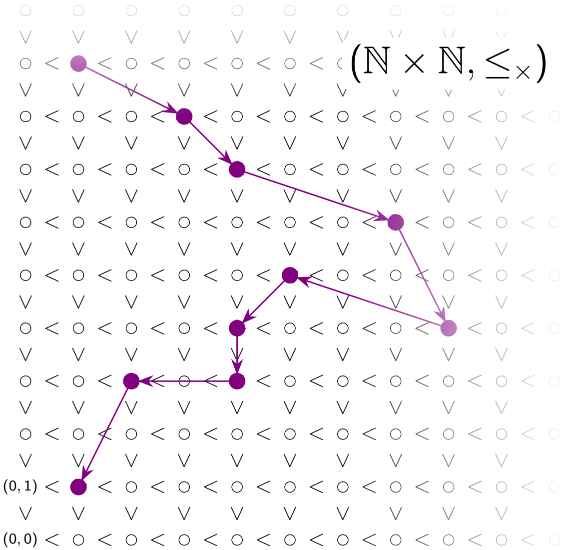
— decreasing sequence

— incomparable sequence (or antichain) i.e. pairwise incomparable

— bad sequence i.e. pairwise non increasing

$(1, 9) \not\leq (3, 8), (4, 7), (7, 5), \ldots$

Definitions: Well Quasi-Order



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WQO

$\Updownarrow$

No infinite antichain
or decreasing seq

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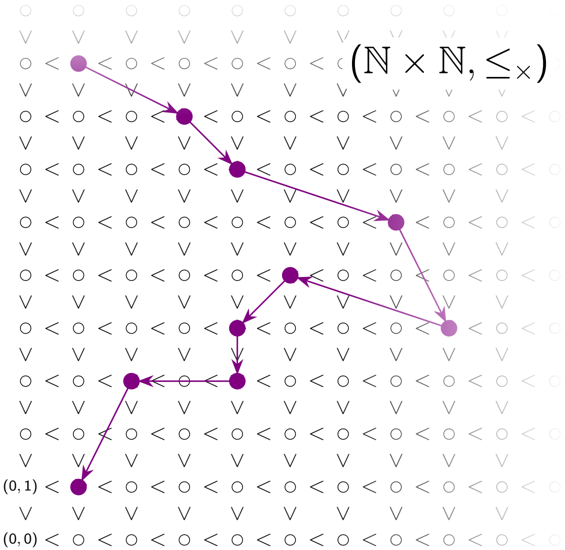

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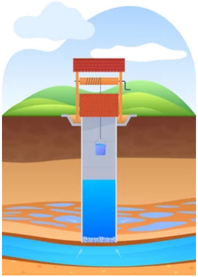

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Blass & Gurevich (2008)

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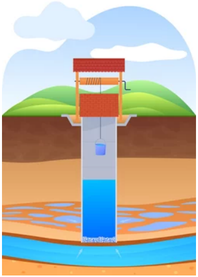

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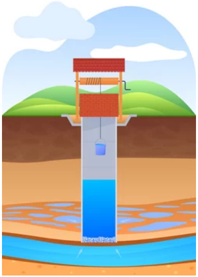

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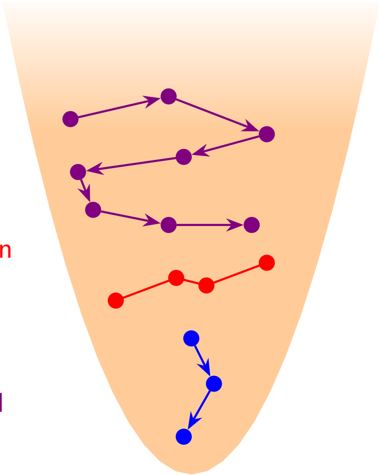

Motivations

♦ Reasons to study wqos

- "It is fun" (Kříž & Thomas (1990))

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♣ Well-structured transition systems

Finkel (1994), Abdulla& Jonsson (1996)

- Set of configurations: WQO
- $\leq$ a simulation relation

$$
\begin{array}{ccc}
t_1 & \longrightarrow & \color{red}{t_2} \\
\rotatebox{90}{$\leq$} & & \color{red}{\rotatebox{90}{$\leq$}} \\
s_1 & \longrightarrow & s_2
\end{array}
$$

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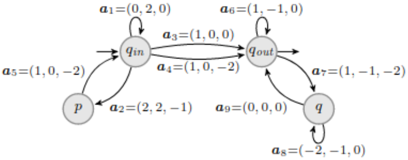
- *Ex: Counter machines, Petri nets, VASS, Lossy channel systems . . .*



Vector Addition Systems with States

♣ Well-structured transition systems

Finkel (1994), Abdulla& Jonsson (1996)

- Set of configurations: WQO
- $\leq$ is *upward-compatible*

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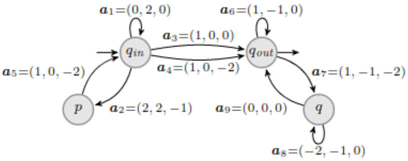


- *Ex: Counter machines, Petri nets, VASS, Lossy channel systems . . .*

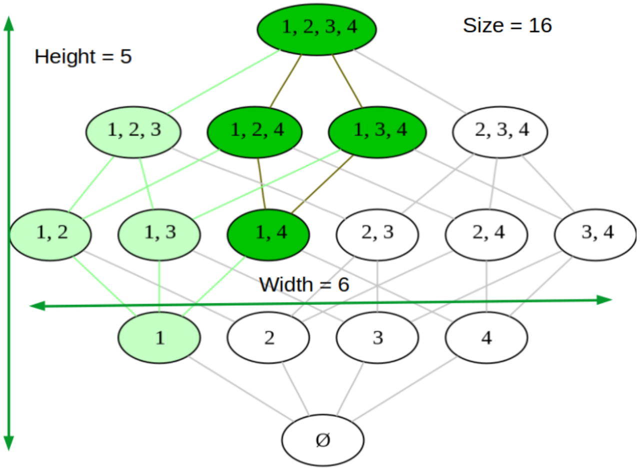
Vector Addition Systems with States

♦ Complexity and expressiveness

Schmitz& Schnoebelen(2011)

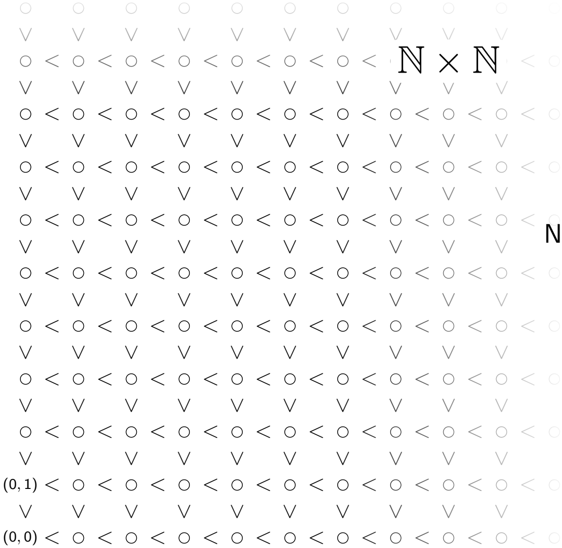
- Controlled bad sequences (even decreasing, or antichains)
- Can we bound the length of controlled sequences by measuring wqo?

♣ Natural notions of measure when finite



Finite subsets of $\{1,2,3,4\}$ ordered by $\subseteq$.

♦ **Let's extend height and width to infinite wqos**

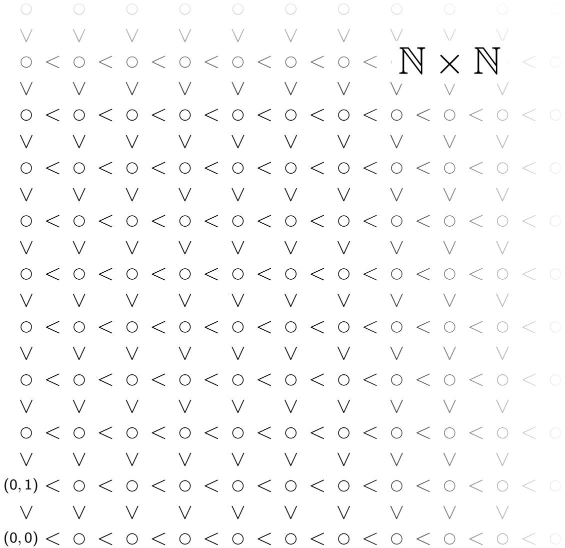


Problem:

No largest decreasing sequence

No largest antichain

♣ Let's extend height and width to infinite wqos with ordinals



Width and height:

at least ω

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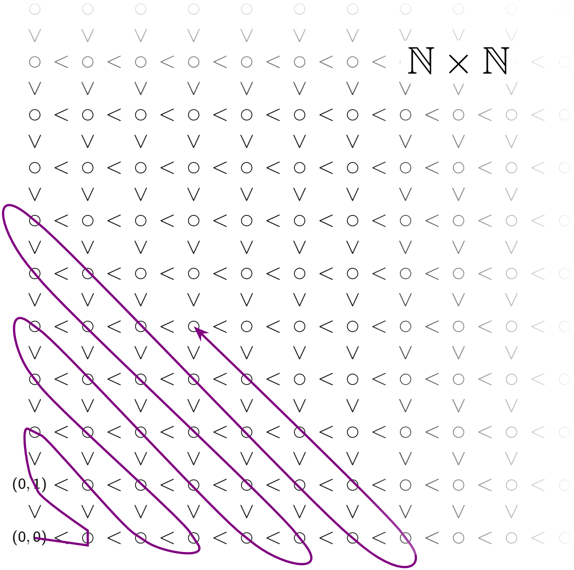

Width and height:

at least ω

Counting elements:

at least ω

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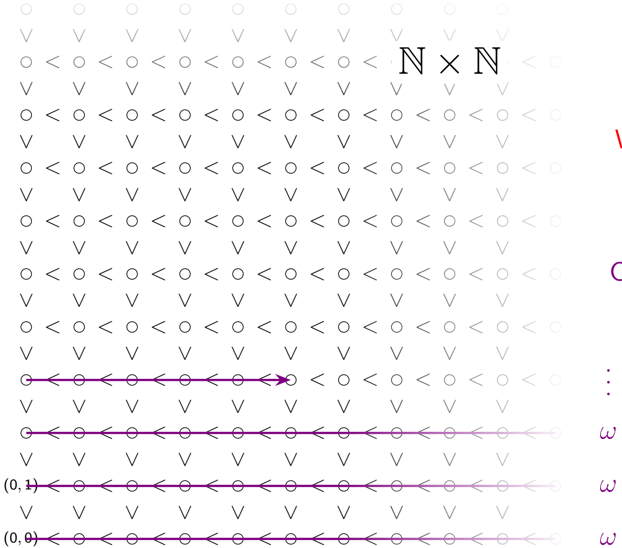


Width and height:

at least ω

Counting elements:

at least ω^2

Definition (Maximal order type, Width and Height)

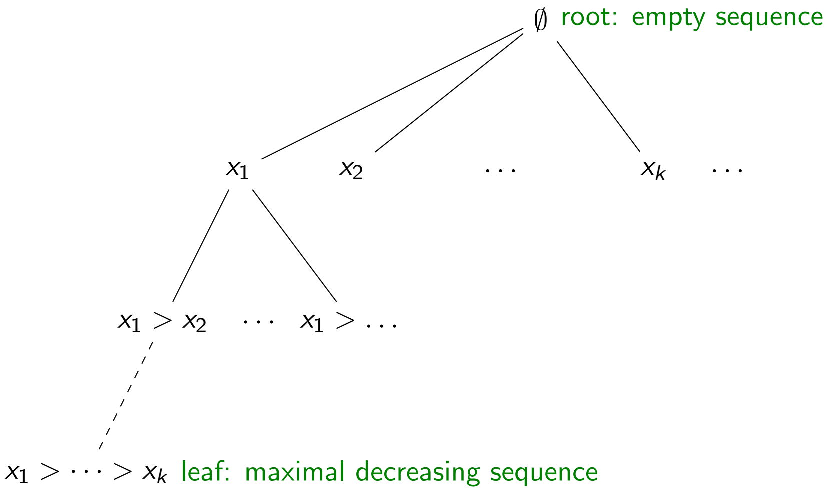
$$\left|\begin{array}{l} \mathrm{o}(X) \\ \mathrm{w}(X) \\ \mathrm{h}(X) \end{array}\right. = \text{ordinal rank of the tree of} \begin{cases} \text{bad sequences} \\ \text{antichain sequences} \\ \text{decreasing sequences} \end{cases} \text{in } X.$$

Definition from Kříž & Thomas(1990) (first definition of ordinal width)

First definition of maximal order type by De Jongh & Parikh(1977)

Definition: Rank of well-founded trees

♣ Ex: Tree of decreasing sequences



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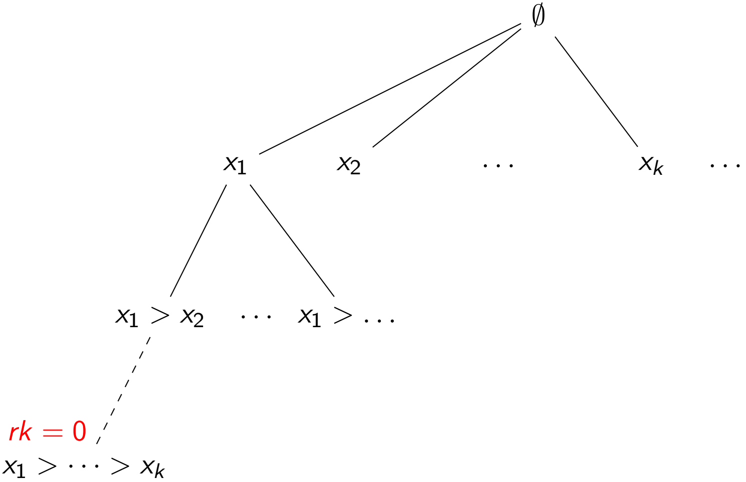

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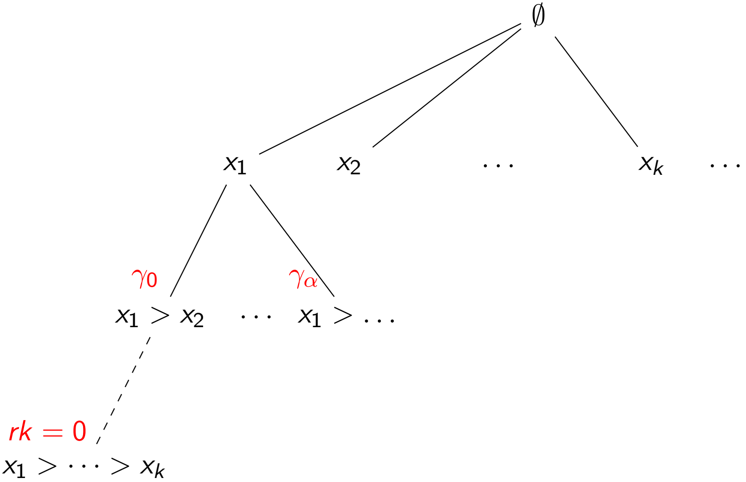

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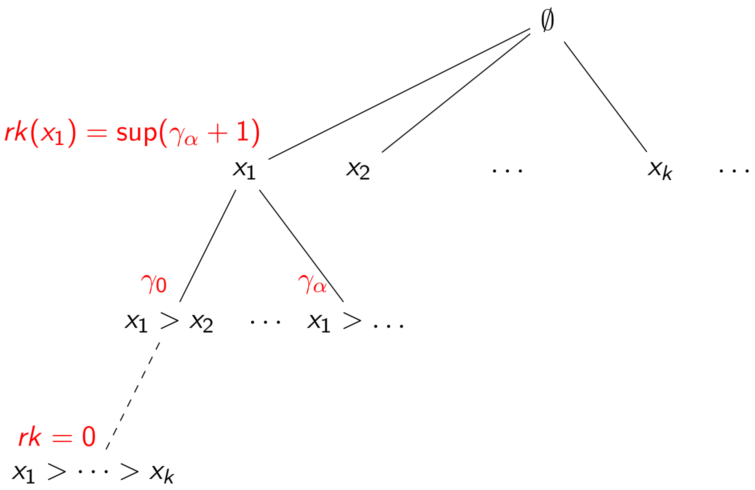

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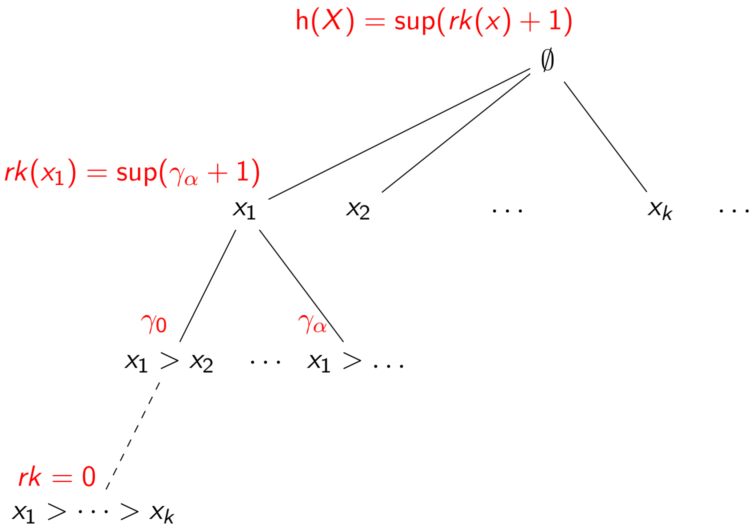
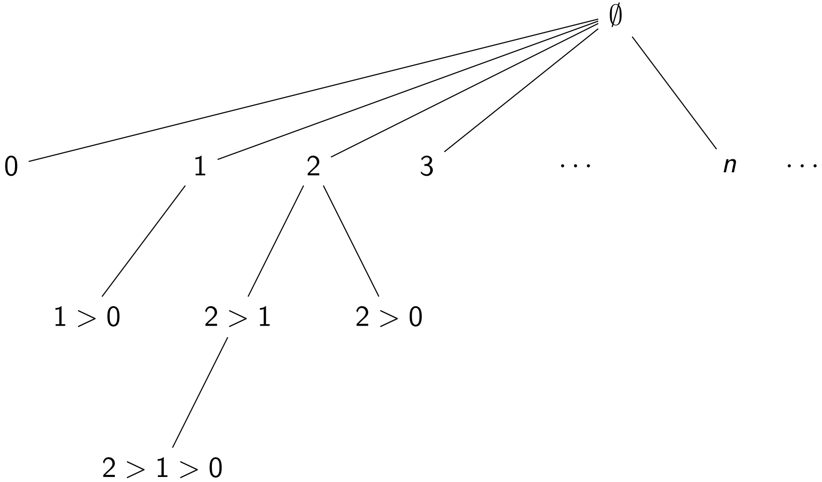


Definition: Rank of well-founded trees

♣ Ex: Height of $\mathbb{N}$



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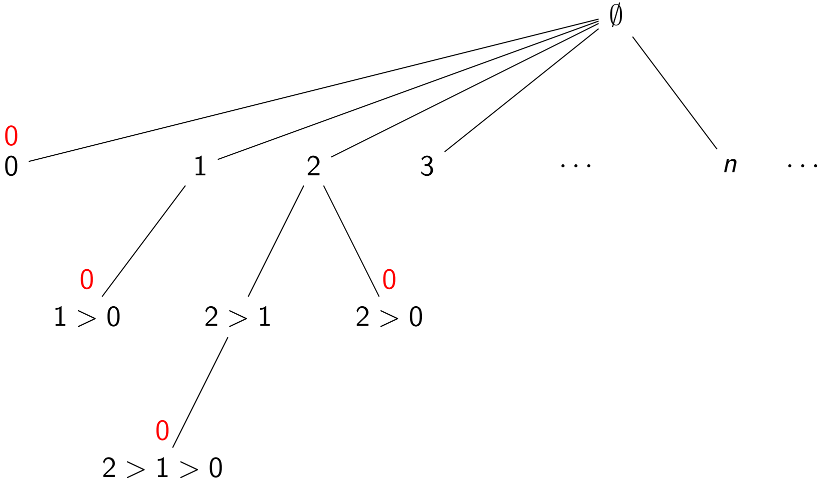

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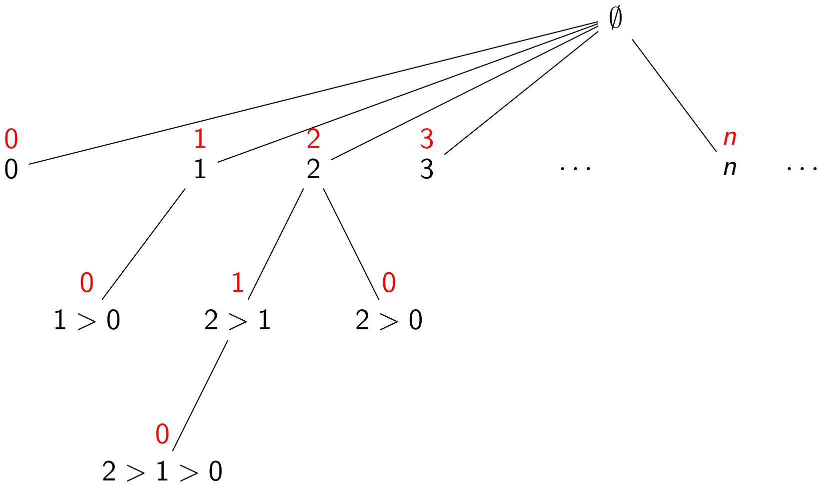

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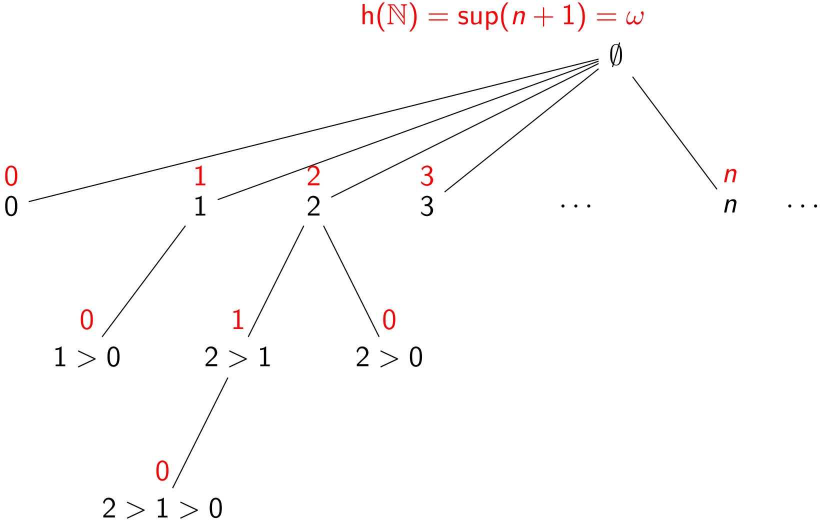
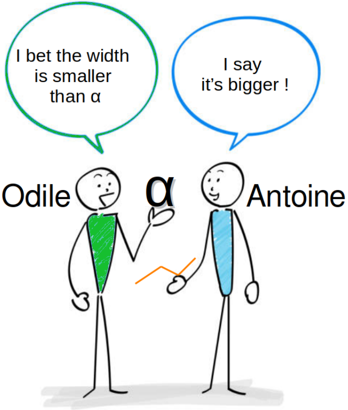


Measuring with games



♦ **Game:** α **vs** $\mathrm{w}(X)$

- Initial configuration:
  - Odile : $\gamma = \alpha$,
  - Antoine : $S = \emptyset$
- Player alternate:
  - Odile picks $\gamma' < \gamma$
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- End: You lose if you cannot play anymore

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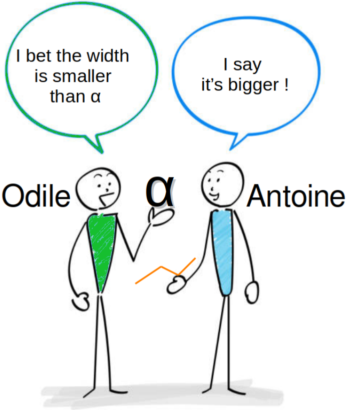

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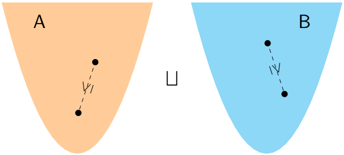
Theorem (Blass & Gurevich (2008))

- Antoine has winning strategy when Odile begins $\Leftrightarrow \alpha \leq \mathrm{w}(X)$
- Odile has winning strategy when Antoine begins $\Leftrightarrow \alpha \geq \mathrm{w}(X)$

Example: Playing on the on disjoint sum



Disjoint sum $A \sqcup B$

Theorem: $o(A \sqcup B) = o(A) \oplus o(B)$ **(De Jongh & Parikh(1977))**

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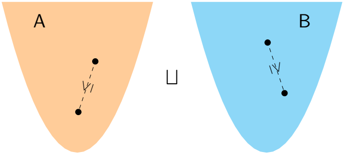

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Ex: $(\omega^\omega + \omega^3) \oplus (\omega^5 + \omega + 1) = \omega^\omega + \omega^5 + \omega^3 + \omega + 1$

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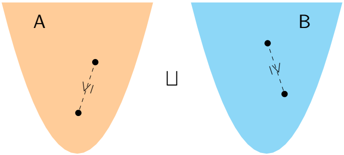

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This theorem is easy to prove with games!

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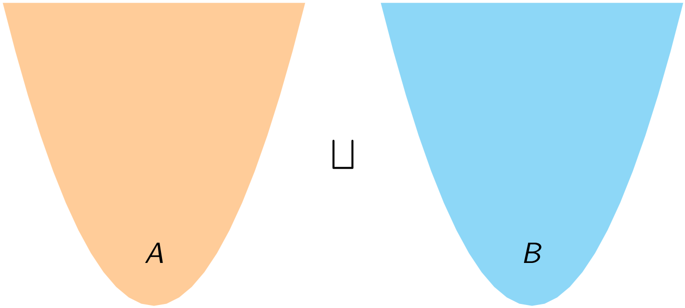
Odile Antoine

$o(A) \oplus o(B)$



$o(A \sqcup B) \leq o(A) \oplus o(B)$ if Odile wins when Antoine begins

$o(A \sqcup B) \geq o(A) \oplus o(B)$ if Antoine wins when Odile begins

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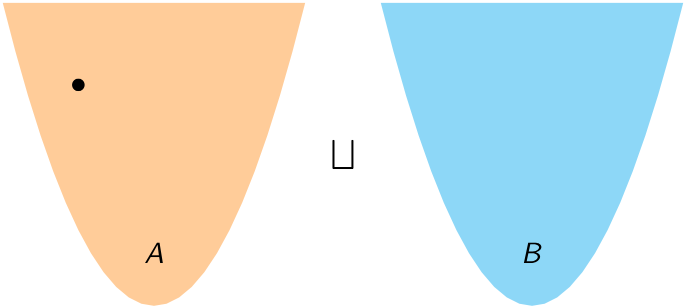

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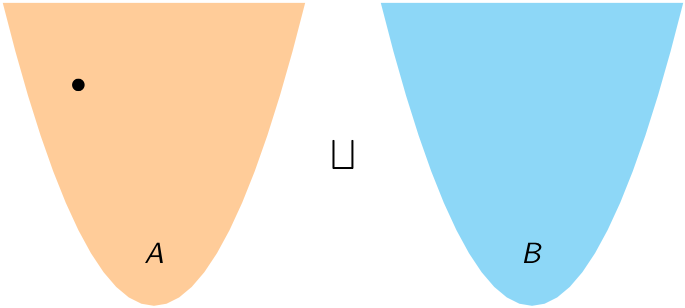

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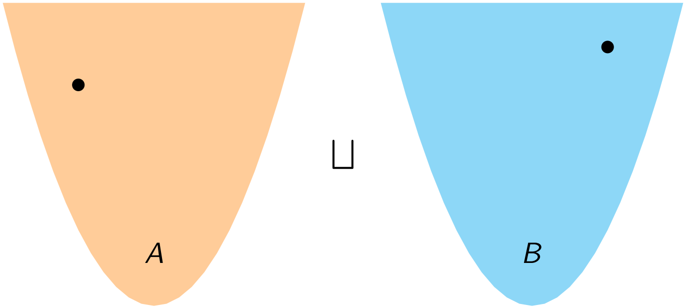

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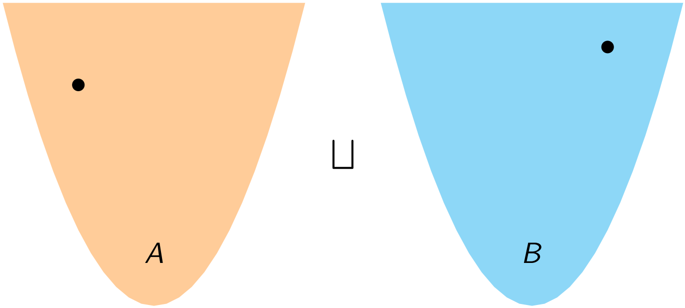

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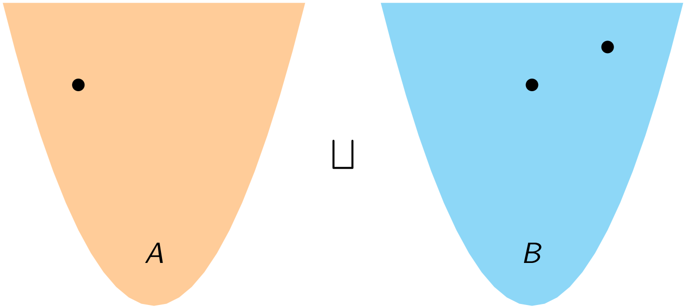

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Example: Playing on the disjoint sum

Odile | Antoine

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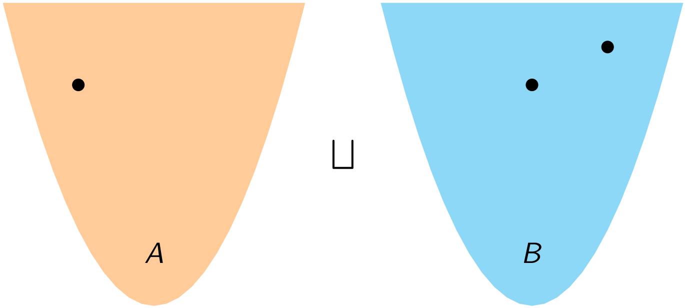
$\alpha_1 \oplus \beta_2$



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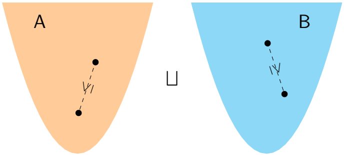
Other classical operations on WQOs



Disjoint sum $A \sqcup B$

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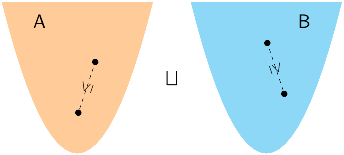

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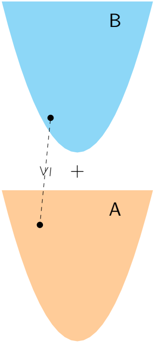

Direct sum $A + B$

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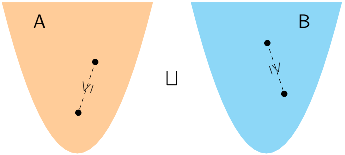

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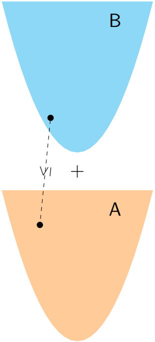

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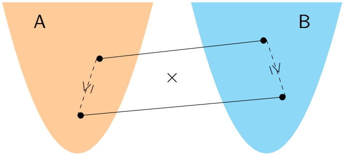
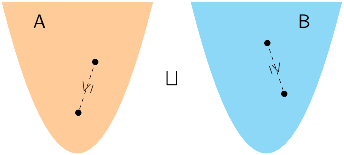


Cartesian product $A \times B$

Other classical operations on WQOs



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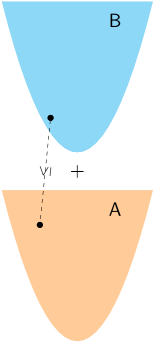

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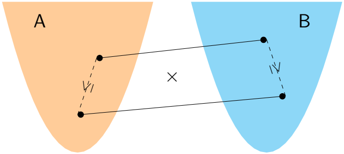

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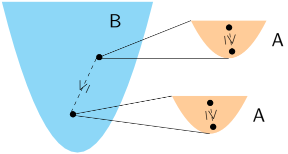


Lexicographic product $A \cdot B$

. . . And their ordinal invariants

	Space	M.O.T.	Height	Width
Disjoint sum	$A \sqcup B$	$o(A) \oplus o(B)$	$\max(h(A), h(B))$	$w(A) \oplus w(B)$
Direct sum	$A + B$	$o(A) + o(B)$	$h(A) + h(B)$	$\max(w(A), w(B))$
Cartesian prod.	$A \times B$	$o(A) \otimes o(B)$	$h(A) \hat{\oplus} h(B)$	**?**
Direct prod.	$A \cdot B$	**?**	$h(A) \cdot h(B)$	$w(A) \odot w(B)$

...And their ordinal invariants

	Space	M.O.T.	Height	Width
Disjoint sum	$A \sqcup B$	$o(A) \oplus o(B)$	$\max(h(A), h(B))$	$w(A) \oplus w(B)$
Direct sum	$A + B$	$o(A) + o(B)$	$h(A) + h(B)$	$\max(w(A), w(B))$
Cartesian prod.	$A \times B$	$o(A) \otimes o(B)$	$h(A) \hat{\oplus} h(B)$	**?**
Direct prod.	$A \cdot B$	**?**	$h(A) \cdot h(B)$	$w(A) \odot w(B)$
Fin. words	A^*	$\omega^{\omega^{(o(A)^{\pm})}}$	$h^*(A)$	$\omega^{\omega^{(o(A)^{\pm})}}$
Fin. multisets	$M^\diamond(A)$	$\omega^{\widehat{o(A)}}$	$h^*(A)$	**?**
	$M^\circ(A)$	$\omega^{o(A)}$	**?**	**?**
Fin. Powerset	$P_f(A)$	**?**	**?**	**?**

Credits to: De Jongh & Parikh(1977), Schmidt(1979), Abraham & Bonnet(1999), Van der Meeren, Rathjen & Weiermann(2009,2015), Džamonja, Schmitz & Schnoebelen(2020)

My contributions

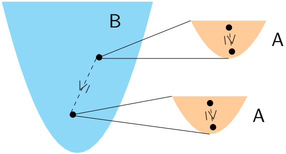
Space	M.O.T.	Height	Width
$A \sqcup B$	$\mathrm{o}(A) \oplus \mathrm{o}(B)$	$\max(\mathrm{h}(A), \mathrm{h}(B))$	$\mathrm{w}(A) \oplus \mathrm{w}(B)$
$A + B$	$\mathrm{o}(A) + \mathrm{o}(B)$	$\mathrm{h}(A) + \mathrm{h}(B)$	$\max(\mathrm{w}(A), \mathrm{w}(B))$
$A \times B$	$\mathrm{o}(A) \otimes \mathrm{o}(B)$	$\mathrm{h}(A) \hat{\oplus} \mathrm{h}(B)$	$\geq \mathrm{w}(\mathrm{o}(A) \times \mathrm{o}(B))$
$A \cdot B$	$\mathrm{o}(A) \cdot pred_k(\mathrm{o}(B)) + \mathrm{o}(A) \otimes k$	$\mathrm{h}(A) \cdot \mathrm{h}(B)$	$\mathrm{w}(A) \odot \mathrm{w}(B)$
A^*	$\omega^{\omega^{(\mathrm{o}(A)^{\pm})}}$	$\mathrm{h}^*(A)$	$\omega^{\omega^{(\mathrm{o}(A)^{\pm})}}$
$\mathrm{M}^\diamond(A)$	$\omega^{\widehat{\mathrm{o}(A)}}$	$\mathrm{h}^*(A)$	$\omega^{\widehat{\mathrm{o}(A)}-1}$
$\mathrm{M}^\circ(A)$	$\omega^{\mathrm{o}(A)}$	$\omega^{\mathrm{h}(A)}$	$\omega^{\mathrm{o}_\perp(A)}$
$\mathrm{P}_\mathrm{f}(A)$	$\leq 2^{\mathrm{o}(A)}$	$\leq 2^{\mathrm{h}(A)}$	$\geq 2^{\mathrm{w}(A)}$

Back in time

Space	M.O.T.	Height	Width
$A \sqcup B$	$o(A) \oplus o(B)$	$\max(h(A), h(B))$	$w(A) \oplus w(B)$
$A + B$	$o(A) + o(B)$	$h(A) + h(B)$	$\max(w(A), w(B))$
$A \times B$	$o(A) \otimes o(B)$	$h(A) \hat{\oplus} h(B)$	**?**
$A \cdot B$	**?**	$h(A) \cdot h(B)$	$w(A) \odot w(B)$
A^*	$\omega^{\omega^{(o(A)^{\pm})}}$	$h^*(A)$	$\omega^{\omega^{(o(A)^{\pm})}}$
$M^\diamond(A)$	$\omega^{\widehat{o(A)}}$	$h^*(A)$	**?**
$M^\circ(A)$	$\omega^{o(A)}$	**?**	**?**
$P_f(A)$	**?**	**?**	**?**

Credits to: De Jongh & Parikh(1977), Schmidt(1979), Abraham & Bonnet(1999), Van der Meeren, Rathjen & Weiermann(2009,2015), Džamonja, Schmitz & Schnoebelen(2020)

Quick look at the direct product

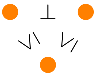


Lexicographic product $A \cdot B$

♦ **I was told that** $o(A \cdot B) = o(A) \cdot o(B)$

. . . but only the lower bound is true: $o(A \cdot B) \geq o(A) \cdot o(B)$ Mistake noticed by Harry Altman (March, 2024)

Quick look at the direct product

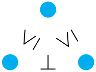


∇

$$o = 3$$

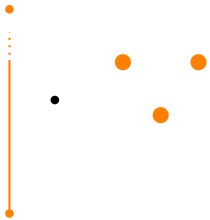
$$h = 2$$

$$w = 2$$

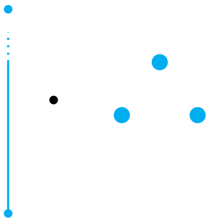


Δ

Quick look at the direct product

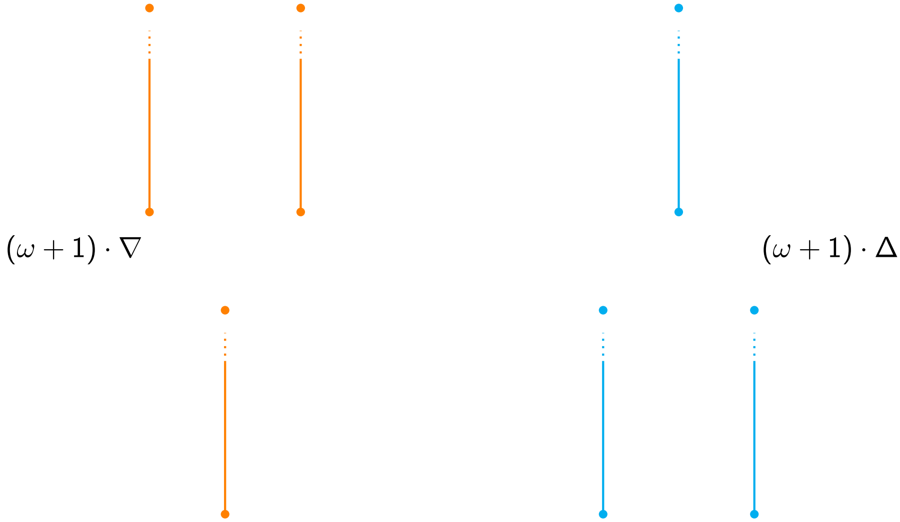


$(\omega + 1) \cdot \nabla$



$(\omega + 1) \cdot \Delta$

Quick look at the direct product



Quick look at the direct product

$$\mathsf{o}((\omega+1)\cdot\nabla) = \begin{array}{c}[(\omega+1) \quad \oplus \quad (\omega+1)] \\ + \\ (\omega+1)\end{array} = \omega\cdot 3+2$$

$$\mathsf{o}((\omega+1)\cdot\Delta) = \begin{array}{c}(\omega+1) \\ + \\ [(\omega+1) \quad \oplus \quad (\omega+1)]\end{array} = \omega\cdot 3+1 = \mathsf{o}(\omega+1)\cdot\mathsf{o}(\nabla)$$

What about the other operations?

Space	M.O.T.	Height	Width
$A \sqcup B$	$\mathrm{o}(A) \oplus \mathrm{o}(B)$	$\max(\mathrm{h}(A), \mathrm{h}(B))$	$\mathrm{w}(A) \oplus \mathrm{w}(B)$
$A + B$	$\mathrm{o}(A) + \mathrm{o}(B)$	$\mathrm{h}(A) + \mathrm{h}(B)$	$\max(\mathrm{w}(A), \mathrm{w}(B))$
$A \times B$	$\mathrm{o}(A) \otimes \mathrm{o}(B)$	$\mathrm{h}(A) \hat{\oplus} \mathrm{h}(B)$	**?**
$A \cdot B$	*Not functional*	$\mathrm{h}(A) \cdot \mathrm{h}(B)$	$\mathrm{w}(A) \odot \mathrm{w}(B)$
A^*	$\omega^{\omega^{(\mathrm{o}(A)^{\pm})}}$	$\mathrm{h}^*(A)$	$\omega^{\omega^{(\mathrm{o}(A)^{\pm})}}$
$\mathrm{M}^\diamond(A)$	$\omega^{\widehat{\mathrm{o}(A)}}$	$\mathrm{h}^*(A)$	**?**
$\mathrm{M}^\circ(A)$	$\omega^{\mathrm{o}(A)}$	**?**	**?**
$\mathrm{P_f}(A)$	**?**	**?**	**?**

Credits to: De Jongh & Parikh(1977), Schmidt(1979), Abraham & Bonnet(1999), Van der Meeren, Rathjen & Weiermann(2009,2015), Džamonja, Schmitz & Schnoebelen(2020)

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$A + B$	$\mathrm{o}(A) + \mathrm{o}(B)$	$\mathrm{h}(A) + \mathrm{h}(B)$	$\max(\mathrm{w}(A), \mathrm{w}(B))$
$A \times B$	$\mathrm{o}(A) \otimes \mathrm{o}(B)$	$\mathrm{h}(A) \hat{\oplus} \mathrm{h}(B)$	*Not functional*
$A \cdot B$	*Not functional*	$\mathrm{h}(A) \cdot \mathrm{h}(B)$	$\mathrm{w}(A) \odot \mathrm{w}(B)$
A^*	$\omega^{\omega^{(\mathrm{o}(A)^{\pm})}}$	$\mathrm{h}^*(A)$	$\omega^{\omega^{(\mathrm{o}(A)^{\pm})}}$
$\mathrm{M}^\diamond(A)$	$\omega^{\widehat{\mathrm{o}(A)}}$	$\mathrm{h}^*(A)$	**?**
$\mathrm{M}^\circ(A)$	$\omega^{\mathrm{o}(A)}$	**?**	**?**
$\mathrm{P_f}(A)$	**?**	**?**	**?**

Credits to: De Jongh & Parikh(1977), Schmidt(1979), Abraham & Bonnet(1999), Van der Meeren, Rathjen & Weiermann(2009,2015), Džamonja, Schmitz & Schnoebelen(2020)

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Space	M.O.T.	Height	Width
$A \sqcup B$	$\mathrm{o}(A) \oplus \mathrm{o}(B)$	$\max(\mathrm{h}(A), \mathrm{h}(B))$	$\mathrm{w}(A) \oplus \mathrm{w}(B)$
$A + B$	$\mathrm{o}(A) + \mathrm{o}(B)$	$\mathrm{h}(A) + \mathrm{h}(B)$	$\max(\mathrm{w}(A), \mathrm{w}(B))$
$A \times B$	$\mathrm{o}(A) \otimes \mathrm{o}(B)$	$\mathrm{h}(A) \hat{\oplus} \mathrm{h}(B)$	*Not functional*
$A \cdot B$	*Not functional*	$\mathrm{h}(A) \cdot \mathrm{h}(B)$	$\mathrm{w}(A) \odot \mathrm{w}(B)$
A^*	$\omega^{\omega^{(\mathrm{o}(A)^{\pm})}}$	$\mathrm{h}^*(A)$	$\omega^{\omega^{(\mathrm{o}(A)^{\pm})}}$
$\mathrm{M}^{\diamond}(A)$	$\omega^{\widehat{\mathrm{o}(A)}}$	$\mathrm{h}^*(A)$	$\omega^{\widehat{\mathrm{o}(A)}-1}$
$\mathrm{M}^{\circ}(A)$	$\omega^{\mathrm{o}(A)}$	**?**	**?**
$\mathrm{P_f}(A)$	**?**	**?**	**?**

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$A \times B$	$\mathrm{o}(A) \otimes \mathrm{o}(B)$	$\mathrm{h}(A) \hat{\oplus} \mathrm{h}(B)$	*Not functional*
$A \cdot B$	*Not functional*	$\mathrm{h}(A) \cdot \mathrm{h}(B)$	$\mathrm{w}(A) \odot \mathrm{w}(B)$
A^*	$\omega^{\omega^{(\mathrm{o}(A)^{\pm})}}$	$\mathrm{h}^*(A)$	$\omega^{\omega^{(\mathrm{o}(A)^{\pm})}}$
$\mathrm{M}^\diamond(A)$	$\omega^{\widehat{\mathrm{o}(A)}}$	$\mathrm{h}^*(A)$	$\omega^{\widehat{\mathrm{o}(A)}-1}$
$\mathrm{M}^\circ(A)$	$\omega^{\mathrm{o}(A)}$	$\omega^{\mathrm{h}(A)}$	**?**
$\mathrm{P_f}(A)$	**?**	**?**	**?**

Credits to: De Jongh & Parikh(1977), Schmidt(1979), Abraham & Bonnet(1999), Van der Meeren, Rathjen & Weiermann(2009,2015), Džamonja, Schmitz & Schnoebelen(2020)

What about the other operations?

Space	M.O.T.	Height	Width
$A \sqcup B$	$o(A) \oplus o(B)$	$\max(h(A), h(B))$	$w(A) \oplus w(B)$
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$A \times B$	$o(A) \otimes o(B)$	$h(A) \hat{\oplus} h(B)$	*Not functional*
$A \cdot B$	*Not functional*	$h(A) \cdot h(B)$	$w(A) \odot w(B)$
A^*	$\omega^{\omega^{(o(A)^{\pm})}}$	$h^*(A)$	$\omega^{\omega^{(o(A)^{\pm})}}$
$M^\diamond(A)$	$\omega^{\widehat{o(A)}}$	$h^*(A)$	$\omega^{\widehat{o(A)}-1}$
$M^\circ(A)$	$\omega^{o(A)}$	$\omega^{h(A)}$	*Not functional*
$P_f(A)$	**?**	**?**	**?**

Credits to: De Jongh & Parikh(1977), Schmidt(1979), Abraham & Bonnet(1999), Van der Meeren, Rathjen & Weiermann(2009,2015), Džamonja, Schmitz & Schnoebelen(2020)

What about the other operations?

Space	M.O.T.	Height	Width
$A \sqcup B$	$o(A) \oplus o(B)$	$\max(h(A), h(B))$	$w(A) \oplus w(B)$
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$A \times B$	$o(A) \otimes o(B)$	$h(A) \hat{\oplus} h(B)$	*Not functional*
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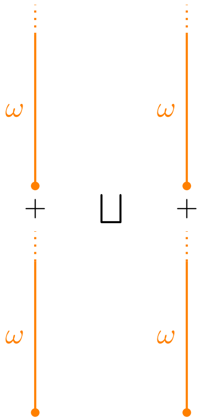
$M^\diamond(A)$	$\omega^{\widehat{o(A)}}$	$h^*(A)$	$\omega^{\widehat{o(A)}-1}$
$M^\circ(A)$	$\omega^{o(A)}$	$\omega^{h(A)}$	*Not functional*
$P_f(A)$	*Not functional*	*Not functional*	*Not functional*

Credits to: De Jongh & Parikh(1977), Schmidt(1979), Abraham & Bonnet(1999), Van der Meeren, Rathjen & Weiermann(2009,2015), Džamonja, Schmitz & Schnoebelen(2020)

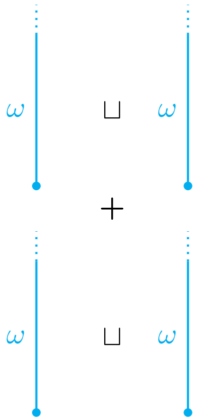
Non functional example for $\mathsf{P_f}$



$$\mathrm{o} = \omega \cdot 4$$

$$\mathrm{h} = \omega \cdot 2$$

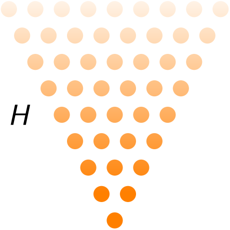
$$\mathrm{w} = 2$$



$$Y_1 = (\omega + \omega) \sqcup (\omega + \omega) \qquad Y_2 = (\omega \sqcup \omega) + (\omega \sqcup \omega)$$

$$f(\mathsf{P_f}(Y_1)) \neq f(\mathsf{P_f}(Y_2)) \text{ for } f = \mathrm{o}, \mathrm{h}, \mathrm{w}$$

Non functional example for Cartesian product and multiset ordering

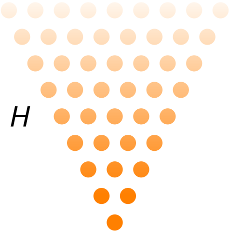


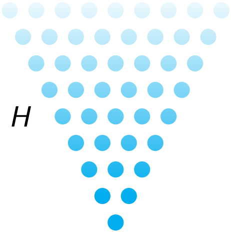
$$\mathsf{o} = \omega \cdot 2$$

$$\mathsf{h} = \omega \cdot 2$$

$$\mathsf{w} = \omega$$

$+$ $\qquad$ $+$





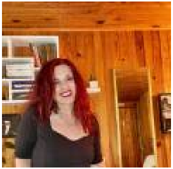
$X_1 = H + H$ $\qquad$ $X_2 = H + \omega$

$\mathsf{w}(X_1 \times \omega) \neq \mathsf{w}(X_2 \times \omega)$ $\qquad$ $\mathsf{w}(\mathsf{M}^{\mathsf{o}}(X_1)) \neq \mathsf{w}(\mathsf{M}^{\mathsf{o}}(X_2))$

Fixing non-functionality

An underrated measure: maximal number of elements

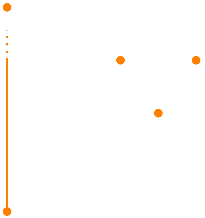
M.o.t. of the direct product



With Mirna Džamonja

Theorem (M.o.t. of the direct product)

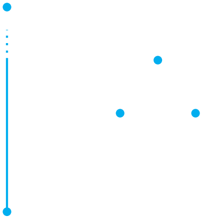
$o(A \cdot B) = o(A) \cdot pred^k(o(B)) + o(A) \otimes k$ if $max\_elt(B) = k$



$\nabla \cdot (\omega + 1)$

$o = \omega \cdot 3 + 2$

$max\_elt = 2$



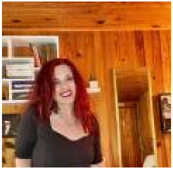
$\Delta \cdot (\omega + 1)$

$o = \omega \cdot 3 + 1$

$max\_elt = 1$

Theorem (M.o.t. of the direct product)

$o(A \cdot B) = o(A) \cdot pred^{k}(o(B)) + o(A) \otimes k$ if $max\_elt(B) = k$



With Mirna Džamonja

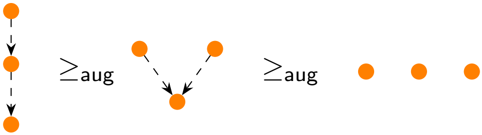
If B has k maximal elements, then $o(B) = \lambda + m$ with $m \geq k$

Then $o(A \cdot B) = o(A) \cdot (\lambda + (m - k)) + o(A) \otimes k$

Tool to compare wqos: Augmentation

$(A, \leq_A)$ is an augmentation of $(B, \leq_B)$ iff

- Same support: $A = B$
- A has more relations: $\leq_B \subseteq \leq_A$



If $A \geq_{\mathrm{aug}} B$ then $\mathrm{o}(A) \leq \mathrm{o}(B)$ and $\mathrm{w}(A) \leq \mathrm{w}(B)$

Tool to compare wqos: Augmentation

$(A, \leq_A)$ is an augmentation of $(B, \leq_B)$ iff

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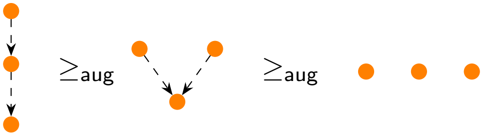


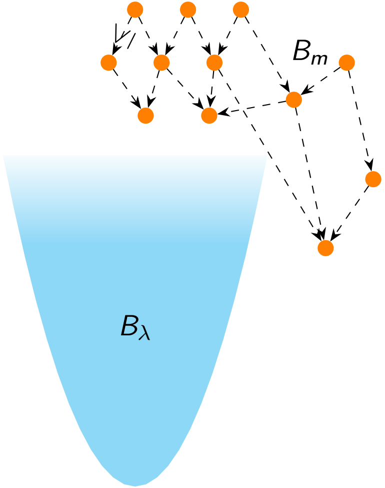
If $A \geq_{\text{aug}} B$ then $\mathrm{o}(A) \leq \mathrm{o}(B)$ and $\mathrm{w}(A) \leq \mathrm{w}(B)$

♣ **If $A \geq_{\text{aug}} B$ then $C(A) \geq_{\text{aug}} C(B)$,**

for most wqo-constructors C

Proof: First separate infinite and finite part

If B has k maximal elements, then $o(B) = \lambda + m$ with $m \geq k$



We can partition B as:

- B_λ, with $o(B_\lambda) = \lambda$
- B_m, with $o(B_m) = m$

Proof: First separate infinite and finite part

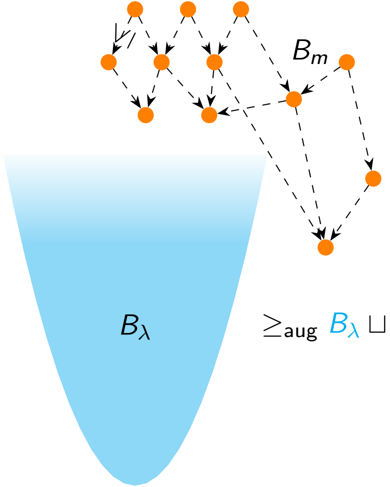
If B has k maximal elements, then $\mathrm{o}(B) = \lambda + m$ with $m \geq k$

We can partition B as:

- B_λ, with $\mathrm{o}(B_\lambda) = \lambda$
- B_m, with $\mathrm{o}(B_m) = m$

  such that

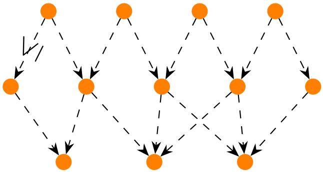
$$\begin{matrix} B_m \\ + \\ B_\lambda \end{matrix} \geq_{\text{aug}}$$



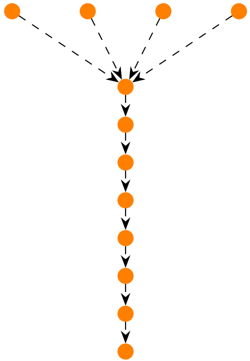
$$\geq_{\text{aug}} B_\lambda \sqcup B_m$$

Hence $\mathrm{o}(A \cdot B_\lambda) + \mathrm{o}(A \cdot B_m) \leq \mathrm{o}(A \cdot B) \leq \mathrm{o}(A \cdot B_\lambda) \oplus \mathrm{o}(A \cdot B_m)$

Focus on $A \cdot B_m$: Lower bound



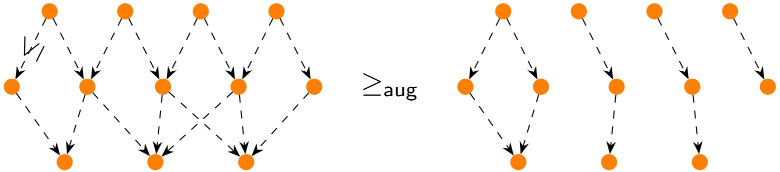
$\leq_{\mathsf{aug}}$



$$o(A \cdot B_m) \geq o(A \cdot ((m-k) + \Gamma_k)) = o(A) \cdot (m-k) + o(A) \otimes k$$

Lemma

If $\text{max\_elt}(B) = 1$ then $B = B' + 1$, then
$\text{o}(A \cdot B) = \text{o}(A \cdot B') + \text{o}(A) = \text{o}(A) \cdot \text{o}(B)$ by induction



$$\text{o}(A \cdot B_m) \leq \text{o}(A) \cdot m_1 \otimes \cdots \otimes \text{o}(A) \cdot m_k = \text{o}(A) \cdot (m-k) + \text{o}(A) \otimes k$$

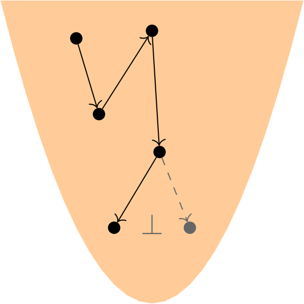
Fixing non-functionality ?

Fixing the multiset ordering : a fourth ordinal invariant!

The fourth ordinal invariant

Definition (Friendly order type)

$$\mathrm{o}_\perp(X) \;=\; \text{rank of the tree of } \textit{open-ended} \text{ bad sequences}$$



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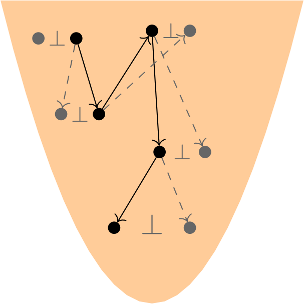

The fourth ordinal invariant

Theorem (Width of M°)

$\mathrm{w}(\mathrm{M}^\circ(X)) = \omega^{\mathrm{o}_\perp(X)}$

Space	$\mathrm{o}, \mathrm{h}, \mathrm{w}$	$\mathrm{o}_\perp$
		?
		?
		?

♦ How to compute the fot?

- Exists $X' \subseteq X$ such that $\mathrm{Bad}(X') \subseteq \mathrm{Bad}_\perp(X)$
- $\mathit{limit\_part}(\mathrm{o}(\mathit{str}(X))) \leq \mathrm{o}_\perp(X) \leq \mathrm{o}(\mathit{str}(X))$ with $\mathit{str}(X) = \{\, x \in X \mid \exists y \in X, y \perp x \,\}$
- $\mathrm{w}(X) - 1 \leq \mathrm{o}_\perp(X)$
- if $\mathrm{w}(A) = \mathrm{o}(A)$ limit, then $\mathrm{o}_\perp(X) = \mathrm{o}(X)$
- $\mathrm{o}_\perp(A \sqcup B) = \mathrm{o}(A) \oplus \mathrm{o}(B)$

Conclusion

Space	M.O.T.	Height	Width
$A \sqcup B$	$\mathrm{o}(A) \oplus \mathrm{o}(B)$	$\max(\mathrm{h}(A), \mathrm{h}(B))$	$\mathrm{w}(A) \oplus \mathrm{w}(B)$
$A + B$	$\mathrm{o}(A) + \mathrm{o}(B)$	$\mathrm{h}(A) + \mathrm{h}(B)$	$\max(\mathrm{w}(A), \mathrm{w}(B))$
$A \times B$	$\mathrm{o}(A) \otimes \mathrm{o}(B)$	$\mathrm{h}(A) \hat{\oplus} \mathrm{h}(B)$	$\geq \mathrm{w}(\mathrm{o}(A) \times \mathrm{o}(B))$
$A \cdot B$	$\mathrm{o}(A) \cdot \mathit{pred}^k(\mathrm{o}(B)) + \mathrm{o}(A) \otimes k$ if $\mathit{max\_elt}(B) = k$	$\mathrm{h}(A) \cdot \mathrm{h}(B)$	$\mathrm{w}(A) \odot \mathrm{w}(B)$
A^*	$\omega^{\omega^{(\mathrm{o}(A)^{\pm})}}$	$\mathrm{h}^*(A)$	$\omega^{\omega^{(\mathrm{o}(A)^{\pm})}}$
$\mathrm{M}^\diamond(A)$	$\omega^{\widehat{\mathrm{o}(A)}}$	$\mathrm{h}^*(A)$	$\omega^{\widehat{\mathrm{o}(A)}-1}$
$\mathrm{M}^\circ(A)$	$\omega^{\mathrm{o}(A)}$	$\omega^{\mathrm{h}(A)}$	$\omega^{\mathrm{o}_\perp(A)}$
$\mathrm{P_f}(A)$	$\leq 2^{\mathrm{o}(A)}$	$\leq 2^{\mathrm{h}(A)}$	$\geq 2^{\mathrm{w}(A)}$

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♣ Measuring well quasi-orders

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$$E := \alpha \geq \omega^\omega \text{mult. indec.} \mid E_1 \sqcup E_2 \mid E_1 \times E_2 \mid \mathsf{M}^\diamond(E) \mid \mathsf{M}^\circ(E) \mid E^* \mid \mathsf{P_f}(E)$$

Wqos that appear in well-structured transition systems!

Conclusion

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♦ Open questions

- New invariants:
  - Computing the fot
  - Is there an invariant that would make CP and $\mathrm{P_f}$ functional?
- New operations: Infinite words, variants of trees, graph minor, ...

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