

Geometry of Interaction for ZX-Diagrams

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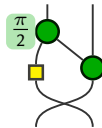
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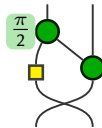
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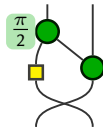
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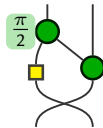
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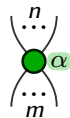
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- Relaxes unitarity
- Is Universal (can encode any linear map)
- Lack a direct operational interpretation (this talk !)

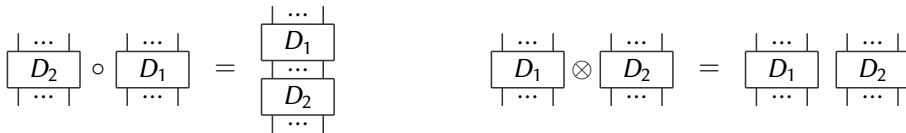
Generators



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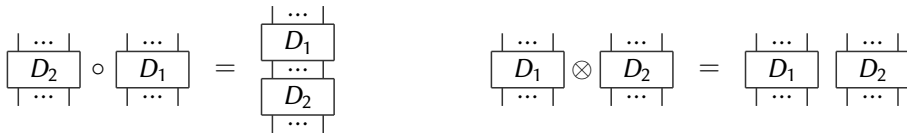
Compositions



Generators



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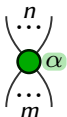



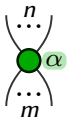

Standard Interpretation

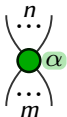
$$[\![\cdot]\!] : \mathbf{ZX} \rightarrow \mathcal{M}(\mathbb{C})$$


• A spider:

$$\begin{array}{c} n \\ \dots \\ \text{---} \\ \text{---} \\ \dots \\ m \end{array} \text{---} \alpha \quad :: \quad 2^m \left\{ \overbrace{\begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ 0 & 0 & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & 0 & 0 \\ 0 & \dots & \dots & 0 & e^{i\alpha} \end{pmatrix}}^{2^n} \right\} = \begin{cases} |0 \dots 0\rangle & \mapsto |0 \dots 0\rangle \\ |1 \dots 1\rangle & \mapsto e^{i\alpha} |1 \dots 1\rangle \\ - & \mapsto 0 \end{cases}$$


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- A change of basis:  $:: \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{cases} |0\rangle & \mapsto \frac{|0\rangle + |1\rangle}{\sqrt{2}} \\ |1\rangle & \mapsto \frac{|0\rangle - |1\rangle}{\sqrt{2}} \end{cases}$


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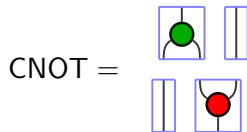
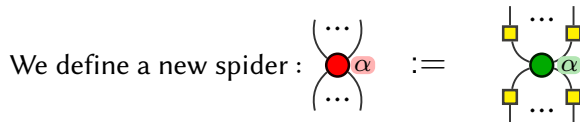
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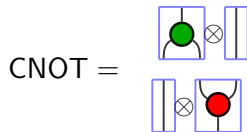
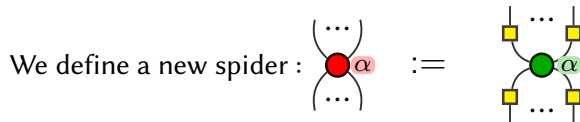
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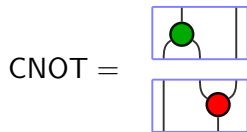
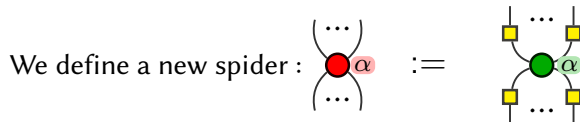
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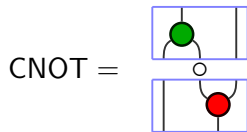
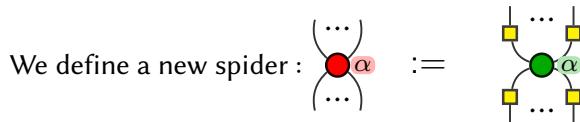
•  $:: \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \alpha \mapsto \alpha |00\rangle + \alpha |11\rangle : \mathbb{C} \rightarrow \mathbb{C}^2$

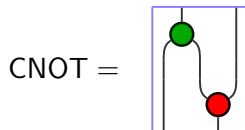
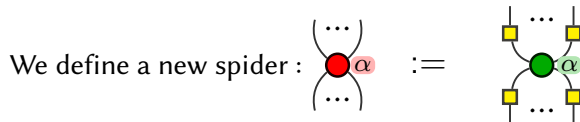
•  $:: (1 \ 0 \ 0 \ 1) = |xy\rangle \mapsto \delta_{x=y} : \mathbb{C}^2 \rightarrow \mathbb{C}$

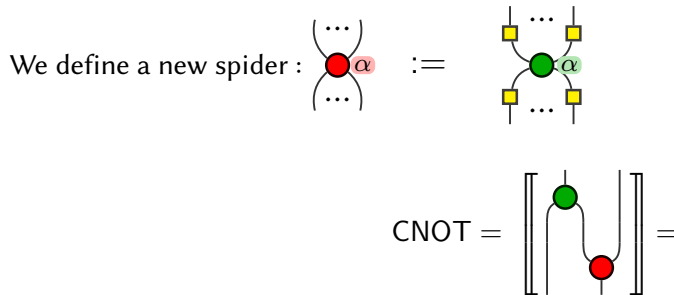



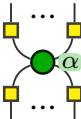


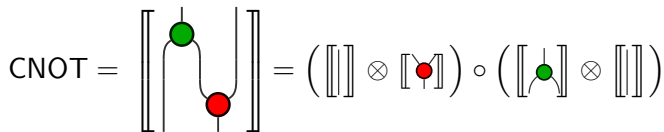



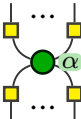






We define a new spider :  $:=$ 

$$\text{CNOT} = \left[\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \right] \otimes \left[\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \right] \circ \left(\left[\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \right] \otimes \left[\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \right] \right)$$


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- Gol as a **Token Machine**
- Proof Nets / ZX seen as graphs
- Tokens travel through the graph
- Proof Nets \Rightarrow Capture computational content of the proof
- Our Token Machine : superposition of tokens, multiple tokens, collisions, ...

\Rightarrow Bring Operational Semantic on ZX-Diagrams.

Already existing work: “The geometry of parallelism. classical, probabilistic, and quantum effects” [Dal Lago, Faggian, Valiron, Yoshimizu]

	Dal Lago et al.	This Work
Superposition	X	✓
Asynchronicity	X	✓
Types	✓	X

Token

3-tuple $(e, d, b) \in \mathcal{E}(D) \times \{\uparrow, \downarrow\} \times \{0, 1\}$

where:

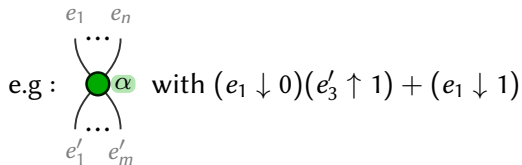
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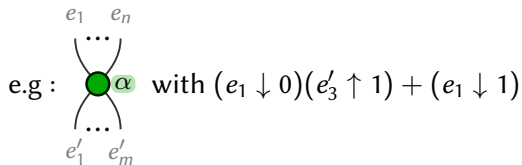
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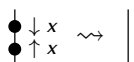
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- The tokens modify the global token state as they move.

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• Diffusions : $\begin{array}{c} \bullet \downarrow x \\ \dots \\ \bullet \end{array} \rightsquigarrow e^{ix\alpha} \begin{array}{c} \bullet \uparrow x \\ \dots \\ \bullet \end{array}$

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$$\begin{array}{c} \bullet \downarrow x \\ \square \end{array} \rightsquigarrow \frac{1}{\sqrt{2}} \left(\begin{array}{c} \square \\ \bullet \downarrow 0 \end{array} + (-1)^x \begin{array}{c} \square \\ \bullet \downarrow 1 \end{array} \right)$$

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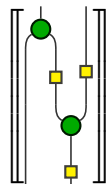
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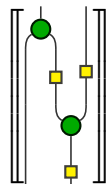
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$$x \downarrow \bullet \rightsquigarrow \bullet \uparrow x \quad \dots$$

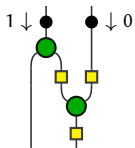
$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$



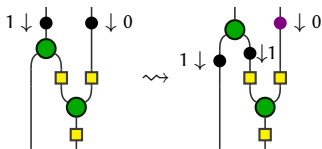
$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \implies |10\rangle \mapsto \frac{1}{\sqrt{2}} |11\rangle$$

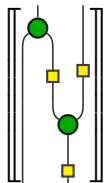


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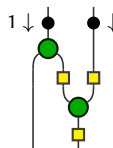


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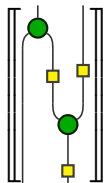


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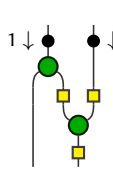


$$\rightsquigarrow^* \frac{1}{\sqrt{2}} \left(\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} \right)$$

The diagram shows a superposition of two states. The first state has a purple dot on the bottom wire of the right node, with arrows labeled 1 and 0. The second state has a black dot on the bottom wire of the right node, with arrows labeled 1 and 1.

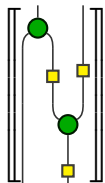


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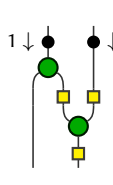
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The diagram shows a superposition of two states. The first state has a black dot with a downward arrow labeled '1' on the left wire and another with a downward arrow labeled '0' on the right wire. The second state has a black dot with a downward arrow labeled '1' on the left wire and a purple dot with a downward arrow labeled '1' on the right wire. Both diagrams have a black dot with an upward arrow labeled '0' on the left wire.



A ZX-diagram representing a CNOT gate. It consists of two vertical wires. The left wire has a green circle (control) and a yellow square (target). The right wire has a green circle (control) and a yellow square (target). The wires are connected by a path of four yellow squares and two green circles.

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \implies |10\rangle \mapsto \frac{1}{\sqrt{2}} |11\rangle$$

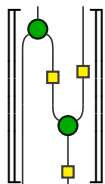


A ZX-diagram with phase labels. The left wire has a black dot with a downward arrow labeled '1' and a black dot with a downward arrow labeled '0'. The right wire has a black dot with a downward arrow labeled '0' and a black dot with a downward arrow labeled '1'. The diagram is connected by a path of four yellow squares and two green circles.

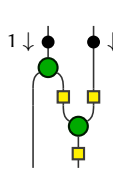
$$\rightsquigarrow^* \frac{1}{\sqrt{2}} \left(\begin{array}{c} \text{Diagram 1} \\ + \\ \text{Diagram 2} \end{array} \right)$$

The two diagrams in the sum are:

- Diagram 1: The left wire has a black dot with a downward arrow labeled '1' and a black dot with a downward arrow labeled '0'. The right wire has a black dot with a downward arrow labeled '0' and a black dot with a downward arrow labeled '1'. A purple dot with a downward arrow labeled '1' is on the left wire between the two green circles.
- Diagram 2: The left wire has a black dot with a downward arrow labeled '1' and a black dot with an upward arrow labeled '1'. The right wire has a black dot with a downward arrow labeled '1' and a black dot with a downward arrow labeled '1'.

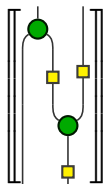


$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \implies |10\rangle \mapsto \frac{1}{\sqrt{2}} |11\rangle$$

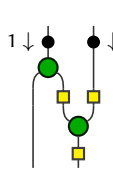


$$\rightsquigarrow^* \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}} \left(\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} \right) - \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} \right) + \begin{array}{c} \text{Diagram 5} \end{array}$$

The diagram shows a decomposition of a ZX-diagram with two green circles and two yellow squares into a sum of three terms. The first term is a product of two diagrams: the first has two black dots with arrows labeled 1 and 0, and the second has two black dots with arrows labeled 0 and 0. The second term is a diagram with two black dots and arrows labeled 1 and 0. The third term is a diagram with a purple dot and arrow labeled 1, and a black dot and arrow labeled 1.



$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \implies |10\rangle \mapsto \frac{1}{\sqrt{2}} |11\rangle$$



$$\rightsquigarrow^* \frac{1}{2} \left(\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \\ \text{Diagram 3} \\ \text{Diagram 4} \end{array} \right)$$

The four diagrams in the sum are:

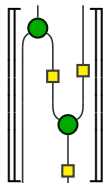
- Diagram 1: Same as the left diagram, but with two purple dots on the left wire between the top and bottom green circles. Arrows point to them with labels '0 down' and '0 up'.
- Diagram 2: Same as the left diagram, but with two black dots on the right wire between the top and bottom green circles. Arrows point to them with labels '1 down' and '0 up'.
- Diagram 3: Same as the left diagram, but with two black dots on the right wire between the top and bottom green circles. Arrows point to them with labels '0 down' and '1 up'.
- Diagram 4: Same as the left diagram, but with two black dots on the right wire between the top and bottom green circles. Arrows point to them with labels '1 down' and '1 up'.

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \implies |10\rangle \mapsto \frac{1}{\sqrt{2}} |11\rangle$$

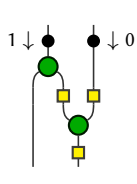
$$\rightsquigarrow^* \frac{1}{2} \left(\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \\ \text{Diagram 3} \\ \text{Diagram 4} \end{array} \right)$$

The four diagrams in the sum are:

- Diagram 1: Same as the left diagram, but with a black dot on the right wire at the top of the lower arc, labeled $\downarrow 0$.
- Diagram 2: Same as the left diagram, but with two purple dots on the right wire at the top of the lower arc, labeled $\downarrow 1$ and $\uparrow 0$.
- Diagram 3: Same as the left diagram, but with two black dots on the right wire at the top of the lower arc, labeled $\downarrow 0$ and $\uparrow 1$.
- Diagram 4: Same as the left diagram, but with two black dots on the right wire at the top of the lower arc, labeled $\downarrow 1$ and $\uparrow 1$.



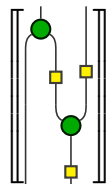
$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \implies |10\rangle \mapsto \frac{1}{\sqrt{2}} |11\rangle$$



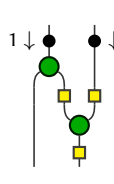
$$\rightsquigarrow^* \frac{1}{2} \left(\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \\ \text{Diagram 3} \\ \text{Diagram 4} \end{array} \right)$$

The diagram shows a ZX-diagram with two vertical lines. The left line has a green node at the top, a black node with a downward arrow labeled '1', and a yellow square at the bottom. The right line has a green node at the bottom, a black node with a downward arrow labeled '0', and a yellow square at the top. The diagram is equal to $\frac{1}{2}$ times the sum of four diagrams:

- Diagram 1: The left line has a green node at the top, a black node with a downward arrow labeled '1', and a yellow square at the bottom. The right line has a green node at the bottom, a black node with a downward arrow labeled '0', and a yellow square at the top.
- Diagram 2: The left line has a green node at the top, a black node with a downward arrow labeled '1', and a yellow square at the bottom. The right line has a green node at the bottom, a black node with a downward arrow labeled '1', and a yellow square at the top.
- Diagram 3: The left line has a green node at the top, a black node with a downward arrow labeled '1', and a yellow square at the bottom. The right line has a green node at the bottom, a black node with an upward arrow labeled '1', and a yellow square at the top.
- Diagram 4: The left line has a green node at the top, a black node with a downward arrow labeled '1', and a yellow square at the bottom. The right line has a green node at the bottom, a black node with an upward arrow labeled '1', and a yellow square at the top.

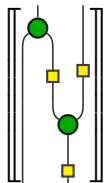


$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \implies |10\rangle \mapsto \frac{1}{\sqrt{2}} |11\rangle$$

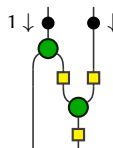


$$\rightsquigarrow^* \frac{1}{2} \left(\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \\ - \text{Diagram 3} \end{array} \right)$$

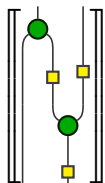
The diagram is a linear combination of three ZX-diagrams, each with two vertical lines. The first diagram has a green circle on the left line and a black dot with a downward arrow on the right line. The second diagram has a green circle on the right line and a black dot with a downward arrow on the left line. The third diagram has a green circle on the right line, two purple circles on the left line, and a black dot with a downward arrow on the right line.



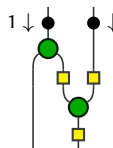
$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \implies |10\rangle \mapsto \frac{1}{\sqrt{2}} |11\rangle$$



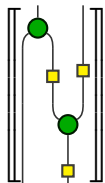
$$\rightsquigarrow^* \frac{1}{2} \left(\begin{array}{c} 1 \downarrow \bullet \quad \bullet \downarrow 0 \\ \text{Diagram 1} \\ \text{Diagram 2} \end{array} \right)$$



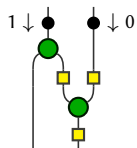
$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \implies |10\rangle \mapsto \frac{1}{\sqrt{2}} |11\rangle$$

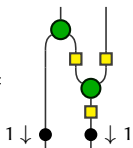


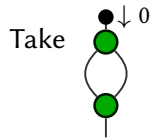
$$\rightsquigarrow^* \frac{1}{2\sqrt{2}} \left(\begin{array}{c} \text{Diagram 1} \\ 1 \downarrow \bullet \quad \bullet \downarrow 0 \end{array} + \begin{array}{c} \text{Diagram 2} \\ 1 \downarrow \bullet \quad \bullet \downarrow 1 \end{array} - \begin{array}{c} \text{Diagram 3} \\ 1 \downarrow \bullet \quad \bullet \downarrow 0 \end{array} + \begin{array}{c} \text{Diagram 4} \\ 1 \downarrow \bullet \quad \bullet \downarrow 1 \end{array} \right)$$

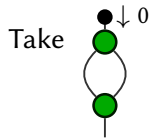


$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \implies |10\rangle \mapsto \frac{1}{\sqrt{2}} |11\rangle$$

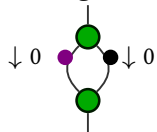


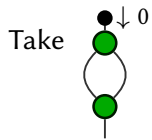
$$\rightsquigarrow^* \frac{1}{\sqrt{2}}$$




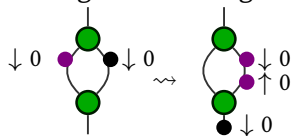


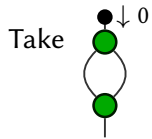
With good rewriting order:



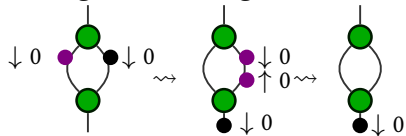


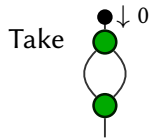
With good rewriting order:



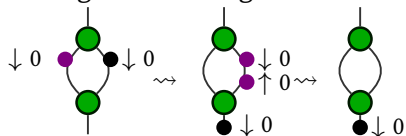


With good rewriting order:

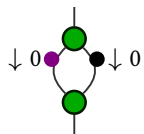


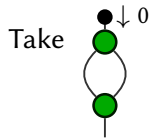


With good rewriting order:

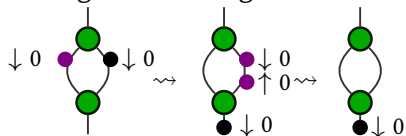


With arbitrary rewriting order:

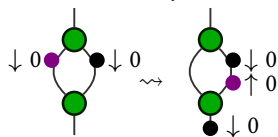


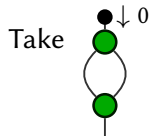


With good rewriting order:

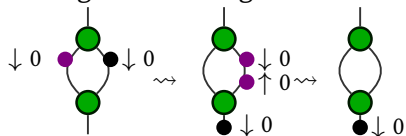


With arbitrary rewriting order:

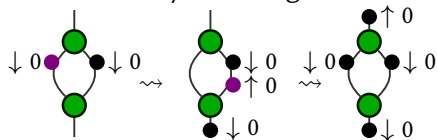




With good rewriting order:



With arbitrary rewriting order:



Rewriting System

We define a transition system \rightsquigarrow as *exactly* one **diffusion rule** rule followed by all possible **collision** rules until none apply

Want to avoid:

- Having multiple tokens on the same edge that don't collide (i.e $(a \downarrow x)(a \downarrow x)$)
- Non-termination

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Two invariants:

- **Well-Formedness** : Avoid bad configuration
- **Cycle-Balancedness** : Termination, Confluence

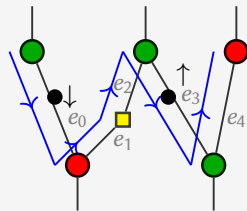
Polarity in a Path

$p = (e_0, e_1, e_2, e_3, e_4)$ is an oriented path.

- If a token follows the path +1
- If it goes against it -1
- If it is not on the path 0

Example:

- Here, polarity 0



Well-Formed Token State

We say that a ZX-Diagram D is **Well-Formed** if for every path p its Polarity $\in \{-1, 0, 1\}$

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Theorem (Characterisation of Well-Formedness)

Well-formed states cannot reach “bad configurations”.

Cycle-Balanced Token State

We say that a ZX-Diagram D is **Cycle-Balanced** if for every cycle c its Polarity = 0

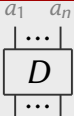
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Theorem (Termination of well-formed, cycle-balanced token state)

Let D be a ZX-Diagram, and s a well-formed, cycle-balanced token state, then s terminates.

Theorem (Simulation of Standard Interpretation)

Let  a ZX-Diagram and let $t = (e \downarrow 0)(e \uparrow 0) + (e \downarrow 1)(e \uparrow 1)$.

t is a well-formed, cycle balanced token state.

We have that $t \rightsquigarrow^* \left[\begin{array}{c} a_1 \quad a_n \\ \vdots \\ \text{---} \\ \vdots \\ b_1 \quad b_m \end{array} \right]$

\Rightarrow we recover the **standard interpretation**.

Conclusion

- The Token Machine gives us a very general framework to study ZX-Calculus.
- More operational approach to ZX-Calculus.
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Future Work

- Stronger relation with Linear Logic and Proof Nets (WIP)
- Hope it can help in defining new extensions of ZX-Calculus such as recursion (Future Work).