Applicative bisimulations for lambda-calculus with continuous probabilities

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Discrete randomized programs

The program can make probabilistic choices at any point during its execution.

Example (Random Walk)

$$n-1 \xleftarrow{(1-p)} n \xrightarrow{p} n+1$$

Faster algorithms

e.g. Randomised sorting algorithms:

Quick-sort:
worst case time complexity $= O(n^2)$.

Randomised Quick-sort:
Expected worst case time complexity $= O(nlogn)$.

In computational cryptography

Can be a necessity in order to achieve security
(e.g. secure encryption in an asymmetric setting.)

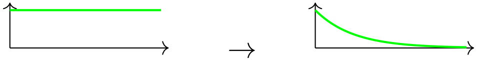
Continuous statistical programming

The program can:

- sample from standard probability distributions (Gaussian distribution, uniform distribution...);
- modify them using a push-forward operation

Example

Build an exponential distribution from a uniform one
let $x =$ sample in $-\log(x)$.



Density functions.

To describe the behaviour of systems:

- with inherent uncertainty
- of whom we have incomplete knowledge

Higher-order Probabilistic Programming

Higher-order languages extended with:

- Discrete randomized algorithms (e.g. randomized sorting);
- continuous probability distributions (e.g. to model physical systems);

$$
\begin{aligned}
M \in \Lambda ::= \; & x \;\big|\; \lambda x^A \cdot M \;\big|\; (MN) \;\big|\; (YN) \\
& \big|\; \text{ifz } (M, N, L) \;\big|\; \text{let } x = M \text{ in } N) \\
& \big|\; M \oplus N \;\big|\; \underline{n} \;\big|\; \text{succ } (M) \;\big|\; \text{pred } (M), \qquad n \in \mathbb{N} \\
& \big|\; \text{sample} \;\big|\; \underline{r} \;\big|\; \underline{f}, \qquad r \in \mathbb{R}, f : \mathbb{R} \to \mathbb{R} \text{ mesureable}
\end{aligned}
$$

Operational Semantics.

Discrete case

$[\![M]\!]$: Values $\to [0,1]$ a discrete distribution

Continuous case

$[\![M]\!] : \Sigma_{\text{Values}} \to [0,1]$ a continuous distribution where:

$$\Sigma_{\text{Values}} \subseteq \text{Parts(Values) a } \sigma\text{-algebra;}$$

Example

- $[\![\mathtt{t} \oplus \mathtt{f}]\!] = \mathscr{D}$ with $\mathscr{D}(\mathtt{t}) = \frac{1}{2}$, $\mathscr{D}(\mathtt{f}) = \frac{1}{2}$;
- $[\![\text{let } x = \text{ sample } \text{ in } (\lambda y.x + y)]\!] = \mathscr{D}$ with

$$\mathscr{D} : A \in \Sigma_{\text{Values}} \mapsto \mu_{\text{Borel}}(\{z \in [0,1] \mid (\lambda y.z + y) \in A\})$$

e.g. $\mathscr{D}(\{\lambda y.r + y \mid r \in [\frac{1}{3}, \frac{2}{3}]\}) = \frac{1}{3}$.

Morris Context Equivalence (1969)

Comparing Two Programs

check whether two programs behaves *the same* no matter how the environment interacts with them (compiler optimisation, verification of a specification...);

When are two programs context equivalent?

- Environments are modelled as **contexts**–i.e. terms with a hole–thus by way of the underlying language
- Two terms are context equivalent if their **observable behaviour** is the same in **any** context.

Definition (Context Equivalence)

$M \ \equiv^{\mathrm{ctx}} N$ when $\forall$ context $\mathcal{C}$ that returns a ground type,

$$[\![\mathcal{C}[M]]\!] = [\![\mathcal{C}[N]]\!] \qquad \text{as distributions.}$$

Contextual Reasoning– Challenges

- $M \not\equiv^{\text{ctx}} N$?
  Find **one** context $\mathcal{C}$ able to distinguish them.
- $M \equiv^{\text{ctx}} N$?
  all contexts should be considered.

Objective:

A characterisation of context equivalence that gets rid of the *universal quantification* on contexts.

Definition

A binary relation R on programs is:

- *sound* when $M\,R\,N \Rightarrow M \equiv^{\text{ctx}} N$
- *complete* when $M \equiv^{\text{ctx}} N \Leftrightarrow M\,R\,N$.

Applicative Bisimilarity [Abramsky93](1)

General idea:

- Expressing interactively the semantics as a Labelled Transition System.
- Obtain an equivalence on programs from the bisimulation on this system.

Programs	**Values**
M	
N	
L	
$\vdots$	

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Programs	**Values**
M	V
N	W
L	U
$\vdots$	$\vdots$

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Programs **Values**

M

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Programs **Values**

$$M \xrightarrow{eval} V$$

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$$\lambda x.N$$

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Programs **Values**

$$M \xrightarrow{eval} V$$

$$N\{L/x\} \xleftarrow{L} \lambda x.N$$

Applicative Bisimilarity [Abramsky93] (2)

$$
\begin{tikzcd}
 & & & \lambda x.L \arrow[dr, bend left, "T"] & & & \\
M \arrow[urrr, bend left, "eval"] & R & N \arrow[dr, bend right, "eval"'] & & L\{T/x\} & R & K\{T/x\} \\
 & & & \lambda x.K \arrow[urrr, bend right, "T"'] & & &
\end{tikzcd}
$$

- **Similarity**: union of all simulations, denoted $\precsim$;
- **Bisimilarity**: union of all bisimulations, denoted $\sim$.

Full Abstraction results (deterministic case)

Untyped pure λ-calculus, where we observe **convergence**

	$\sim \subseteq \equiv^{ctx}$	$\sim = \equiv^{ctx}$	$\precsim \subseteq \leq^{ctx}$	$\precsim = \leq^{ctx}$
CBN	✓	✓	✓	✓
CBV	✓	✓	✓	✓

[Abramsky1990,Howe1993]

Bisimilarity for discrete probabilistic systems (LMC)

Definition (Labelled Markov Chain)

a triple $\mathscr{M} = (\mathcal{S}, \mathcal{A}, \{h_a \mid a \in \mathcal{A}\})$, where

- $\mathcal{S}$ is a countable set of *states*, $\mathcal{A}$ a countable set of *labels*;
- h_a is a *transition probability matrix*, i.e., a function $h_a : \mathcal{S} \times \mathcal{S} \to [0,1]$ such that $\forall s, a,\ h_a(s,\cdot)$ is a (sub)-distribution;

Definition ([Larsen-Skou'91])

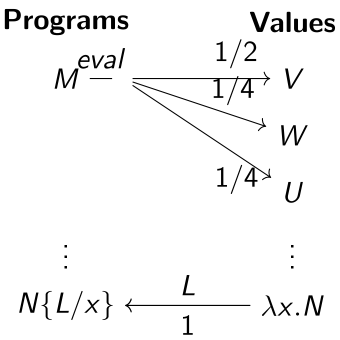
A symmetric relation R on $\mathcal{S}$ is a bisimulation when:

$$s\,R\,t \quad \Rightarrow \quad \forall a \in \mathcal{L}, \forall X \subseteq \mathcal{S}\ R\text{-closed}, h_a(s,X) = h_a(t,X).$$

Theorem (Logical caracterisation[Breugel et al'05])

$$\phi ::= \top \;\Big|\; < a >_p \cdot\phi \;\Big|\; \phi \wedge \phi$$

A Labelled Markov Chain for $\Lambda_\oplus$ [Dal Lago et al'14]



Theorem (Dal Lago et al'14,Crubillé et al'14)

	$\sim \subseteq \equiv^{ctx}$	$\equiv^{ctx} \subseteq \sim$	$\precsim \subseteq \leq^{ctx}$	$\leq^{ctx} \subseteq \precsim$
CBN	✓	✗	✓	✗
CBV	✓	✓	✓	✗

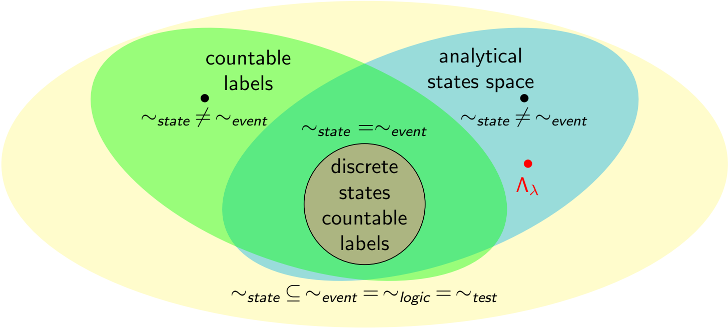
The subject of this talk: the continuous case

Definition (Labelled Markov Process)

A triple $(\mathcal{S}, \mathcal{A}, \{h_a \mid a \in \mathcal{A}\})$, where $\mathcal{S}$ is measurable, $\mathcal{A}$ is an arbitrary set, and for every $a \in \mathcal{A}$ the map $h_a : \mathcal{S} \times \Sigma_{\mathcal{S}} \to [0, 1]$ is a sub-probability kernel.

From the literature

Two distincts notions of bisimulations exist for LMPs:



State bisimulation

Definition (for Labelled Markov Chain)

A symmetric relation R on $\mathcal{S}$ is a bisimulation when:

$$s \, R \, t \quad \Rightarrow \quad \forall a \in \mathcal{L}, \forall X \; R\text{-closed}, h_a(s, X) = h_a(t, X).$$

Proposition (Dal Lago-Gavazzo'19)

Applicative state bisimulation is sound (w.r.t. context equivalence) for Λ_λ.

State bisimulation

Definition (for Labelled Markov Process)

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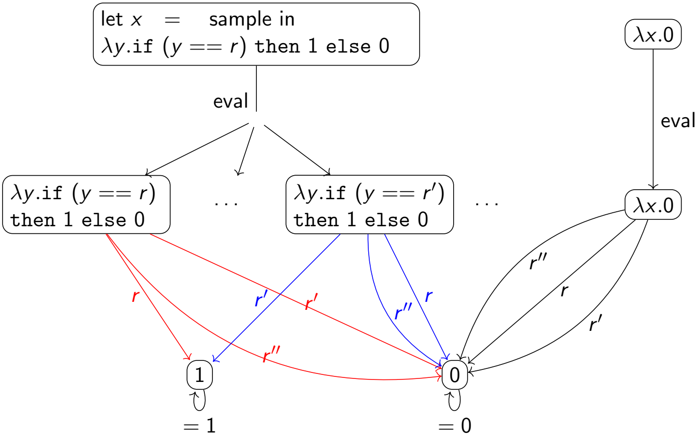
Proposition (Dal Lago-Gavazzo'19)

Applicative state bisimulation is sound (w.r.t. context equivalence) for Λ_λ.

Theorem

Applicative state bisimulation is not complete for Λ_λ.

Proof.



M and N are context equivalent [Staton et al'21], but not bisimilar.

Event bisimulation for Λ_λ

Definition

An event bisimulation on a LMP $(\mathscr{M}, \Sigma, \{h_a :| a \in \mathcal{A}\})$ is a sub-σ-algebra Λ of Σ,such that $(\mathscr{M}, \Lambda, \{h_a \mid a \in \mathcal{A}\})$ is a LMP.

Proposition

logical caracterisation [Breugel et al'05]

Theorem

Applicative event bisimulation is complete, but not sound.

Proof.

- completeness proof: uses the logical caracterisation
- counter-example for soundness:

$$M := \text{let } x = \text{sample in } (\lambda y.((\text{if } x == y \text{ then } 1 \text{ else } 0) \oplus x)),$$
$$N := \text{let } x = \text{sample in } (\lambda y.(0 \oplus x)),$$
$$C = (\text{ let } z = [\,] \text{ in } z(z1)).$$

M and N are event bisimilar, but not context equivalent.

Λ_λ with only continuous primitive functions

$$
\begin{aligned}
M \in \Lambda_{\lambda,c} ::= \; & x \;\big|\; \lambda x^A \cdot M \;\big|\; (MN) \;\big|\; (YN) \\
& \big|\; \text{ifz } (M, N, L) \;\big|\; \text{let}(x, M, N) \\
& \big|\; M \oplus N \;\big|\; \underline{n} \;\big|\; \text{succ } (M) \;\big|\; \text{pred } (M), \qquad n \in \mathbb{N} \\
& \big|\; \text{sample} \;\big|\; \underline{r} \;\big|\; \underline{f}, \qquad r \in \mathbb{R}, f : \mathbb{R} \to \mathbb{R} \textbf{ continuous}
\end{aligned}
$$

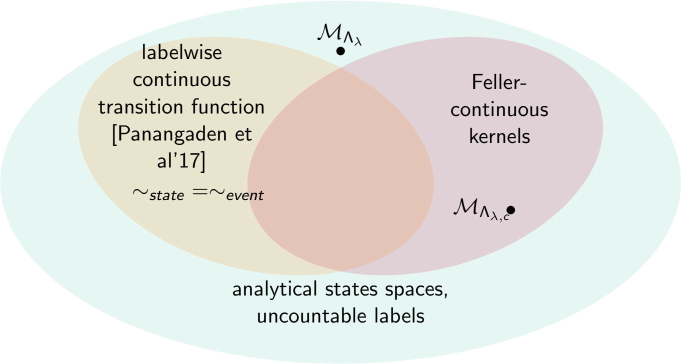
Previous counter-examples cannot be written in this language.

Question:

Can we recover $\sim_{event} = \sim_{context} = \sim_{state}$?

Our demarch

Build a class of LMP with **uncountable labels** such that the two bisimulations coincide.



Feller Continuous LMPs

Definition

X a polish space, $(\mu_n)_{n\in\mathbb{N}}$ a sequence of measures over X.
$(\mu_n)_{n\in\mathbb{N}}$ *converges weakly* toward μ when $\forall f : X \to \mathbb{R}$ bounded and continuous function:

$$\lim_{n\to\infty} \int_X f.d\mu_n = \int_X f.d\mu.$$

Definition

A LMP $\mathcal{M} = (\mathcal{S}, \mathcal{A}, \{h_a \mid a \in \mathcal{A}\})$ with $\mathcal{S}, \mathcal{A}$ polish spaces. $\mathcal{M}$ is *Feller continuous* when:

- for every $a \in \mathcal{A}$, the map $h_a : \mathcal{S} \to \mathrm{Distrs}(\mathcal{S})$ is continuous;
- for every $s \in \mathscr{M}$ the map $h_{(.)}(s) : a \in \mathcal{A} \mapsto h_a(s) \in \mathrm{Distrs}(\mathcal{S})$ is continuous.

Bisimulations for Feller continuous LMPs

Theorem

For Feller continuous LMPs, state bisimulation and event bisimulation coincide.

Proof.

Uses a result from optimal transport [Villani'08]. □

Theorem

$\Lambda_{\lambda,c}$ *is Feller continuous, thus* $\sim_{event} = \sim_{context} = \sim_{state}$.

Conclusion:

Contribution

- an extensive picture of the full abstraction problem for applicative similarité on Λ_λ
- the definition of a new class of LMP (Feller-continuous LMPs) where state and event bisimilarity coincide.

Perspectives

- bisimulation for a language with continuous probabilities and bayesian reasonning...
- quantitative reasonning (i.e. distances) for a continuous language.