A model theoretic approach to sparsity

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joint work with J. Gajarský, S. Kreutzer, J. Nešetřil, Mi. Pilipczuk, R. Rabinovich, S. Siebertz, and S. Toruńczyk

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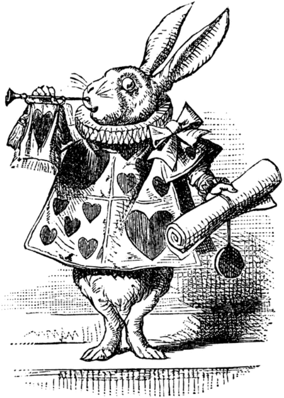
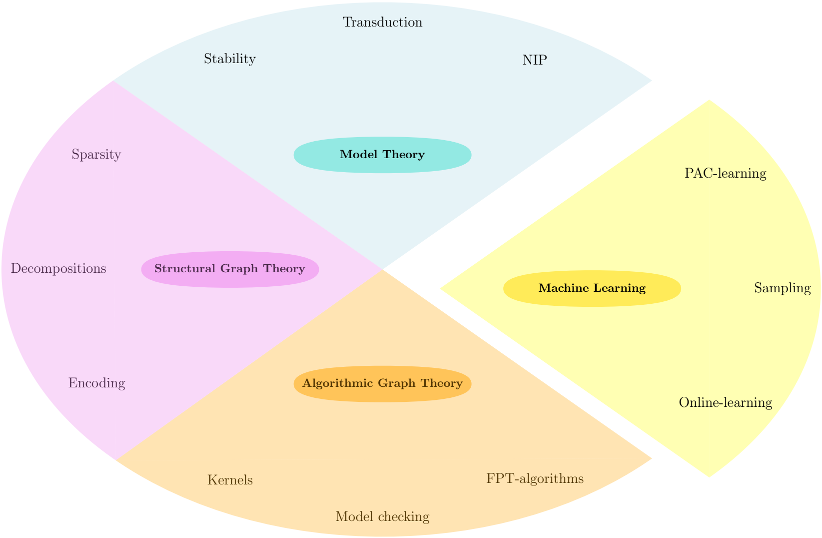
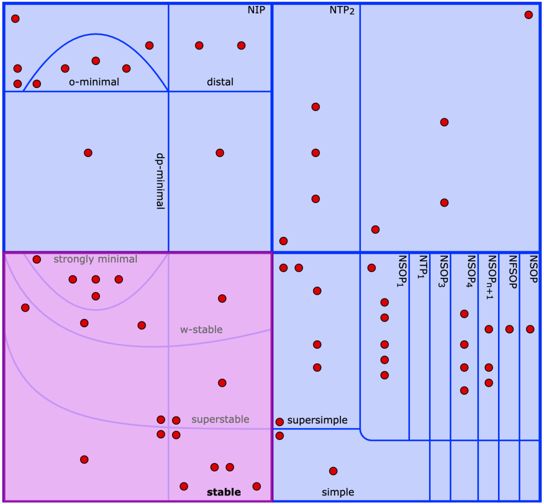
Introduction



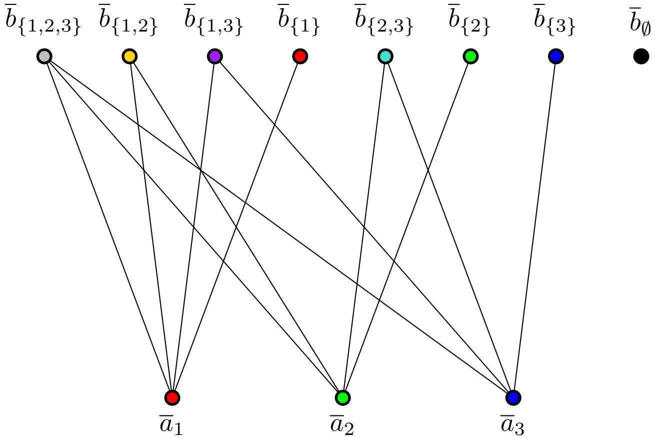
General View



The model theoretic universe

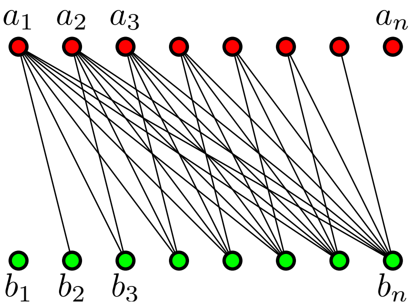


NIP and VC-dimension



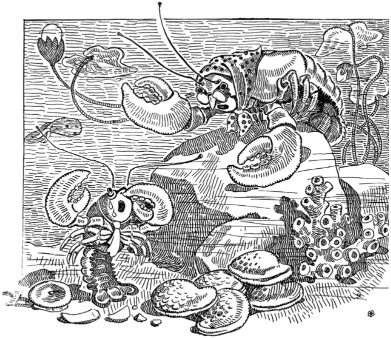
$$G \models \phi(\bar{a}_i, \bar{b}_I) \quad \Longleftrightarrow \quad i \in I$$

Stability and Order property



$$G \models \phi(\bar{a}_i, \bar{b}_j) \iff i < j$$

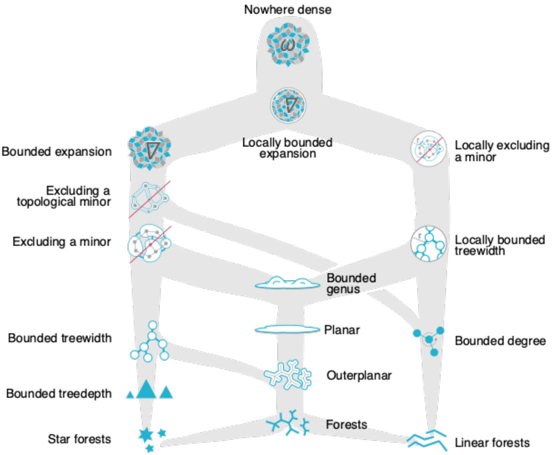
Computational learning theory



PAC learning $\rightsquigarrow$ NIP (Laskovski '92)

Online learning $\rightsquigarrow$ Stability (Chase & Freitag '18)

Structural graph theory



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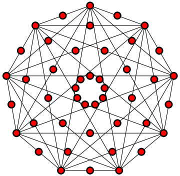
Adler & Adler '14

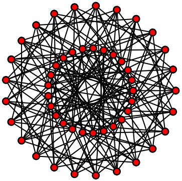
For a monotone class $\mathcal{C}$ tfae:

1. $\mathcal{C}$ is nowhere dense,
2. $\mathcal{C}$ is NIP,
3. $\mathcal{C}$ is stable.

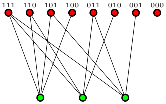
Special Generic Structures



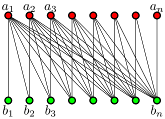
Nowhere dense



Bounded expansion



NIP



Stability

Computational complexity

Theorem (Grohe, Kreutzer, Siebertz '14)

For every nowhere dense class $\mathcal{C}$ and every $\epsilon > 0$, every property of graphs definable in first-order logic can be decided in time $O(n^{1+\epsilon})$ on $\mathcal{C}$.

Theorem (Dvořák, Kráľ, Thomas '10; Kreutzer '11)

if a monotone class $\mathcal{C}$ is somewhere dense, then deciding first-order properties of graphs in $\mathcal{C}$ is not fixed-parameter tractable (unless FPT = W[1]).

Computational complexity

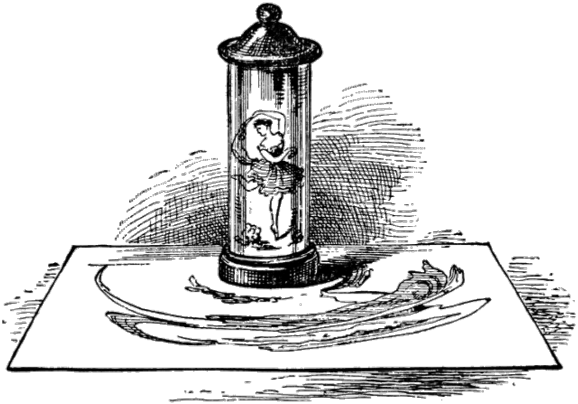
Theorem (Gajarskỳ, Hliněný, Lokshtanov, Ramanujan ’16)

Let $\mathcal{D}$ be a graph class interpretable in a bounded degree class. Then $\mathcal{D}$ has an FO model checking algorithm in FPT.

Conjecture (Gajarskỳ *et al.* ’16)

Let $\mathcal{C}$ be a nowhere dense class and $\mathcal{D}$ a graph class interpretable in $\mathcal{C}$. Then $\mathcal{D}$ has an FO model checking algorithm in FPT.

Transductions



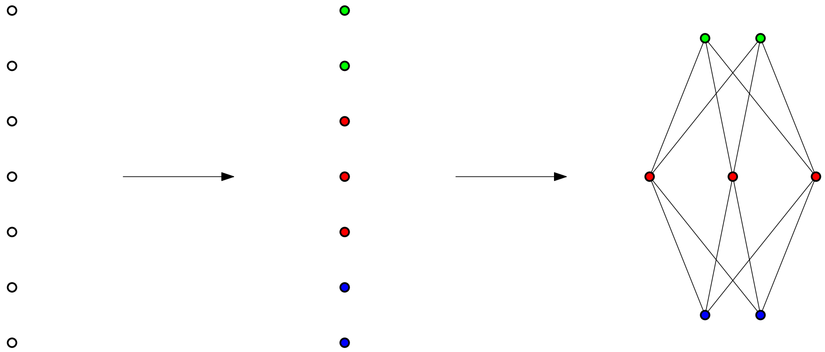
Transductions

How to encode graphs in a structure?

- Use a formula $\nu(x)$ to select the vertices,
- Use a formula $\eta(x, y)$ to define the edges,
- Use colors to encode several graphs in a same graph.

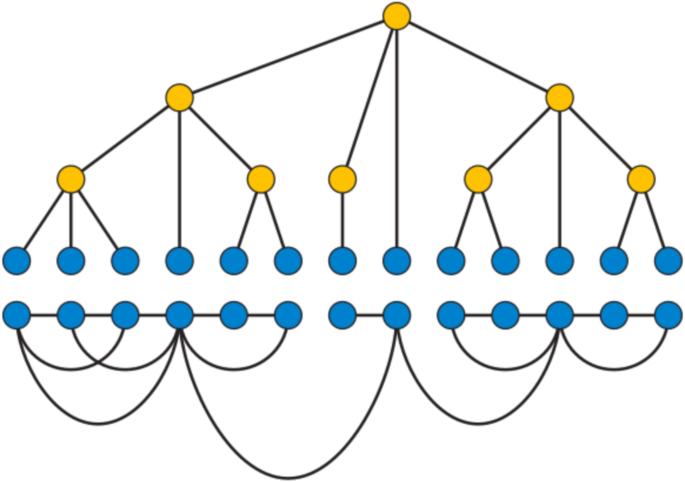
$$\mathcal{C} \twoheadrightarrow \mathcal{D}$$

Example 1: blowing



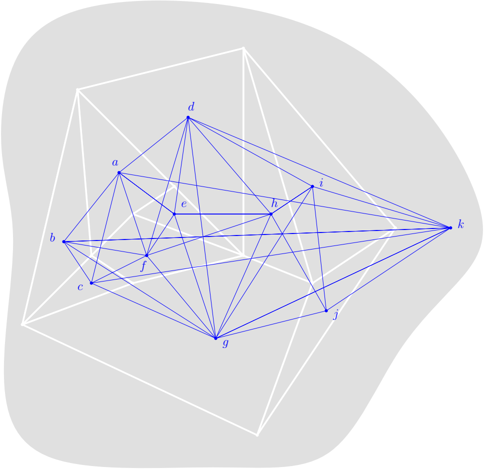
Edgeless $\longrightarrow\!\!\!\!\!\rightarrow$ Blowing of F

Example 2: k-leaf power



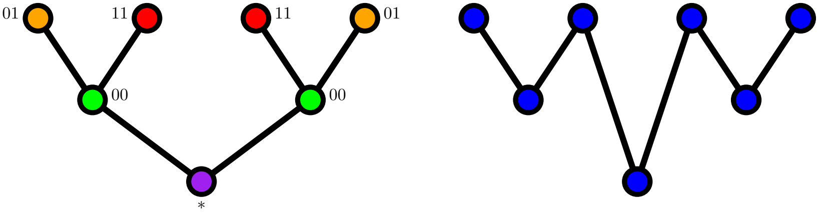
Trees $\longrightarrow\!\!\!\!\!\!\rightarrow$ k-leaf powers

Example 3: map graph



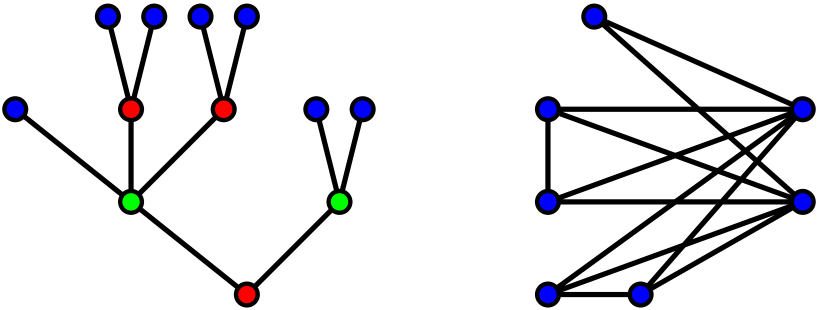
Planar quadrangulations $\longrightarrow\!\!\!\!\!\rightarrow$ Map graphs

Example 4: bounded tree-depth



Bounded height trees $\twoheadrightarrow$ Bounded tree-depth graphs

Example 5: cograph



Tree orders ⟶ Cographs

Monotone closure

Proposition

$$\chi_{\mathrm{st}}(\mathcal{C}) < \infty \quad\Longrightarrow\quad \mathcal{C} \twoheadrightarrow \mathrm{Monotone}(\mathcal{C})$$

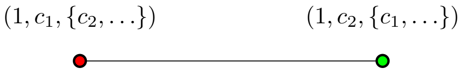
Let $\mathcal{C}$ be a class with star chromatic number at most k. Then there exists a transduction T mapping each graph $G \in \mathcal{C}$ to the set of all the subgraphs of G.

Proof

Proof

Let $\gamma : V(G) \to [k]$ be a star coloring of G, and let $H \subseteq G$. Color $v \in V(G)$ by

$$c(v) = (\mathbf{1}_{V(H)}(v), \gamma(v), \{\gamma(x) \mid xv \in E(H)\})$$



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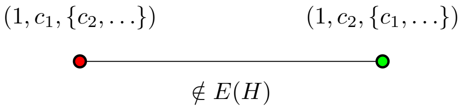

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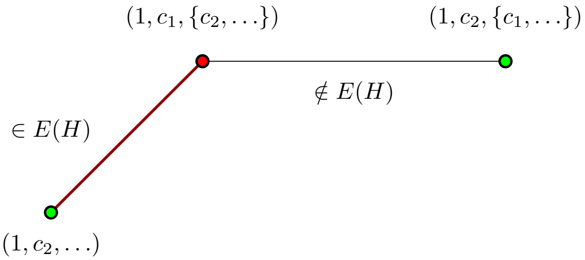

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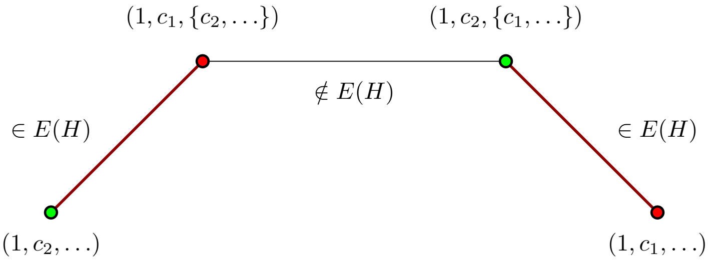

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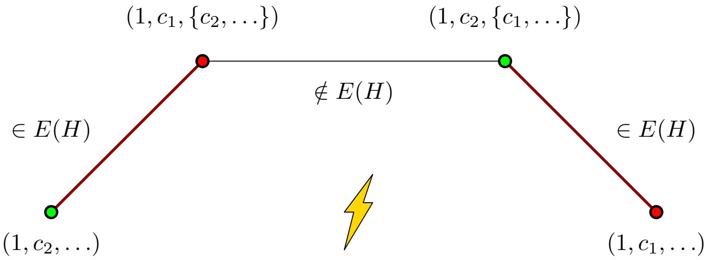


$\square$

Monadic dependence and stability

- $\mathcal{C}$ is monadically NIP iff every definable class (in a monadic lift) has bounded VC-dimension

Theorem (Baldwin, Shelah '85)

$$\mathcal{C} \text{ monadically NIP} \iff \mathcal{C} \not\twoheadrightarrow \mathcal{G}$$

- $\mathcal{C}$ is monadically stable iff every definable class (in a monadic lift) has bounded Littlestone dimension

Theorem (Anderson '90; Baldwin, Shelah '85)

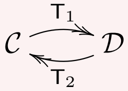
$$\mathcal{C} \text{ monadically stable} \iff \mathcal{C} \not\twoheadrightarrow \mathsf{LO}$$

Sparsification & Decomposition

Sparsification

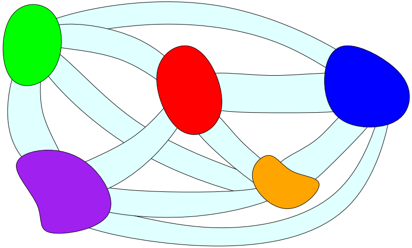
Problem

Find a $K_{s,s}$-free class $\mathcal{D}$ and transductions T_1, T_2 with $T_2 \circ T_1 = \mathrm{Id}$ and

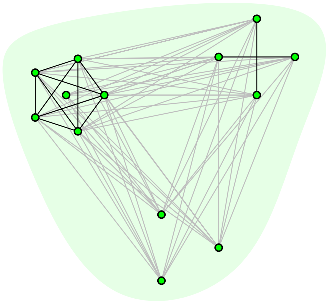


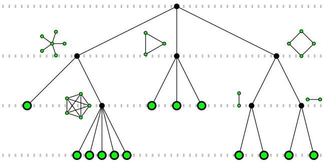
- If $\mathcal{C}$ is NIP then $\mathcal{D}$ is nowhere dense
- Model checking on $\mathsf{T}_1(G)$ can be transported on G.

Sparsification

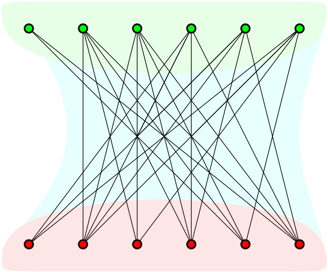


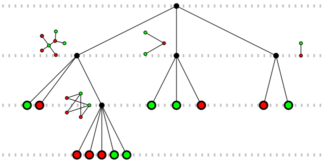
Vertex bloc: bounded depth cographs



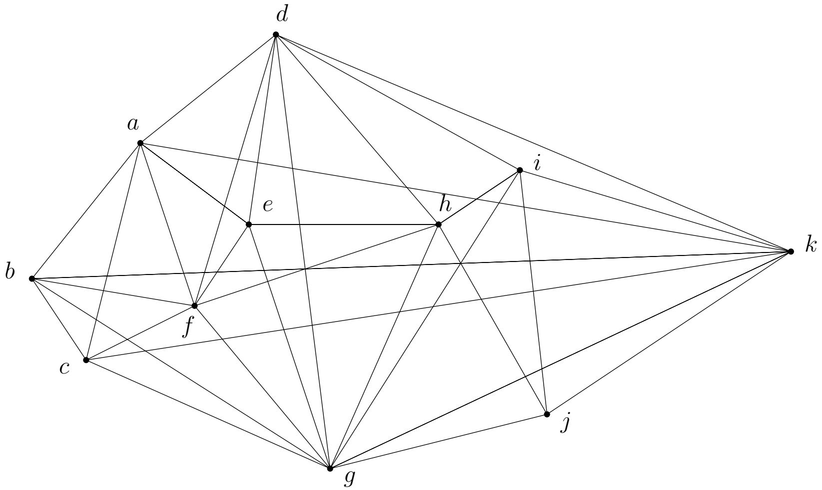


Edge bloc: bounded depth bi-cographs

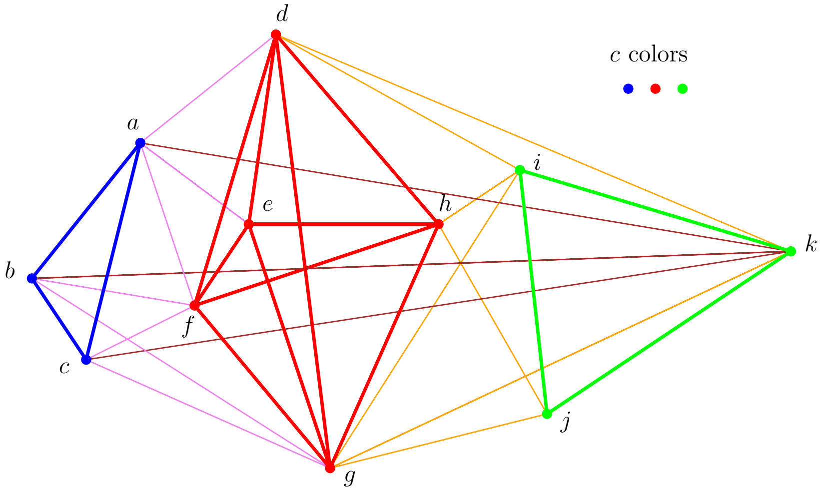




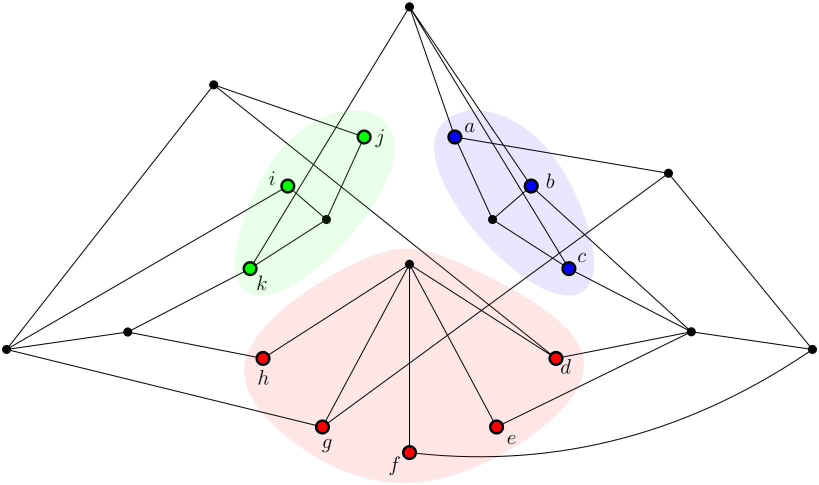
(c, d)-fold coloring



(c, d)-fold coloring



Sparsification: Cut & Paste



Structural Sparsity

Theorem (Gajarský, Kreutzer, Kwon, Nešetril, POM, Pilipczuk, Siebertz, Toruńczyk '18)

For a class of graphs $\mathcal{C}$ with (c, d)-fold coloring the following are equivalent:

- $\mathcal{C}$ has low shrub-depth decompositions
- Sparsify$(\mathcal{C})$ has tree-depth decompositions;
- Sparsify$(\mathcal{C})$ has bounded expansion.
- $\mathcal{C}$ has structurally bounded expansion;

If (c, d)-fold colorings can be computed in time $F(n)$ for $G \in \mathcal{C}$ then checking a first-order sentence ϕ on $\mathcal{C}$ can be done in time

$$F(n) + C(\phi, \mathcal{C})n.$$

Decompositions

Low rank-width decomposition $\Rightarrow$ χ-bounded

(Kwon, Pilipczuk, Siebertz '17)

$\Uparrow$

Low linear rank-width decomposition

$\Uparrow$

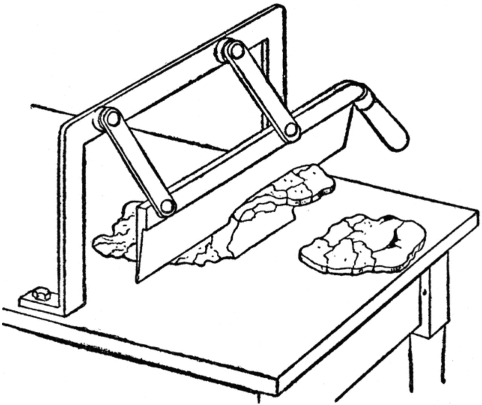
SBE $\Longleftrightarrow$ Low shrub-depth decomposition $\Rightarrow$ linearly χ-bounded

$\Updownarrow$ $\Uparrow$

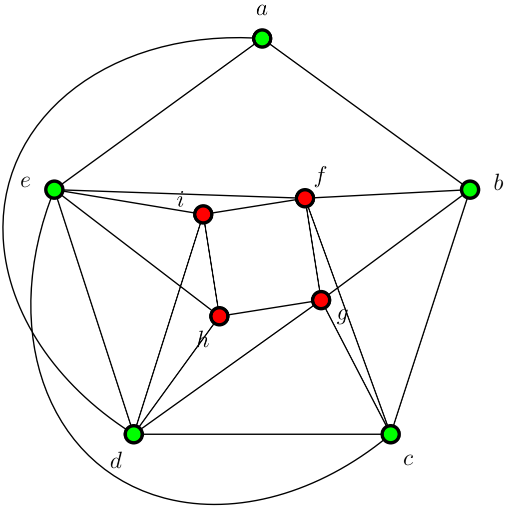
BE $\Longleftrightarrow$ Low tree-depth decomposition

Monadically stable

Rank-width



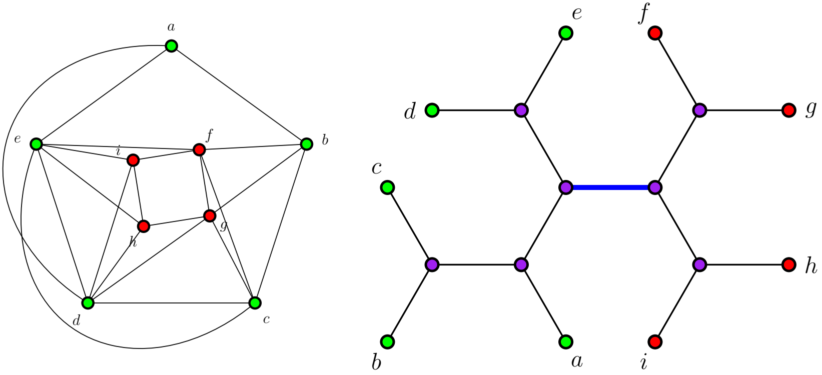
Rank-width



	a	b	c	d	e
f	0	1	1	0	1
g	0	1	1	1	0
h	0	0	0	1	1
i	0	0	0	1	1

	a	b	c	d	e
f	0	1	1	0	1
g	0	1	1	1	0
h	0	0	0	1	1
i	0	0	0	1	1

Rank-width



Rank-width and Linear rank-width

Theorem (from Colcombet '07)

- A class of finite graphs has bounded rank-width if and only it is a transduction of the class TO of finite tree orders

$$\mathsf{TO} \longrightarrow\!\!\!\!\!\rightarrow \mathcal{C}$$

- A class of finite graphs has bounded linear rank-width if and only it is a transduction of the class LO of finite linear orders

$$\mathsf{LO} \longrightarrow\!\!\!\!\!\rightarrow \mathcal{C}$$

Order without order

What happens if $\mathsf{LO} \twoheadrightarrow \mathcal{C} \not\twoheadrightarrow \mathsf{LO}$?

Order without order

What happens if $\mathsf{LO} \twoheadrightarrow \mathcal{C} \not\twoheadrightarrow \mathsf{LO}$?

Is it true that there is a standard class like $\mathcal{PW}_n$ such that

$$\mathsf{LO} \underset{\not\twoheadleftarrow}{\overset{\twoheadrightarrow}{\rightleftarrows}} \mathcal{PW}_n \twoheadrightarrow \mathcal{C} \ ?$$

Order without order

What happens if $\mathsf{LO} \twoheadrightarrow \mathcal{C} \not\twoheadrightarrow \mathsf{LO}$?
Is it true that there is a standard class like $\mathcal{PW}_n$ such that

$$\mathsf{LO} \twoheadrightarrow \mathcal{PW}_n \not\twoheadrightarrow \mathsf{LO}, \quad \mathcal{PW}_n \twoheadrightarrow \mathcal{C} \text{ ?}$$

And does $\mathsf{TO} \twoheadrightarrow \mathcal{C} \not\twoheadrightarrow \mathsf{LO}$ imply

$$\mathsf{TO} \twoheadrightarrow \mathcal{TW}_n, \quad \mathsf{TO} \twoheadrightarrow \mathsf{LO}, \quad \mathcal{TW}_n \not\twoheadrightarrow \mathsf{LO}, \quad \mathcal{TW}_n \twoheadrightarrow \mathcal{C} \text{ ?}$$

Rank-width and stability

Theorem (Nešetřil, POM, Rabinovich, Siebertz '19+)

Let $\mathcal{C}$ be a class of graphs. The following are equivalent:

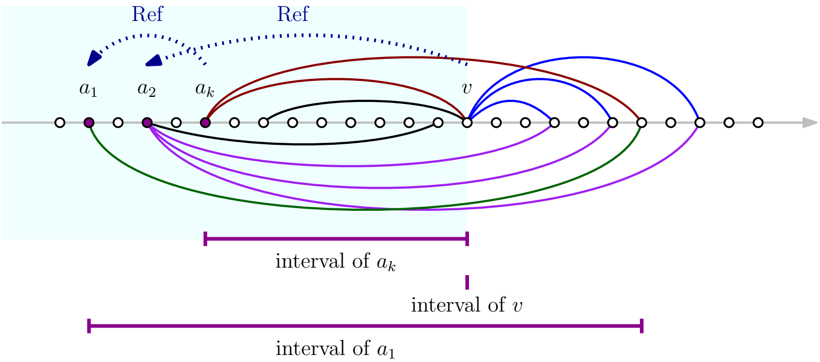
1. $\mathcal{C}$ has bounded linear rank-width and excludes some semi-induced half-graph,
2. $\mathcal{C}$ is a transduction of a class with bounded pathwidth.

Corollary

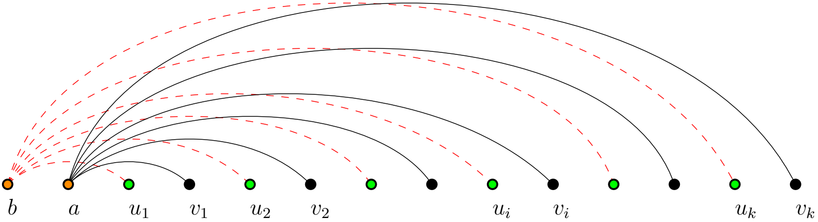
Let $\mathcal{C}$ be a class of graphs. The following are equivalent:

1. $\mathcal{C}$ is monadically stable and has low linear rank-width decompositions,
2. $\mathcal{C}$ has structurally bounded expansion.

Hint



Hint



More?

Conjecture

Let $\mathcal{C}$ be a class of graphs. The following are equivalent:

1. $\mathcal{C}$ has bounded rank-width and is monadically stable,
2. $\mathcal{C}$ is a transduction of a class with bounded treewidth.

If true, the following are equivalent:

1. $\mathcal{C}$ is monadically stable and has low rank-width decompositions,
2. $\mathcal{C}$ has structurally bounded expansion.

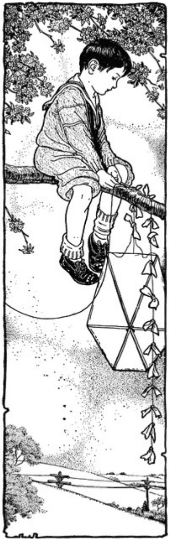
Dualities?

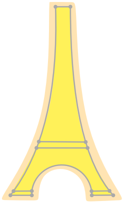
Conjecture

A class of graphs $\mathcal{C}$ has bounded shrub-depth if and only there is no surjective transduction from $\mathcal{C}$ to the class of all finite paths.

This would corresponds to a duality between bounded height trees and paths:

$$(\exists n)\ \ \mathcal{Y}_n \longrightarrow\!\!\!\!\!\rightarrow \mathcal{C} \qquad \Longleftrightarrow \qquad \mathcal{C} \not\longrightarrow\!\!\!\!\!\rightarrow \mathcal{P}$$





Thank you for your attention.