A model theoretic approach to sparsity

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Introduction
General View

Model Theory
Stability
NIP
Transduction
Sparsity
Decompositions
Structural Graph Theory
Encoding
Algorithmic Graph Theory
Kernels
FPT-algorithms
Model checking
PAC-learning
Sampling
Online-learning
The model theoretic universe

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NIP and VC-dimension

\[
G \models \phi(\bar{a}_i, \bar{b}_I) \iff i \in I
\]
Stability and Order property

\[ G \models \phi(\overline{a_i}, \overline{b_j}) \iff i < j \]
Computational learning theory

PAC learning $\iff$ NIP

Online learning $\iff$ Stability

(Laskovski ’92)
(Chase & Freitag ’18)
Adler & Adler ’14

For a monotone class $C$ tfae:

1. $C$ is nowhere dense,
2. $C$ is NIP,
3. $C$ is stable.

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Special Generic Structures

- NIP
- Stability
- Nowhere dense
- Bounded expansion
Computational complexity

Theorem (Grohe, Kreutzer, Siebertz ’14)
For every nowhere dense class $C$ and every $\epsilon > 0$, every property of graphs definable in first-order logic can be decided in time $O(n^{1+\epsilon})$ on $C$.

Theorem (Dvořák, Kráľ, Thomas ’10; Kreutzer ’11)
if a monotone class $C$ is somewhere dense, then deciding first-order properties of graphs in $C$ is not fixed-parameter tractable (unless $\text{FPT} = \text{W}[1]$).
Introduction

Computational complexity

Theorem (Gajarský, Hliněný, Lokshtanov, Ramanujan ’16)

Let $\mathcal{D}$ be a graph class interpretable in a bounded degree class. Then $\mathcal{D}$ has an FO model checking algorithm in FPT.

Conjecture (Gajarský et al. ’16)

Let $\mathcal{C}$ be a nowhere dense class and $\mathcal{D}$ a graph class interpretable in $\mathcal{C}$. Then $\mathcal{D}$ has an FO model checking algorithm in FPT.
Transductions
Transductions

How to encode graphs in a structure?

- Use a formula $\nu(x)$ to select the vertices,
- Use a formula $\eta(x, y)$ to define the edges,
- Use colors to encode several graphs in a same graph.

$C \longrightarrow D$
Example 1: blowing

Edgeless $\longrightarrow$ Blowing of $F$
Example 2: $k$-leaf power

Trees $\rightarrow k$-leaf powers
Example 3: map graph

Planar quadrangulations $\longrightarrow$ Map graphs
Example 4: bounded tree-depth

Bounded height trees $\rightarrow$ Bounded tree-depth graphs
Example 5: cograph

Tree orders → Cographs
Monotone closure

Proposition

\[ \chi_{st}(C) < \infty \implies C \rightarrow Monotone(C) \]

Let \( C \) be a class with star chromatic number at most \( k \). Then there exists a transduction \( T \) mapping each graph \( G \in C \) to the set of all the subgraphs of \( G \).
Proof

Let $\gamma : V(G) \rightarrow [k]$ be a star coloring of $G$, and let $H \subseteq G$. Color $v \in V(G)$ by

$$c(v) = (\mathbf{1}_{V(H)}(v), \gamma(v), \{\gamma(x) \mid xv \in E(H)\})$$

$$= (1, c_1, \{c_2, \ldots \}) \quad (1, c_2, \{c_1, \ldots \})$$
Proof

Let $\gamma : V(G) \to [k]$ be a star coloring of $G$, and let $H \subseteq G$. Color $v \in V(G)$ by

$$c(v) = (\mathbf{1}_{V(H)}(v), \gamma(v), \{\gamma(x) | xv \in E(H)\})$$

$$(1, c_1, \{c_2, \ldots\}) \\ (1, c_2, \{c_1, \ldots\})$$

$\notin E(H)$
Proof

Let $\gamma : V(G) \to [k]$ be a star coloring of $G$, and let $H \subseteq G$. Color $v \in V(G)$ by

$$c(v) = (1_{V(H)}(v), \gamma(v), \{\gamma(x) \mid xv \in E(H)\})$$

$$\in E(H)$$

$$\notin E(H)$$

$(1, c_1, \{c_2, \ldots\})$               $(1, c_2, \{c_1, \ldots\})$
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Proof

Let \( \gamma : V(G) \to [k] \) be a star coloring of \( G \), and let \( H \subseteq G \). Color \( v \in V(G) \) by

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c(v) = (1_{V(H)}(v), \gamma(v), \{\gamma(x) \mid xv \in E(H)\})
\]

\((1, c_1, \{c_2, \ldots\})\) \hspace{2cm} (1, c_2, \{c_1, \ldots\})

\(\in E(H)\) \hspace{2cm} \notin E(H) \hspace{2cm} \in E(H)

(1, c_2, \ldots) \hspace{2cm} (1, c_1, \ldots)\]
Monadic dependence and stability

• $\mathcal{C}$ is monadically NIP iff every definable class (in a monadic lift) has bounded VC-dimension

Theorem (Baldwin, Shelah ’85)

$$\mathcal{C} \text{ monadically NIP } \iff \mathcal{C} \quad \mathcal{G}$$

• $\mathcal{C}$ is monadically stable iff every definable class (in a monadic lift) has bounded Littlestone dimension

Theorem (Anderson ’90; Baldwin, Shelah ’85)

$$\mathcal{C} \text{ monadically stable } \iff \mathcal{C} \quad \mathcal{LO}$$
Sparsification & Decomposition
Sparsification

Problem

Find a $K_{s,s}$-free class $\mathcal{D}$ and transductions $T_1, T_2$ with $T_2 \circ T_1 = \text{Id}$ and

\[
\begin{array}{c}
\mathcal{C} \\
\downarrow \quad T_1 \\
\mathcal{D}
\end{array}
\quad \quad \quad
\begin{array}{c}
\mathcal{D} \\
\downarrow \quad T_2 \\
\mathcal{C}
\end{array}
\]

- If $\mathcal{C}$ is NIP then $\mathcal{D}$ is nowhere dense
- Model checking on $T_1(G)$ can be transported on $G$. 
Sparsification
Vertex bloc: bounded depth cographs
Edge bloc: bounded depth bi-cographs
(c, d)-fold coloring
(c, d)-fold coloring
Sparsification: Cut & Paste
Structural Sparsity

Theorem (Gajarský, Kreutzer, Kwon, Nešetril, POM, Pilipczuk, Siebertz, Toruńczyk ’18)

For a class of graphs $\mathcal{C}$ with $(c, d)$-fold coloring the following are equivalent:

- $\mathcal{C}$ has low shrub-depth decompositions
- $\text{Sparsify}(\mathcal{C})$ has tree-depth decompositions;
- $\text{Sparsify}(\mathcal{C})$ has bounded expansion.
- $\mathcal{C}$ has structurally bounded expansion.

If $(c, d)$-fold colorings can be computed in time $F(n)$ for $G \in \mathcal{C}$ then checking a first-order sentence $\phi$ on $\mathcal{C}$ can be done in time

$$F(n) + C(\phi, \mathcal{C})n.$$
Decompositions

Low rank-width decomposition $\Rightarrow$ $\chi$-bounded
(Kwon, Pilipczuk, Siebertz '17)

Low linear rank-width decomposition

Low shrub-depth decomposition $\Rightarrow$ linearly $\chi$-bounded

Low tree-depth decomposition

Monadically stable

SBE $\iff$ Low shrub-depth decomposition
BE $\iff$ Low tree-depth decomposition
Rank-width
Rank-width and Linear rank-width

Theorem (from Colcombet ’07)

• A class of finite graphs has bounded rank-width if and only if it is a transduction of the class $\text{TO}$ of finite tree orders

$\text{TO} \rightarrow \mathcal{C}$

• A class of finite graphs has bounded linear rank-width if and only if it is a transduction of the class $\text{LO}$ of finite linear orders

$\text{LO} \rightarrow \mathcal{C}$
Order without order

What happens if $\text{LO} \rightarrow C \rightarrow \text{LO}$?
Order without order

What happens if $\text{LO} \rightarrow C \rightarrow \text{LO}$?
Is it true that there is a standard class like $\mathcal{PW}_n$ such that

$$\text{LO} \leftrightarrow \mathcal{PW}_n \rightarrow C$$
Order without order

What happens if \( \text{LO} \rightarrow \mathcal{C} \rightarrow \text{LO} \)?

Is it true that there is a standard class like \( \mathcal{PW}_n \) such that

\[
\text{LO} \leftrightarrow \mathcal{PW}_n \rightarrow \mathcal{C} ?
\]

And does \( \text{TO} \rightarrow \mathcal{C} \rightarrow \text{LO} \) imply

\[
\text{TO} \rightarrow \mathcal{TW}_n \rightarrow \mathcal{C} ?
\]
Theorem (Nešetřil, POM, Rabinovich, Siebertz ’19+)

Let $\mathcal{C}$ be a class of graphs. The following are equivalent:

1. $\mathcal{C}$ has bounded linear rank-width and excludes some semi-induced half-graph,
2. $\mathcal{C}$ is a transduction of a class with bounded pathwidth.

Corollary

Let $\mathcal{C}$ be a class of graphs. The following are equivalent:

1. $\mathcal{C}$ is monadically stable and has low linear rank-width decompositions,
2. $\mathcal{C}$ has structurally bounded expansion.
**Hint**

- Interval of $a_k$
- Interval of $v$
- Interval of $a_1$

Diagram showing nodes labeled $a_1$, $a_2$, $a_k$, and $v$ with labeled intervals.
Hint
Conjecture

Let $\mathcal{C}$ be a class of graphs. The following are equivalent:

1. $\mathcal{C}$ has bounded rank-width and is monadically stable,
2. $\mathcal{C}$ is a transduction of a class with bounded treewidth.

If true, the following are equivalent:

1. $\mathcal{C}$ is monadically stable and has low rank-width decompositions,
2. $\mathcal{C}$ has structurally bounded expansion.
Conjecture

A class of graphs $\mathcal{C}$ has bounded shrub-depth if and only there is no surjective transduction from $\mathcal{C}$ to the class of all finite paths.

This would corresponds to a duality between bounded height trees and paths:

$$\left( \exists n \right) \mathcal{Y}_n \longrightarrow \mathcal{C} \iff \mathcal{C} \longrightarrow \mathcal{P}$$
Thank you for your attention.