On recent results by Zuiddam and Bukh & Cox on the Shannon capacity

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with Sven Polak

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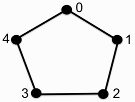

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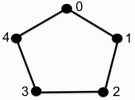

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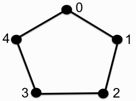

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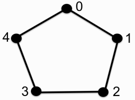

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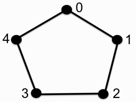

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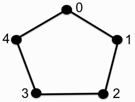

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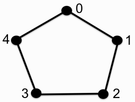

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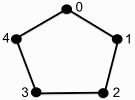

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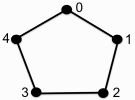

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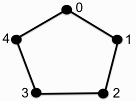

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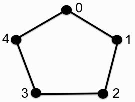

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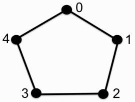

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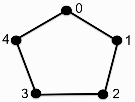

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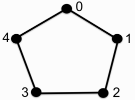

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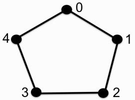

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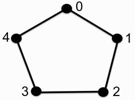

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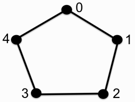

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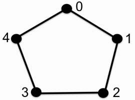

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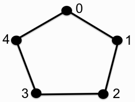

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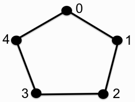

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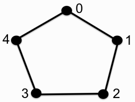

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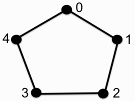

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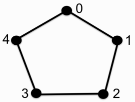

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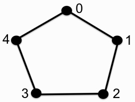

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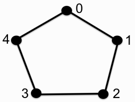

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On the Shannon capacity of a graph

by

L. Lovász (Szeged)

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It is easily seen that

$$\Theta(G) \overset{def}{=} \sup_k \sqrt[k]{\alpha(G^k)} = \lim_{k\to\infty} \sqrt[k]{\alpha(G^k)} .$$

This number was introduced by Shannon [3], and is called the <u>Shannon capacity</u> of the graph G. The previous consideration shows that $\alpha(G) \leqq \Theta(G)$ and that, in general, equality does not hold.

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Proof. The "diagonal" in $G\cdot\overline{G}$ is independent, hence

$$\Theta(G\cdot\overline{G}) \geq \alpha(G\cdot\overline{G}) \geq |V(G)|.$$

On the other hand, we have by Theorems 1, 6 and 7 that

$$\Theta(G\cdot\overline{G}) \leqq \vartheta(G\cdot\overline{G}) = \vartheta(G)\vartheta(\overline{G}) = |V(G)|.$$

If G is self-complementary then

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This proves the theorem. The proof also shows that in these cases $\Theta = \vartheta$.

Theorem 12. Let $n \geq 2r$ and let the graph $K(n,r)$ be defined as the graph whose vertices are the r-subsets of an n-element set S, two being adjacent iff they are disjoint. Then

$$\Theta(K(n,r)) = \binom{n-1}{r-1}.$$

Corollary (Erdős-Ko-Rado Theorem)

$$\alpha(K(n,r)) = \binom{n-1}{r-1}.$$

Proof. The r-subsets containing a specified element of S form an independent set of points in $K(n,r)$; hence

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Theorem 12. Let $n \geq 2r$ and let the graph $K(n,r)$ be defined as the graph whose vertices are the r-subsets of an n-element set S, two being adjacent iff they are disjoint. Then

$$\Theta(K(n,r)) = \binom{n-1}{r-1}.$$

Corollary (Erdős-Ko-Rado Theorem)

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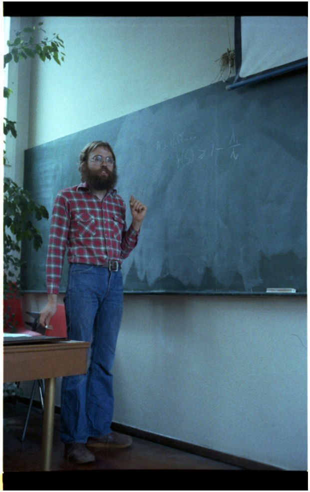


Willem Haemers 28/2/78.

Let G be a graph on n vertices. Let A be a symmetric matrix of size n such that $(A)_{ii} = 1$, $(A)_{ij} = 0$ iff i and j correspond to nonadjacent vertices of G.

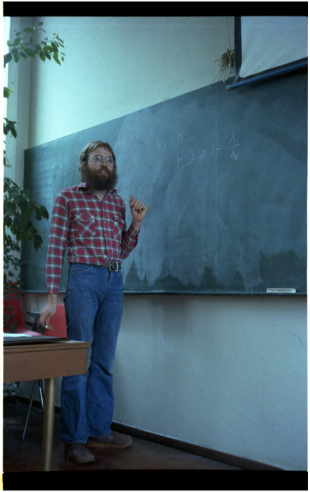
<u>Theorem</u> $\Theta(G) \leq \text{rank}(A)$

<u>Proof</u> 1°. $\text{rank}(A^{\circ k}) = (\text{rank}(A))^k$.
2°. $A^{\circ k}$ has submatrix I of size $\alpha(A^{\circ k})$, hence $\text{rank}(A^{\circ k}) \geqslant \alpha(A^{\circ k})$. We now have

$$\Theta(G) = \sup \sqrt[k]{\alpha(A^{\circ k})} \leqslant \sup \sqrt[k]{\text{rank}(A^{\circ k})} = \text{rank}(A) \quad \square$$

<u>Example</u> G is the Schläfli graph on 27 vertices. The eigenvalues of its (0,1) adjacency matrix B are $1, -5, 10$ with multiplicities $20, 6$ and 1 resp. Put $A := I - B$, then $\text{rank}(A) = 7$, so $\Theta(G) \leq 7$.
G and $\bar{G}$ are edge transitive, so we have $\vartheta(G) = 27 \frac{5}{15} = 9$, $\vartheta(\bar{G}) = \frac{27}{9} = 3$, in fact $\Theta(\bar{G}) = 3$ because $\bar{G}$ admits a 3-coclique.

So in the above example we have seen:
$\Theta(G) \neq \vartheta(G)$ and $\Theta(G) \cdot \Theta(\bar{G}) \leq 21 < 27$.
This answers problem 2,3 and part of problem 1 of Lovász' paper.



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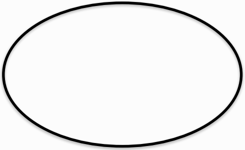

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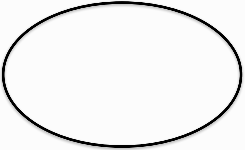

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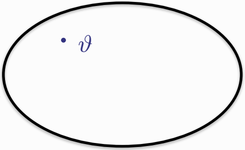

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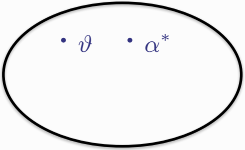

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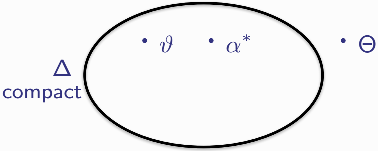

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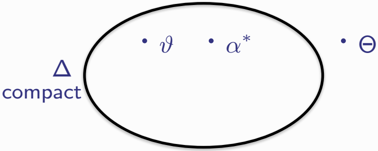

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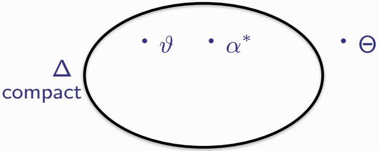

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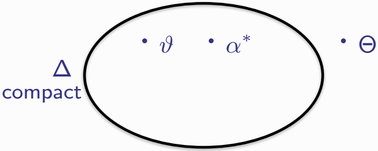

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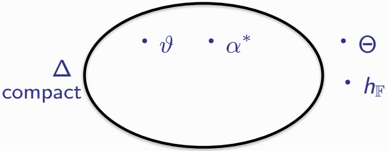

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The fractional Haemers bound

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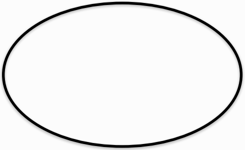

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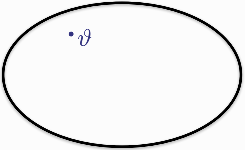

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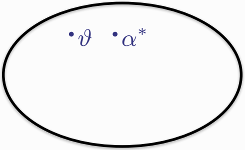

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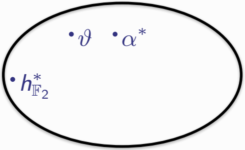

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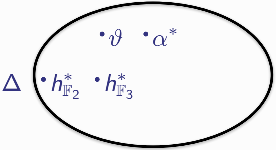

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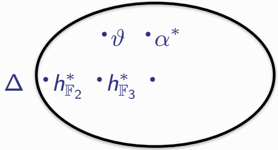

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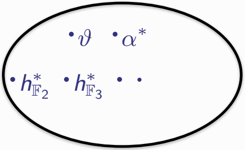

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Let $\mathbb{F}$ be a field and $G = (V, E)$ be a graph.

Haemers bound:

$$\Theta(G) \leq h_{\mathbb{F}}(G) := \min\{\mathrm{rank}(X) \mid X \in \mathbb{F}^{V \times V}, \forall v : X_{v,v} = 1, \ \forall u \not\simeq v : X_{u,v} = 0\}.$$

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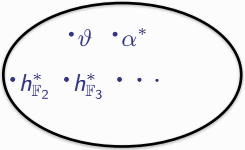
Bukh & Cox 2018: Theorem: $h^*_{\mathbb{F}}$ is multiplicative

$\implies h^*_{\mathbb{F}}$ belongs to Δ.

Δ



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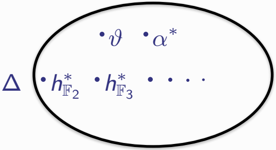

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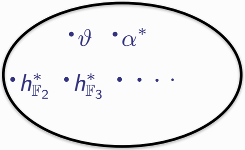

$\lim_{p \to \infty} h^*_{\mathbb{F}_p} = h^*_{\mathbb{Q}}$?

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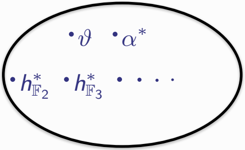

$\lim_{p\to\infty} h^*_{\mathbb{F}_p} = h^*_{\mathbb{Q}}$? $\lim_{p\to\infty} h^*_{\overline{\mathbb{F}_p}} = h^*_{\overline{\mathbb{Q}}}$?

“Right convergence”

"Right convergence"

Recall: $\Delta := \{f : \{\text{graphs}\} \to \mathbb{R} \mid$ for all graphs G, H :
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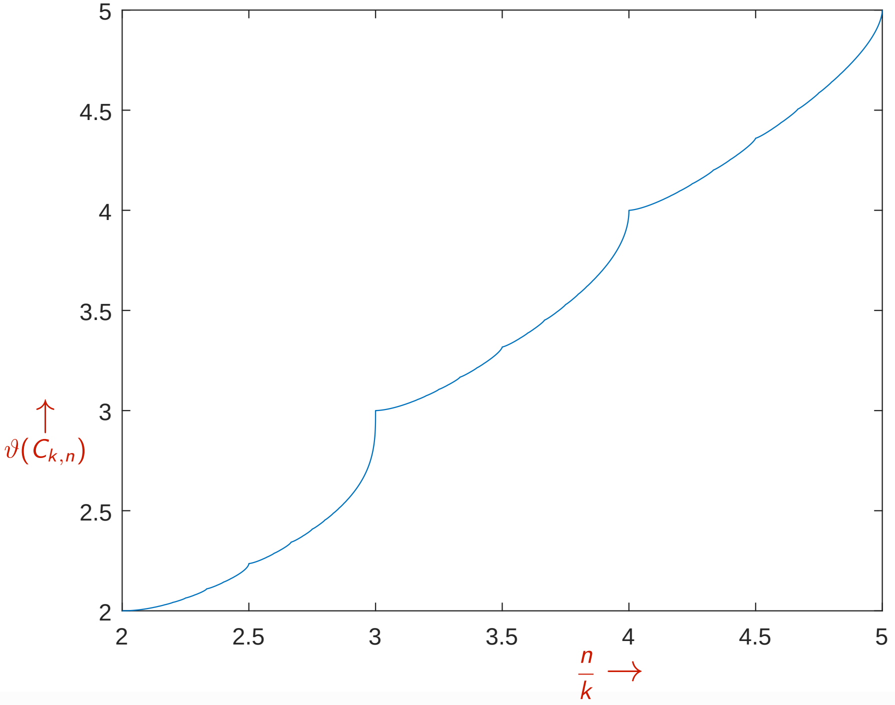

5
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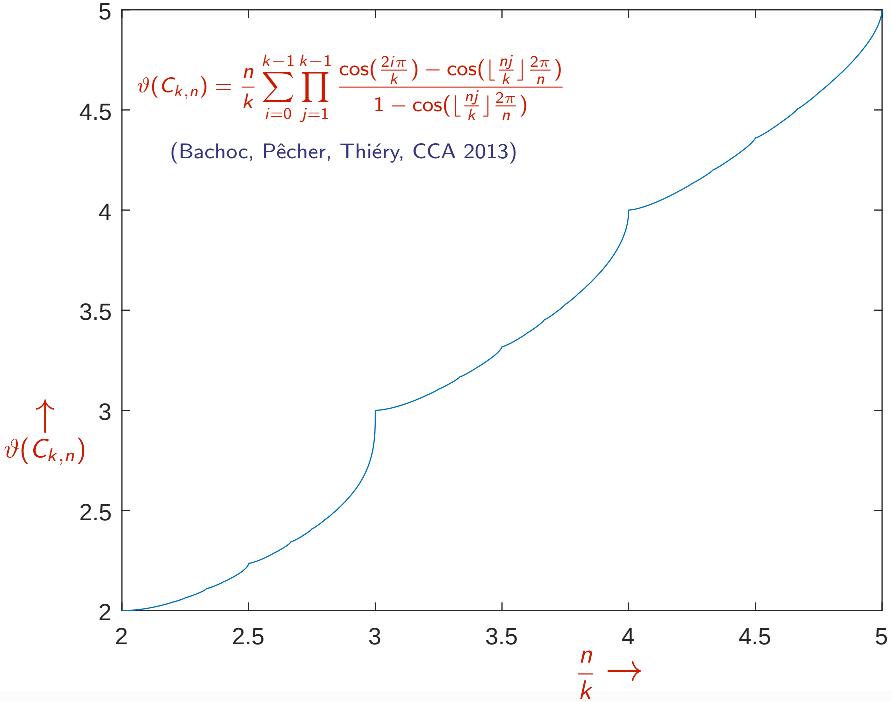
5
$\vartheta(C_{k,n})$ ↑
$\frac{n}{k}$ →



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(Bachoc, Pêcher, Thiéry, CCA 2013)
$\uparrow$ $\vartheta(C_{k,n})$
$\frac{n}{k} \longrightarrow$
2
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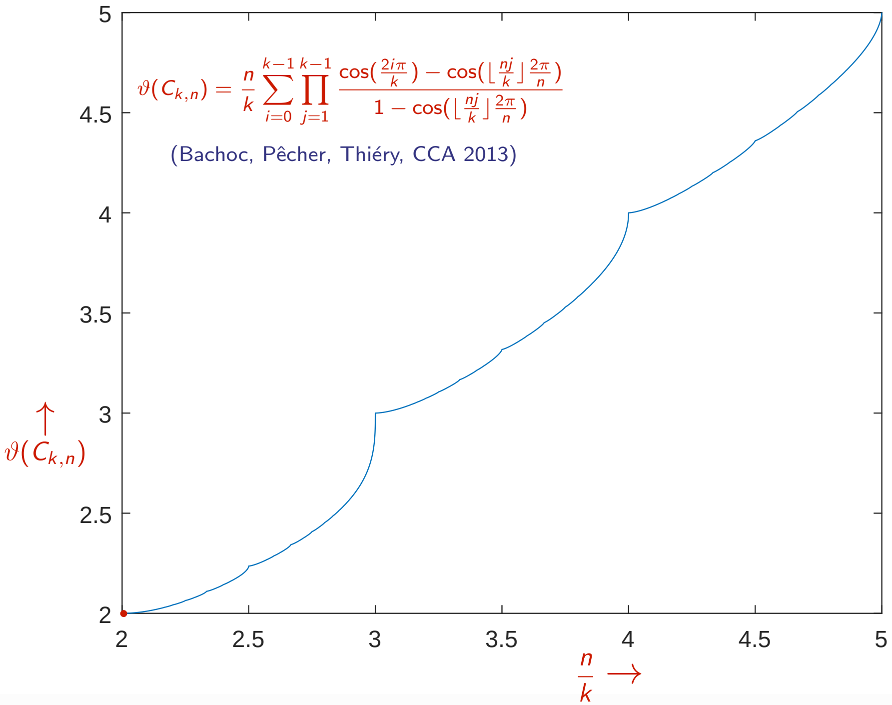
5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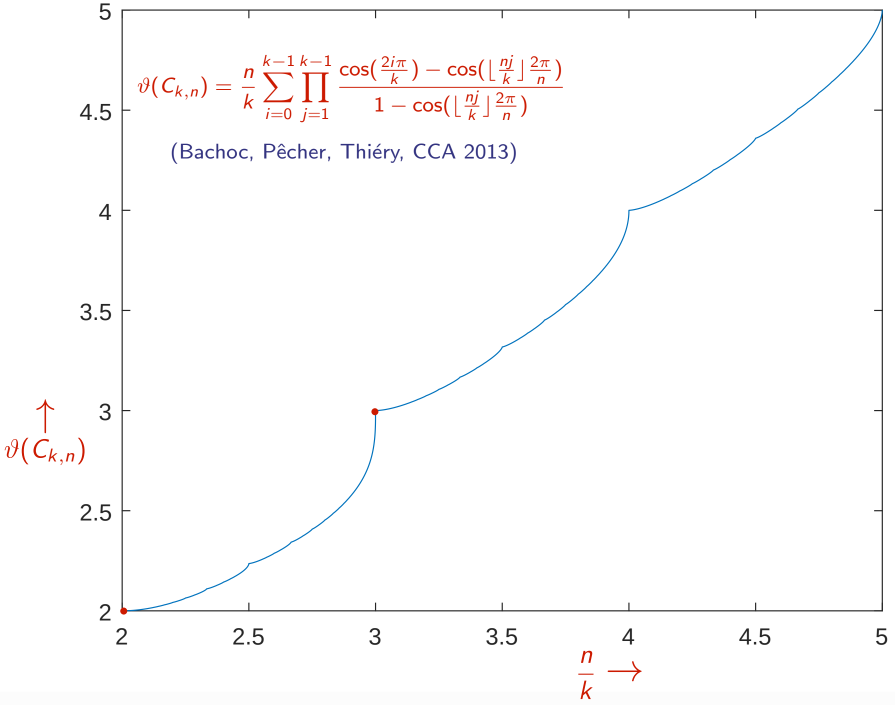

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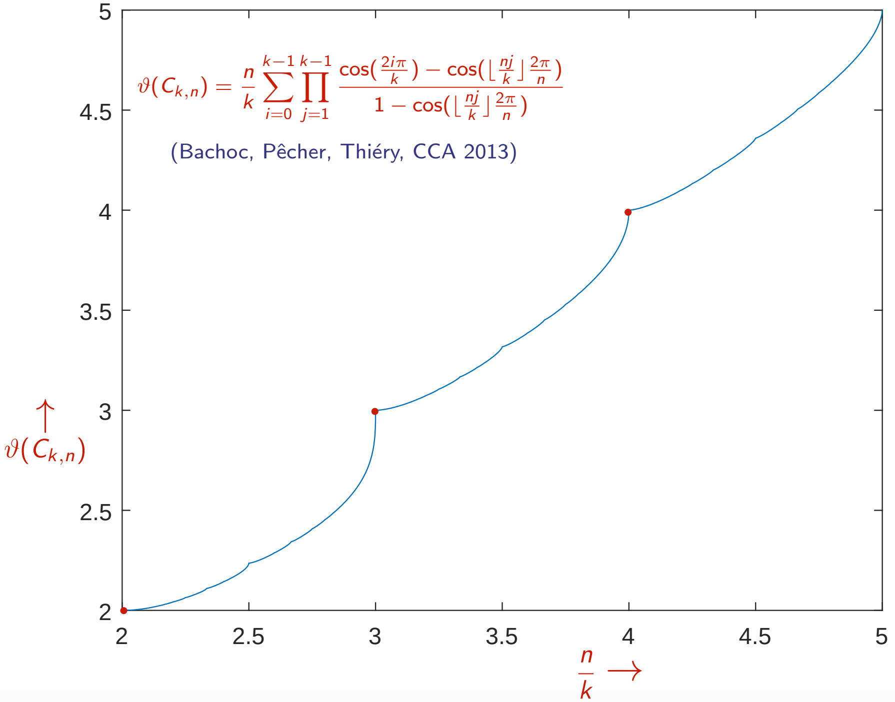

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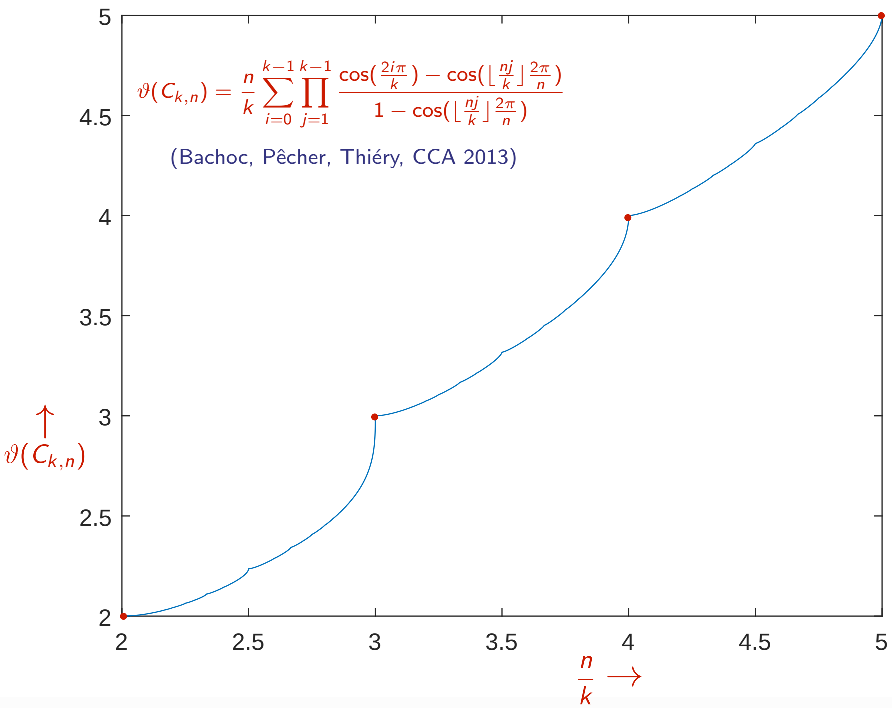
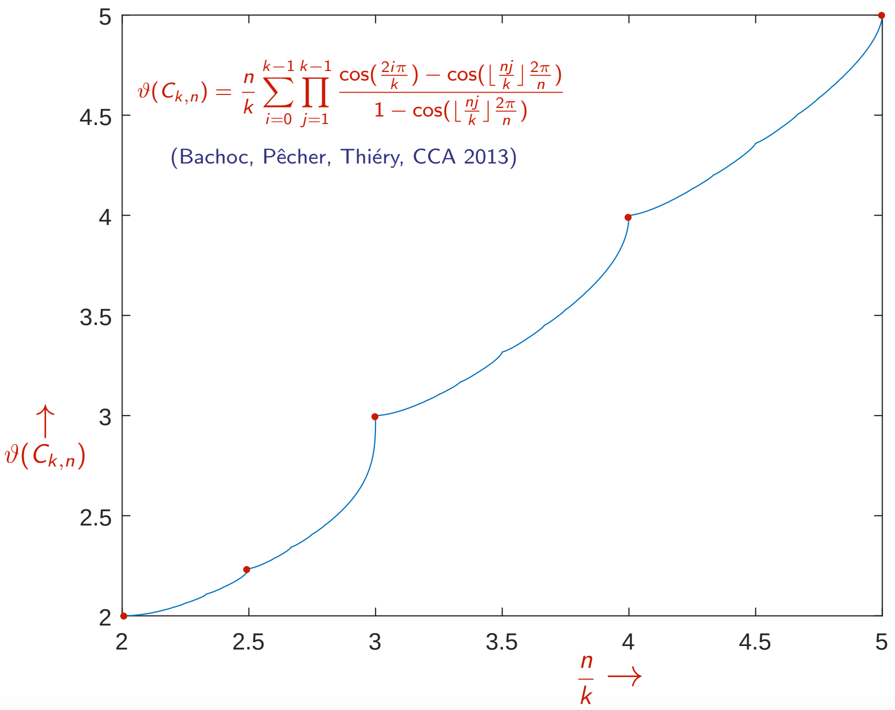
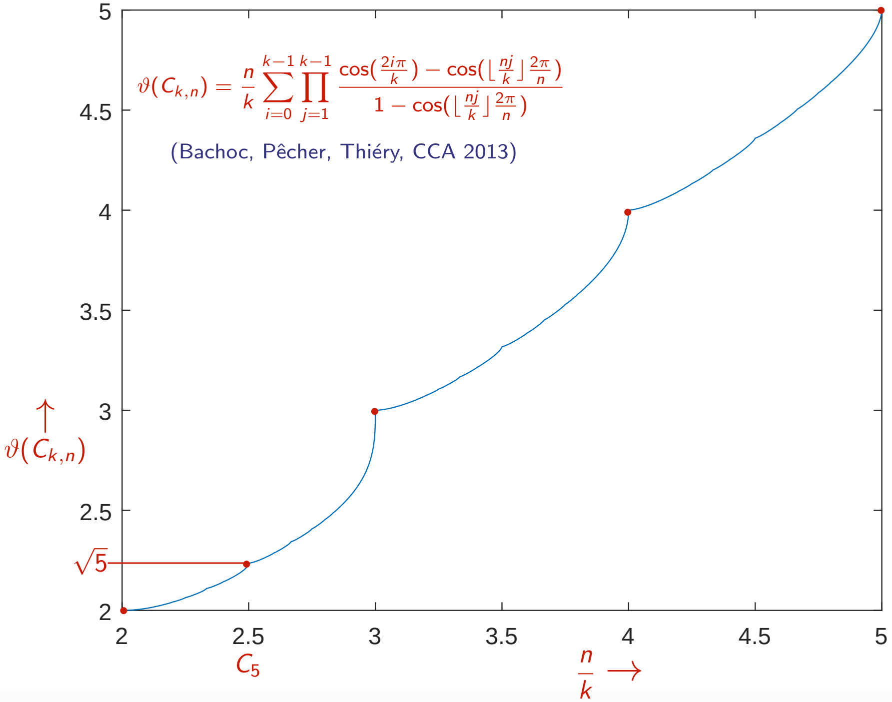
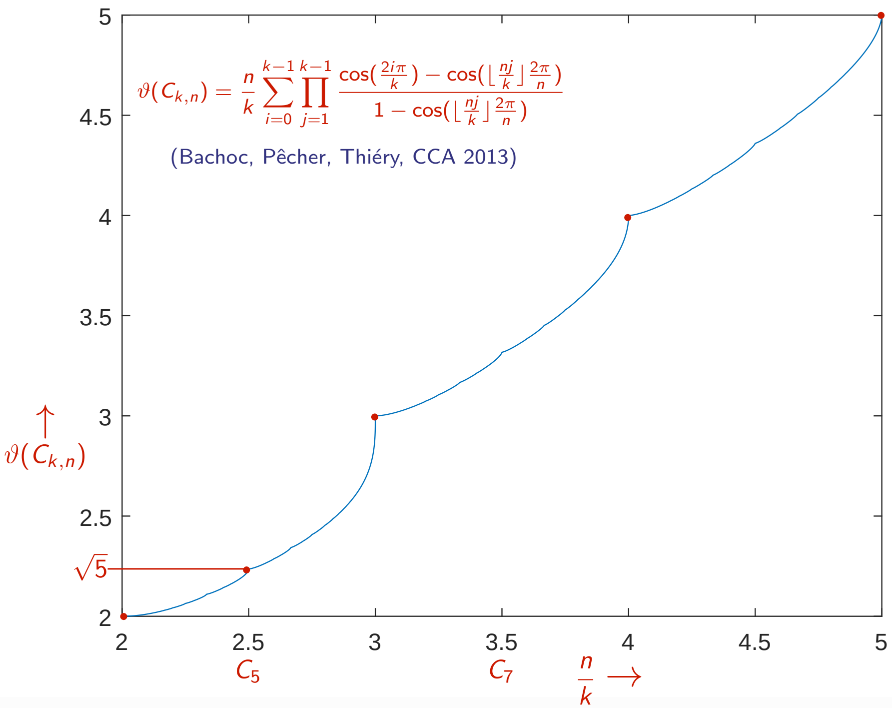
2
2.5
3
3.5
4
4.5
5

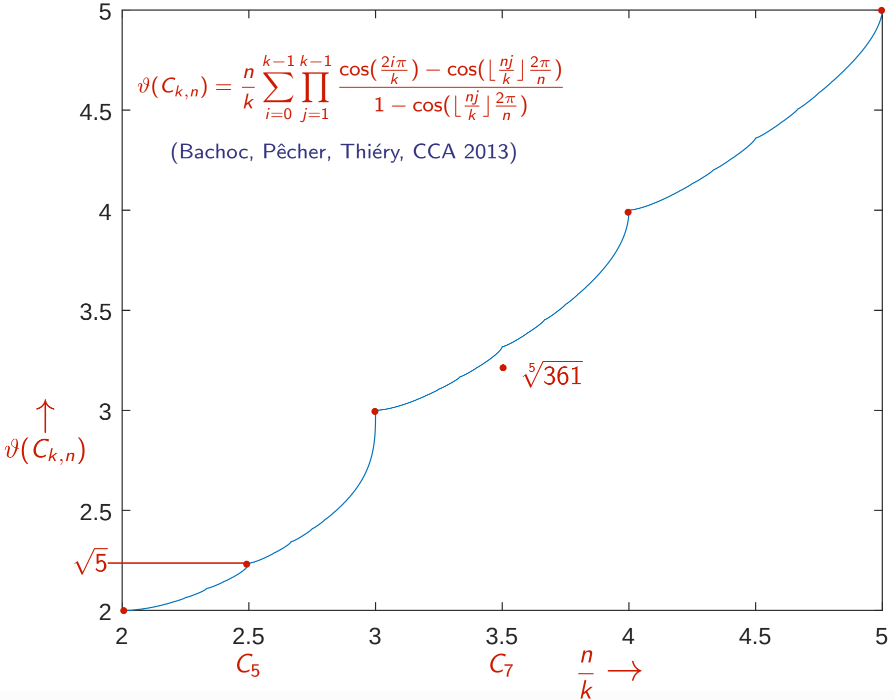


$\vartheta(C_{k,n}) = \frac{n}{k} \sum_{i=0}^{k-1} \prod_{j=1}^{k-1} \frac{\cos(\frac{2i\pi}{k}) - \cos(\lfloor \frac{nj}{k} \rfloor \frac{2\pi}{n})}{1 - \cos(\lfloor \frac{nj}{k} \rfloor \frac{2\pi}{n})}$
(Bachoc, Pêcher, Thiéry, CCA 2013)
$\uparrow \vartheta(C_{k,n})$
$\frac{n}{k} \rightarrow$
5
4.5
4
3.5
3
2.5
2
2.5
3
3.5
4
4.5
5

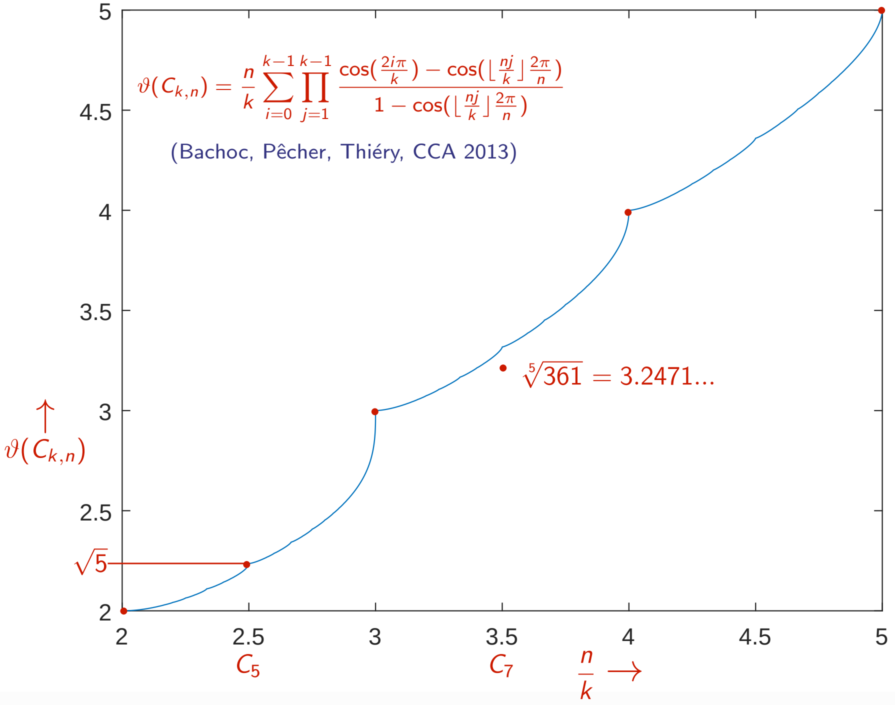

$\vartheta(C_{k,n}) = \frac{n}{k} \sum_{i=0}^{k-1} \prod_{j=1}^{k-1} \frac{\cos(\frac{2i\pi}{k}) - \cos(\lfloor \frac{nj}{k} \rfloor \frac{2\pi}{n})}{1 - \cos(\lfloor \frac{nj}{k} \rfloor \frac{2\pi}{n})}$
(Bachoc, Pêcher, Thiéry, CCA 2013)
$\uparrow$ $\vartheta(C_{k,n})$
$\frac{n}{k} \longrightarrow$
2
2.5
3
3.5
4
4.5
5


$\vartheta(C_{k,n}) = \frac{n}{k} \sum_{i=0}^{k-1} \prod_{j=1}^{k-1} \frac{\cos(\frac{2i\pi}{k}) - \cos(\lfloor \frac{nj}{k} \rfloor \frac{2\pi}{n})}{1 - \cos(\lfloor \frac{nj}{k} \rfloor \frac{2\pi}{n})}$
(Bachoc, Pêcher, Thiéry, CCA 2013)
$\uparrow$ $\vartheta(C_{k,n})$
$\sqrt{5}$
C_5
$\frac{n}{k} \longrightarrow$
5
4.5
4
3.5
3
2.5
2
2
2.5
3
3.5
4
4.5
5

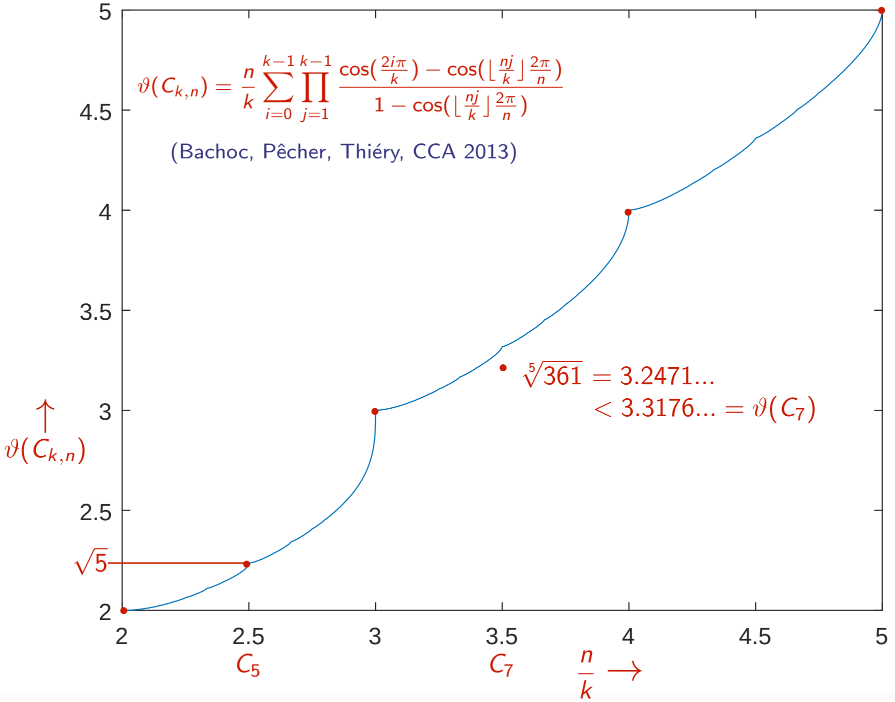

$\vartheta(C_{k,n}) = \frac{n}{k}\sum_{i=0}^{k-1}\prod_{j=1}^{k-1}\frac{\cos(\frac{2i\pi}{k}) - \cos(\lfloor\frac{nj}{k}\rfloor\frac{2\pi}{n})}{1-\cos(\lfloor\frac{nj}{k}\rfloor\frac{2\pi}{n})}$
(Bachoc, Pêcher, Thiéry, CCA 2013)
$\uparrow$ $\vartheta(C_{k,n})$
$\sqrt{5}$
C_5
C_7
$\frac{n}{k} \longrightarrow$
5
4.5
4
3.5
3
2.5
2
2
2.5
3
3.5
4
4.5
5



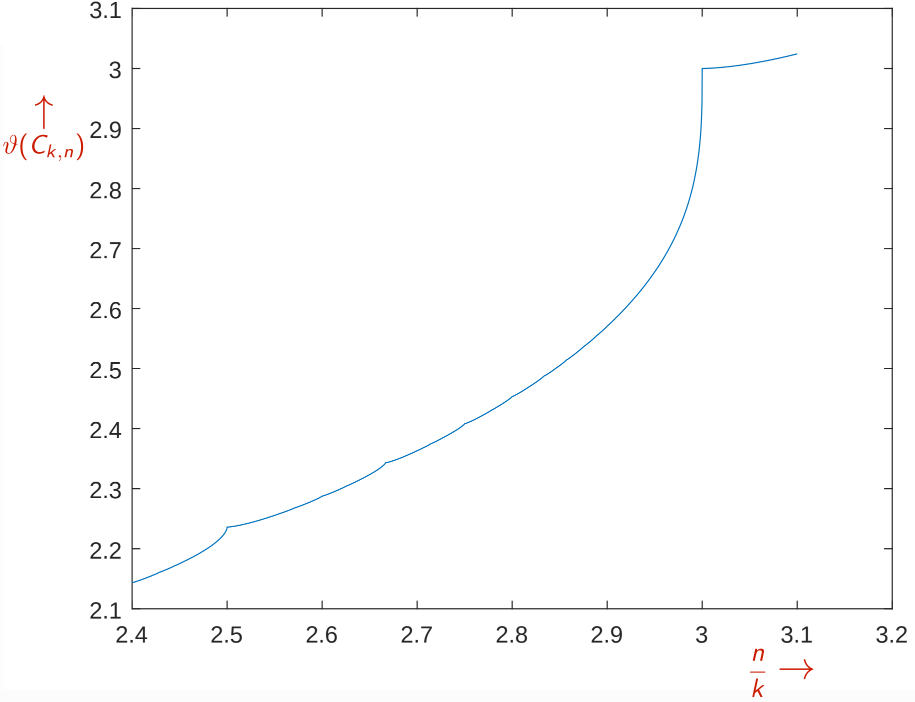
$\vartheta(C_{k,n}) = \frac{n}{k} \sum_{i=0}^{k-1} \prod_{j=1}^{k-1} \frac{\cos(\frac{2i\pi}{k}) - \cos(\lfloor \frac{nj}{k} \rfloor \frac{2\pi}{n})}{1 - \cos(\lfloor \frac{nj}{k} \rfloor \frac{2\pi}{n})}$
(Bachoc, Pêcher, Thiéry, CCA 2013)
$\sqrt[5]{361}$
$\sqrt{5}$
$\uparrow$ $\vartheta(C_{k,n})$
C_5
C_7
$\frac{n}{k} \longrightarrow$
5
4.5
4
3.5
3
2.5
2
2
2.5
3
3.5
4
4.5
5



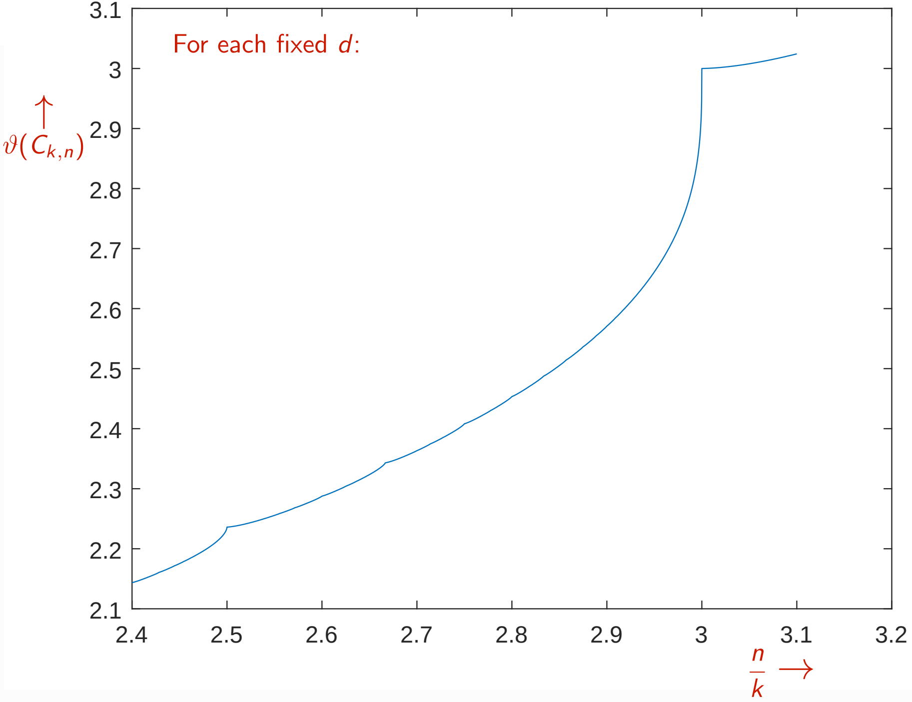
$\vartheta(C_{k,n}) = \frac{n}{k}\sum_{i=0}^{k-1}\prod_{j=1}^{k-1}\frac{\cos(\frac{2i\pi}{k}) - \cos(\lfloor\frac{nj}{k}\rfloor\frac{2\pi}{n})}{1-\cos(\lfloor\frac{nj}{k}\rfloor\frac{2\pi}{n})}$
(Bachoc, Pêcher, Thiéry, CCA 2013)
$\sqrt[5]{361} = 3.2471\ldots$
$\sqrt{5}$
$\uparrow \vartheta(C_{k,n})$
$\frac{n}{k} \longrightarrow$
C_5
C_7
2
2.5
3
3.5
4
4.5
5



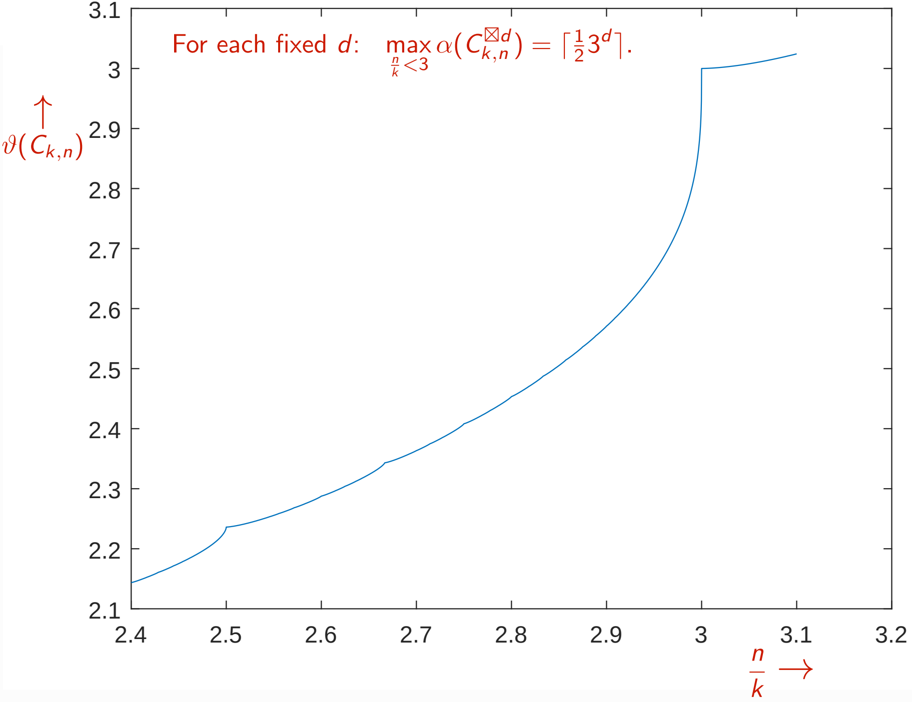
$\vartheta(C_{k,n}) = \frac{n}{k}\sum_{i=0}^{k-1}\prod_{j=1}^{k-1}\frac{\cos(\frac{2i\pi}{k}) - \cos(\lfloor\frac{nj}{k}\rfloor\frac{2\pi}{n})}{1-\cos(\lfloor\frac{nj}{k}\rfloor\frac{2\pi}{n})}$
(Bachoc, Pêcher, Thiéry, CCA 2013)
$\sqrt[5]{361} = 3.2471...$
$< 3.3176... = \vartheta(C_7)$
$\uparrow$ $\vartheta(C_{k,n})$
$\sqrt{5}$
5
4.5
4
3.5
3
2.5
2
2
2.5
3
3.5
4
4.5
5
C_5
C_7
$\frac{n}{k} \longrightarrow$



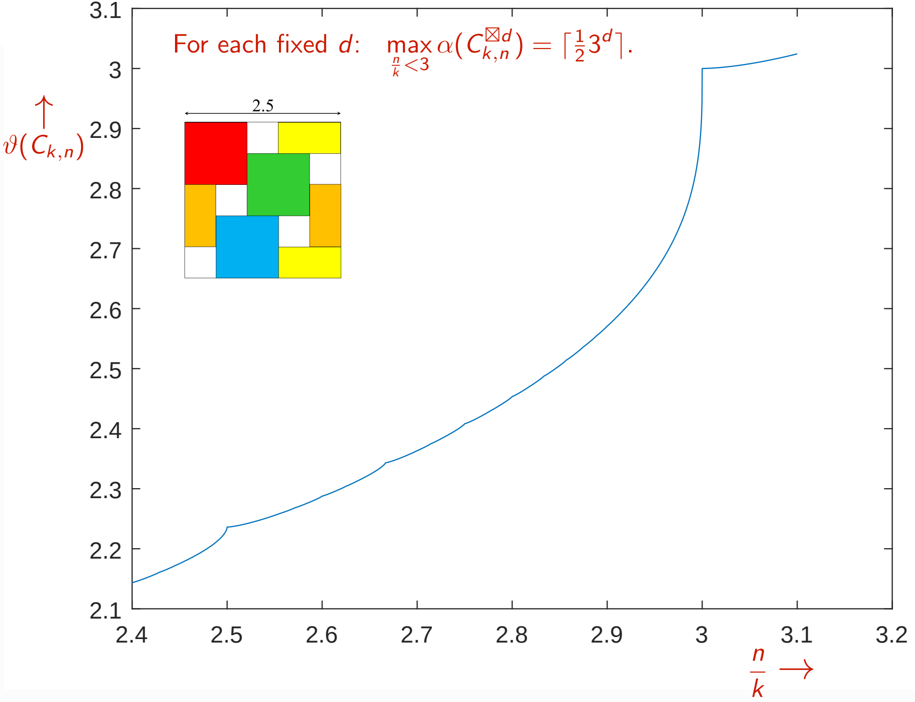
3.1
3
2.9
2.8
2.7
2.6
2.5
2.4
2.3
2.2
2.1
$\vartheta(C_{k,n})$ ↑
2.4
2.5
2.6
2.7
2.8
2.9
3
3.1
3.2
$\frac{n}{k}$ →

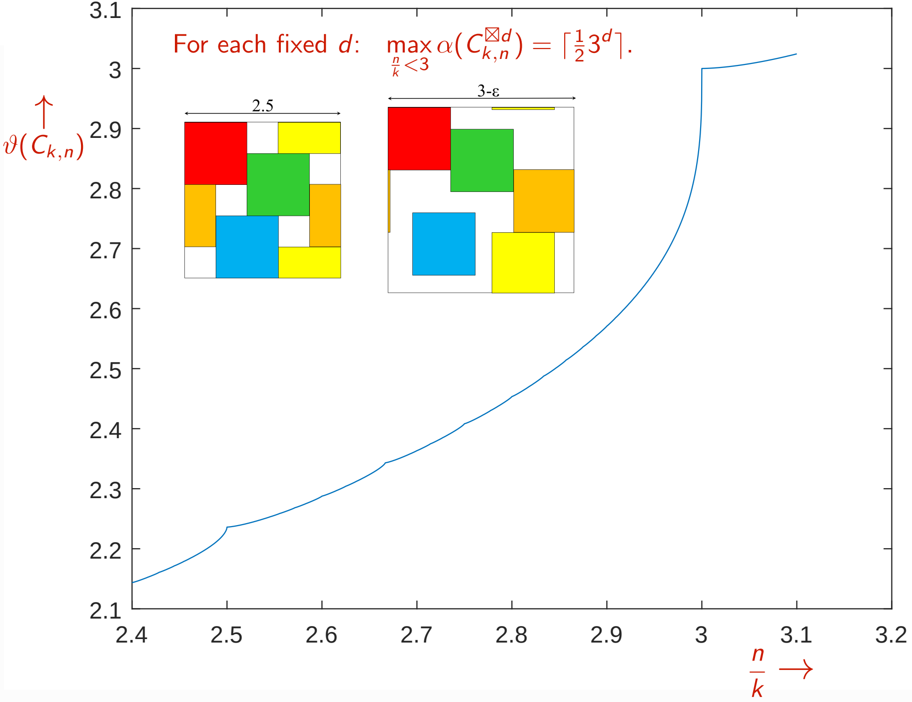


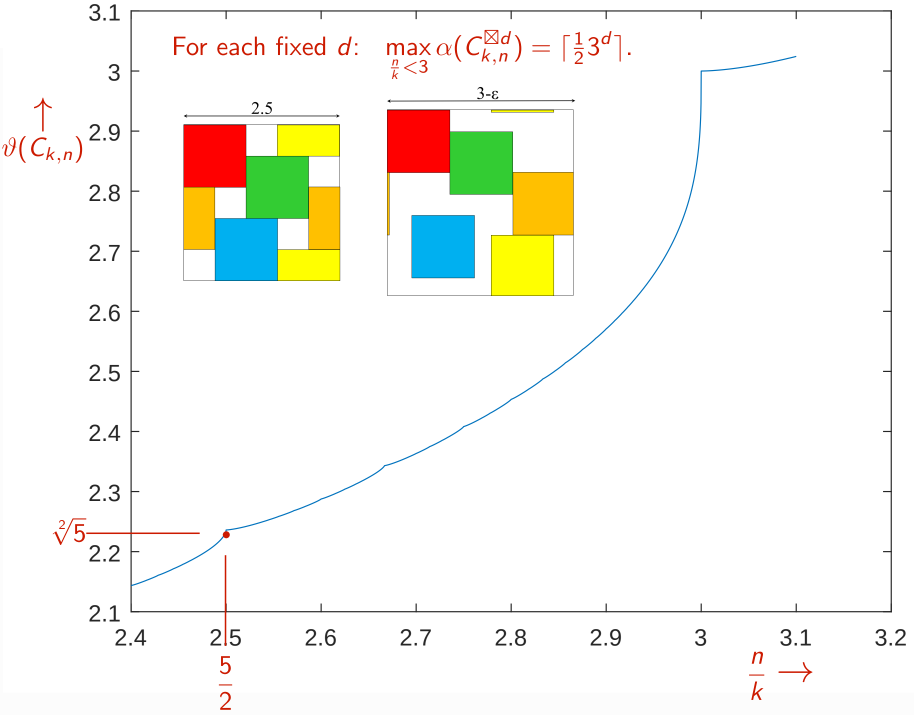
For each fixed d:
$\uparrow$
$\vartheta(C_{k,n})$
3.1
3
2.9
2.8
2.7
2.6
2.5
2.4
2.3
2.2
2.1
2.4
2.5
2.6
2.7
2.8
2.9
3
3.1
3.2
$\frac{n}{k} \longrightarrow$

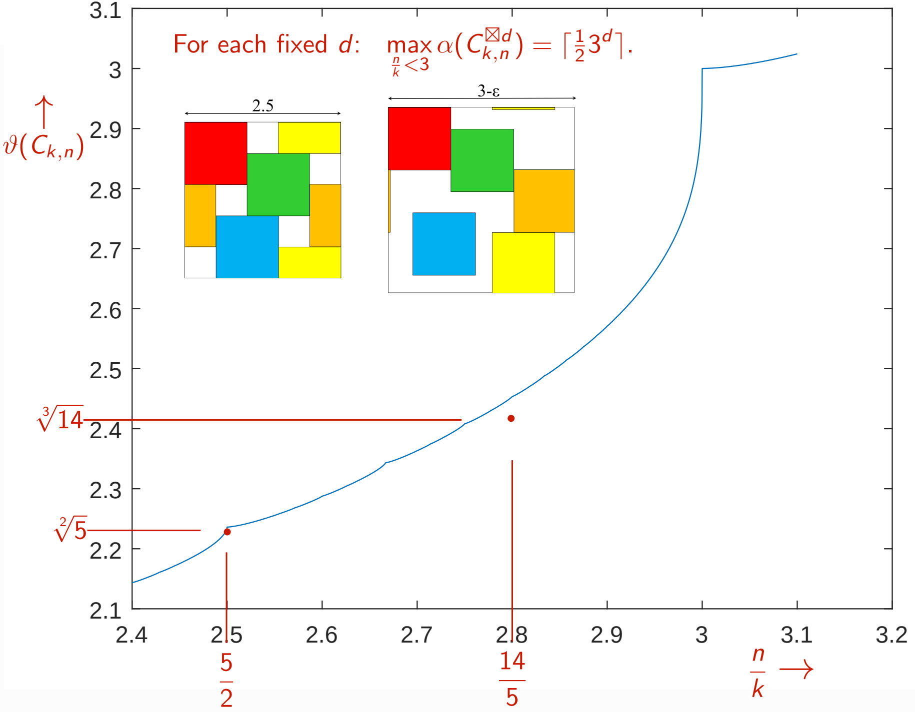


For each fixed d: $\max_{\frac{n}{k}<3} \alpha(C_{k,n}^{\boxtimes d}) = \lceil \frac{1}{2} 3^d \rceil$.
$\uparrow$ $\vartheta(C_{k,n})$
3.1
3
2.9
2.8
2.7
2.6
2.5
2.4
2.3
2.2
2.1
2.4
2.5
2.6
2.7
2.8
2.9
3
3.1
3.2
$\frac{n}{k} \longrightarrow$

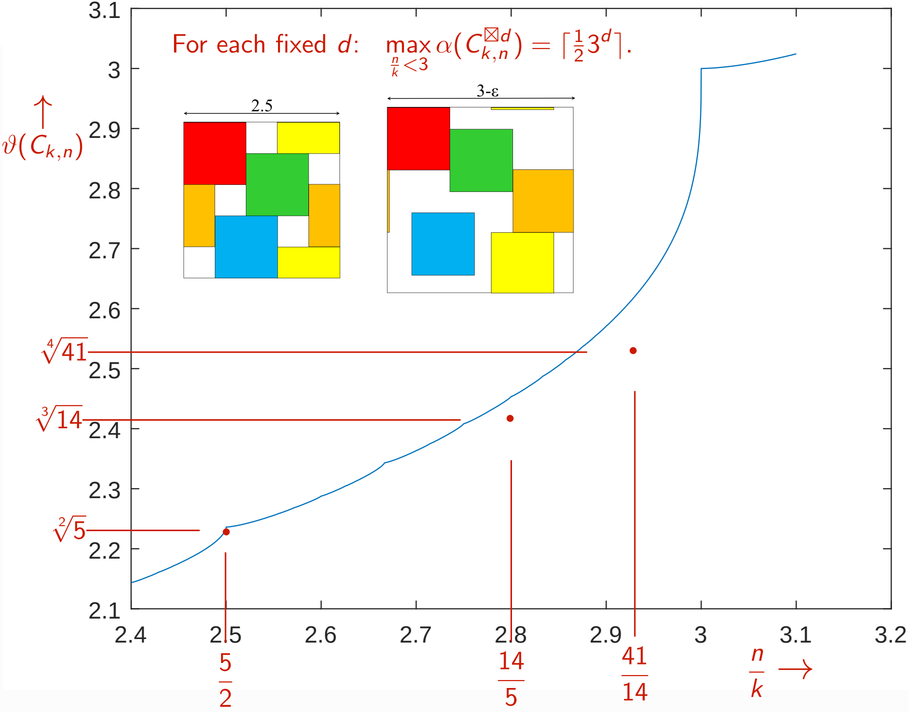

For each fixed d: $\max_{\frac{n}{k}<3} \alpha(C_{k,n}^{\boxtimes d}) = \lceil \frac{1}{2} 3^d \rceil$.
2.5
$\uparrow$
$\vartheta(C_{k,n})$
3.1
3
2.9
2.8
2.7
2.6
2.5
2.4
2.3
2.2
2.1
2.4
2.5
2.6
2.7
2.8
2.9
3
3.1
3.2
$\frac{n}{k} \longrightarrow$


For each fixed d: $\max_{\frac{n}{k}<3} \alpha(C_{k,n}^{\boxtimes d}) = \lceil \frac{1}{2} 3^d \rceil$.
$\uparrow$ $\vartheta(C_{k,n})$
2.5
3-ε
3.1
3
2.9
2.8
2.7
2.6
2.5
2.4
2.3
2.2
2.1
2.4
2.5
2.6
2.7
2.8
2.9
3
3.1
3.2
$\frac{n}{k} \rightarrow$

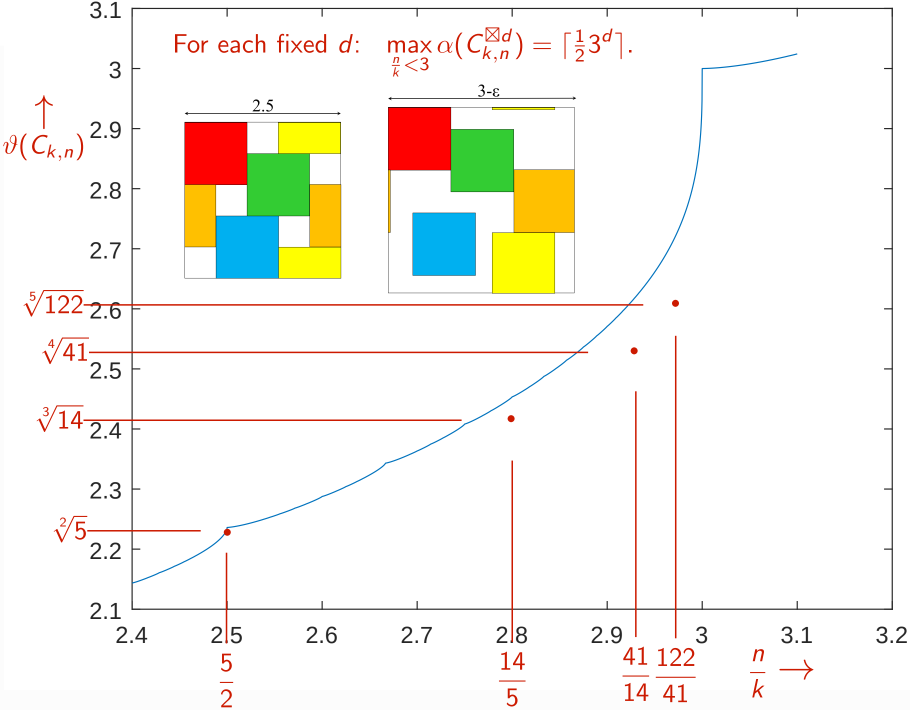

For each fixed d: $\max_{\frac{n}{k}<3} \alpha(C_{k,n}^{\boxtimes d}) = \lceil \frac{1}{2} 3^d \rceil$.
2.5
3-ε
$\vartheta(C_{k,n})$
$\sqrt[2]{5}$
$\frac{5}{2}$
$\frac{n}{k}$
3.1
3
2.9
2.8
2.7
2.6
2.5
2.4
2.3
2.2
2.1
2.4
2.5
2.6
2.7
2.8
2.9
3
3.1
3.2

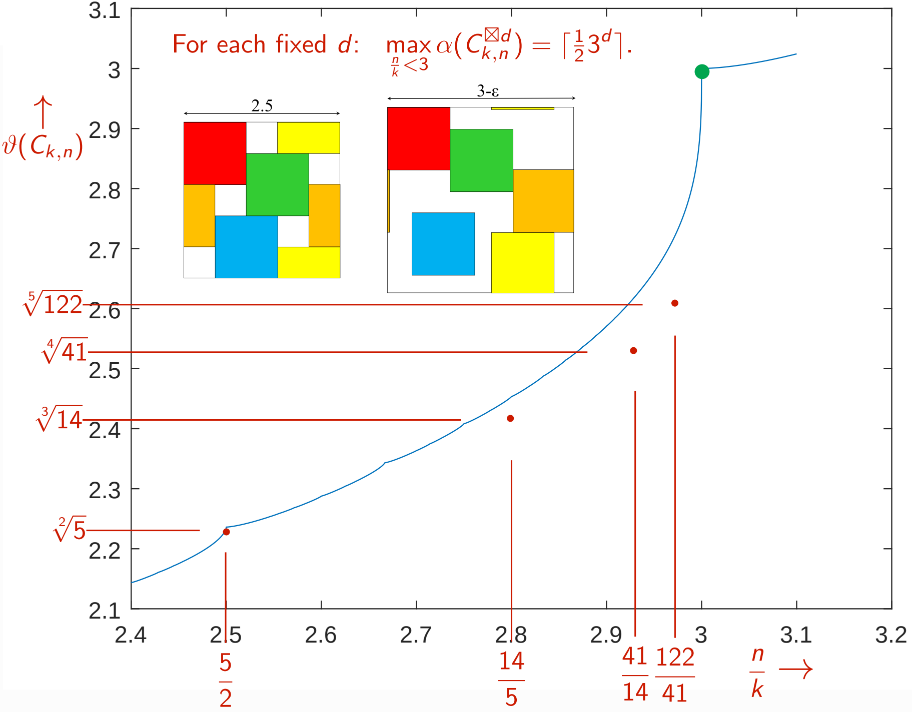


For each fixed d: $\max_{\frac{n}{k}<3} \alpha(C_{k,n}^{\boxtimes d}) = \lceil \frac{1}{2} 3^d \rceil$.
2.5
3-ε
$\uparrow$
$\vartheta(C_{k,n})$
3.1
3
2.9
2.8
2.7
2.6
2.5
2.4
2.3
2.2
2.1
$\sqrt[3]{14}$
$\sqrt[2]{5}$
2.4
2.5
2.6
2.7
2.8
2.9
3
3.1
3.2
$\frac{5}{2}$
$\frac{14}{5}$
$\frac{n}{k} \longrightarrow$



For each fixed d: $\max_{\frac{n}{k}<3} \alpha(C_{k,n}^{\boxtimes d}) = \lceil \frac{1}{2} 3^d \rceil$.
2.5
3-ε
$\vartheta(C_{k,n})$
$\sqrt[4]{41}$
$\sqrt[3]{14}$
$\sqrt[2]{5}$
3.1
3
2.9
2.8
2.7
2.6
2.5
2.4
2.3
2.2
2.1
2.4
2.5
2.6
2.7
2.8
2.9
3
3.1
3.2
$\frac{5}{2}$
$\frac{14}{5}$
$\frac{41}{14}$
$\frac{n}{k}$


For each fixed d: $\max_{\frac{n}{k}<3} \alpha(C_{k,n}^{\boxtimes d}) = \lceil \frac{1}{2} 3^d \rceil$.
2.5
3-ε
$\uparrow$ $\vartheta(C_{k,n})$
3.1
3
2.9
2.8
2.7
2.6
2.5
2.4
2.3
2.2
2.1
$\sqrt[5]{122}$
$\sqrt[4]{41}$
$\sqrt[3]{14}$
$\sqrt[2]{5}$
2.4
2.5
2.6
2.7
2.8
2.9
3
3.1
3.2
$\frac{5}{2}$
$\frac{14}{5}$
$\frac{41}{14}$
$\frac{122}{41}$
$\frac{n}{k} \rightarrow$



For each fixed d: $\max_{\frac{n}{k}<3} \alpha(C_{k,n}^{\boxtimes d}) = \lceil \frac{1}{2} 3^d \rceil$.
2.5
3-ε
$\vartheta(C_{k,n})$
$\sqrt[5]{122}$
$\sqrt[4]{41}$
$\sqrt[3]{14}$
$\sqrt[2]{5}$
3.1
3
2.9
2.8
2.7
2.6
2.5
2.4
2.3
2.2
2.1
2.4
2.5
2.6
2.7
2.8
2.9
3
3.1
3.2
$\frac{5}{2}$
$\frac{14}{5}$
$\frac{41}{14}$
$\frac{122}{41}$
$\frac{n}{k}$

And now for something completely different:

And now for something completely different:

Conjecture: *Each simple, k-regular, properly k-edge-coloured graph*

And now for something completely different:

Conjecture: *Each simple, k-regular, properly k-edge-coloured graph contains a rainbow path of length $k-1$.*

And now for something completely different:

Conjecture: *Each simple, k-regular, properly k-edge-coloured graph contains a rainbow path of length $k-1$.*

all colours different

And now for something completely different:

Conjecture: *Each simple, k-regular, properly k-edge-coloured graph contains a rainbow path of length $k-1$.*

(rainbow: all colours different)

Proved for $k \leq 11$.

And now for something completely different:

Conjecture: *Each simple, k-regular, properly k-edge-coloured graph contains a rainbow path of length $k-1$.*

↖ all colours different

Proved for $k \leq 11$.

MERCI BIEN!