INSTITUT
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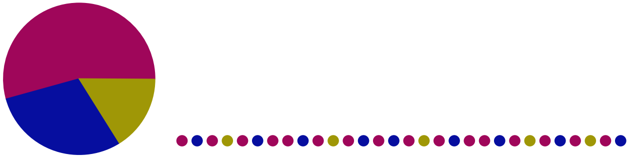
Université Paris Cité

A dynamical view on fair assignments

Valérie Berthé

Évaluation Hcéres, 2023

The chairman assignment problem



- We are given k states which form a union.
- Every year a union chairman has to be selected.
- At any time the accumulated number of chairmen from each state has to be proportional to its weight.

How to get in an effective way a fair assignment?

From assignments to symbolic discrepancy

Take an infinite word $u = (u_n)_n$ with values in a finite alphabet.

The frequency α_a of the letter a in u is defined as the following limit, if it exists

$$\alpha_a = \lim_{n \to \infty} \frac{1}{n} \mathrm{Card}\{k, 0 \le k \le n-1, u_k = a\}$$

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Examples

$$0\,1\,0\,0\,1\,0\,1\,0\,0\,1\,0\,0\,1\,0\,1\,0\,0\,1\,0\,1\,0 \cdots$$

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The discrepancy of $u = (u_n)_n$ is defined as

$$\Delta_{\boldsymbol{\alpha}}(u) = \max_{a\in\mathcal{A}} \sup_{n\in\mathbb{N}} |\mathrm{Card}\{k, 0 \leq k \leq n-1, u_k = a\} - n\alpha_a|$$

The discrepancy measures the difference between the accumulated number and the expected value.

How small can the discrepancy be?

We are given a finite alphabet $\mathcal{A}$, and a vector $\boldsymbol{\alpha}$ of frequencies for the letters of $\mathcal{A}$.

Theorem [Meijer,Tijdeman] Let d stand for the cardinality of $\mathcal{A}$. Let $d \geq 2$. One has

$$D_d = \sup_{\boldsymbol{\alpha}} \inf_{u} \Delta_{\boldsymbol{\alpha}}(u) = 1 - \frac{1}{2d-2}.$$

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When $d = 2$, $D_2 = 1/2$.

How to construct such sequences?

The two-letter case

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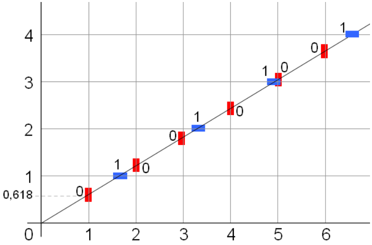
Sturmian words are codings of discrete lines.



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The words having the smallest discrepancy on a two-letter alphabet are the Sturmian words.

Sturmian words are codings of trajectories of dynamical systems.

A trajectory for a discrete-time dynamical system

We consider orbits/trajectories of points of X under the action of the map $T : X \to X$

$$\{ T^n x \mid n \in \mathbb{N} \}$$

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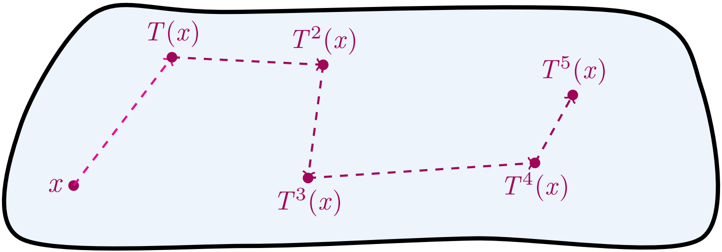

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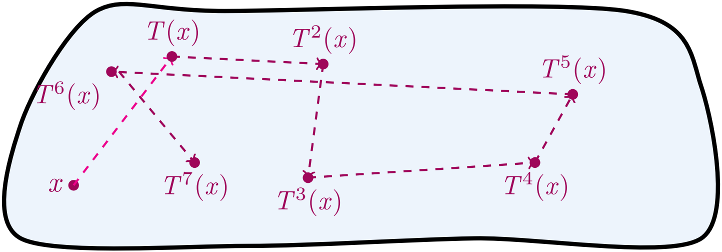
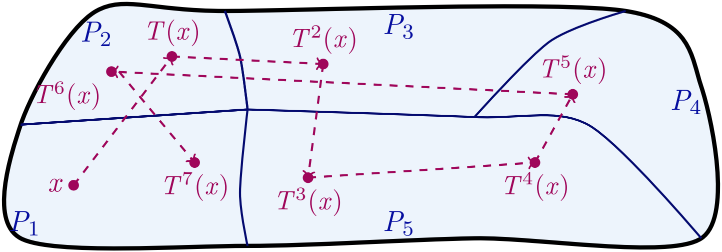


And a coding of a trajectory



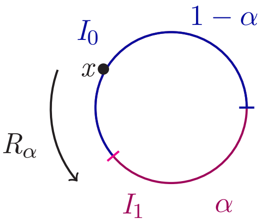
The coding works as follows

$$u_n = i \text{ if and only if } T^n(x) \in P_i$$

$$u = (u_n)_n = 12355421\cdots$$

Symbolic codings of circle rotations

Sturmian words are codings of the orbits of the rotation
$R_\alpha \colon x \mapsto x + \alpha \bmod 1$ w.r.t. 2 intervals.



$u_n = i$ iff $x + n\alpha \in I_i$ mod. 1

A natural measure of order: factor complexity

What kind of information can the dynamical viewpoint offer here?

$$u = 0100101\underbrace{00}10010100101001$$

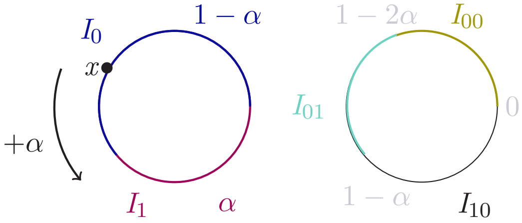
00 is a factor, 11 is not a factor

A natural measure of order: factor complexity

What kind of information can the dynamical viewpoint offer here?

$$u = 0100101001001010010100 1\cdots$$

Does the factor 00 occur? Does it have a frequency? Does it have bounded discrepancy?



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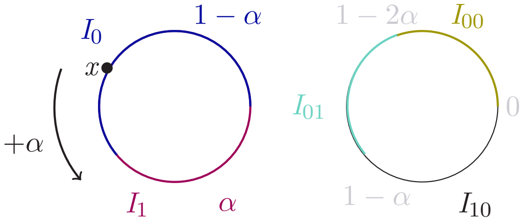

The factors of length n of u are in one-to-one correspondence with the $n+1$ intervals of $\mathbb{T}$ whose end-points are given by

$$-k\alpha \bmod 1 \qquad \text{for } 0 \leq k \leq n$$

By uniform distribution of $(k\alpha)_k$ modulo 1, the frequency of a factor w of a Sturmian word is equal to the length of I_w.

Fair assignments in general dimension

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- Given a frequency vector $\boldsymbol{\alpha} = (\alpha_a)_{a \in \mathcal{A}}$, R. Tijdeman ('80) has given an algorithmic way, to construct a sequence u with $\Delta_{\boldsymbol{\alpha}}(u) \leq 1 - \frac{1}{2d-2}$.

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Theorem [B.-Carton-Chevallier-Steiner-Yassawi] Let u be a Tijdeman sequence with a frequency vector $\boldsymbol{\alpha}$ which has rationally independent coordinates.
Then, the sequence u has factor complexity of order n^{d-1}.

The sequence u is a symbolic coding of a translation $R_{\boldsymbol{\alpha}}$ via a partition of a fundamental domain of $\mathbb{T}^{d-1}$ into d finite unions of polytopes such that $R_{\boldsymbol{\alpha}}$ is a translation by a vector on each of the polytopes.

Evenly distributed sequences

Let $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_d) \in [0,1]^d$ such that $\sum_{i=1}^d \alpha_i = 1$.

How to construct sequences u over the alphabet $\{1, 2, \cdots, d\}$ satisfying the following conditions

- the letter frequencies in u are given by $(\alpha_1, \cdots, \alpha_d)$
- u has discrepancy smaller than or equal to D_d
- u has linear complexity function
- u has bounded discrepancy for factors

Let us start from the dynamical system given by the translation $R_{\boldsymbol{\alpha}} : \mathbf{x} \mapsto \boldsymbol{x} + \boldsymbol{\alpha}$ modulo 1. How to find a good partition?

The ubiquitous Fibonacci word

Take the golden ratio $\alpha = \frac{\sqrt{5}+1}{2}$ and the dynamical system

$$x \mapsto x + \alpha \text{ modulo } 1$$

$$\alpha^2 = \alpha + 1 \rightsquigarrow \text{self-similarity}$$

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And the Fibonacci substitution

$$\sigma(u) = u \text{ with } \sigma : 1 \mapsto 12, \;\; 2 \mapsto 1$$

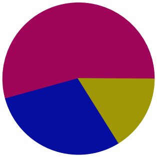
$$u = \sigma^{\infty}(1) = 121121211211212 \cdots$$

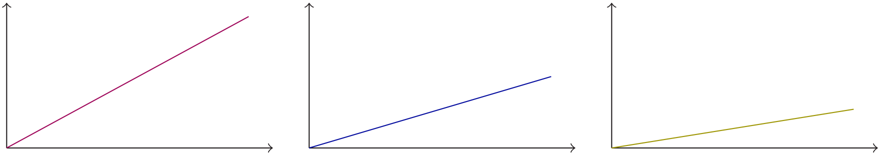
Theorem The symbolic dynamical system (X_σ, S) is isomorphic to the geometric dynamical system $(\mathbb{T}, R_{\frac{1+\sqrt{5}}{2}})$ where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$

A few milestones

- 1898, Hadamard Geodesic flows on surfaces of negative curvature.
- 1912 Prouhet-Thue-Morse substitution $\sigma : a \mapsto ab,\ b \mapsto ba$
- 1940, Morse-Hedlund Symbolic dynamics.
- 30's Skolem-Mahler-Lech theorem and linear recurrences.
- 60's Tilings, substitutions and the domino problem.
- 1984 Quasicrystals, quasiperiodic order and the Pisot conjecture.
- 80's Rauzy fractal and Thurston's tile for the Tribonacci numeration.
- 80's Reachability problems and linear recurrences.
- 2023 The Einstein monotile.

Discrepancy





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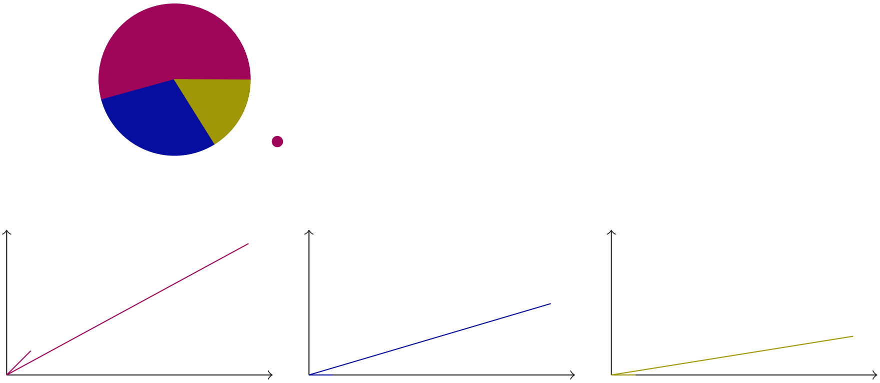

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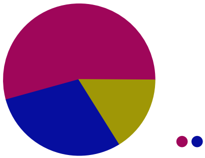

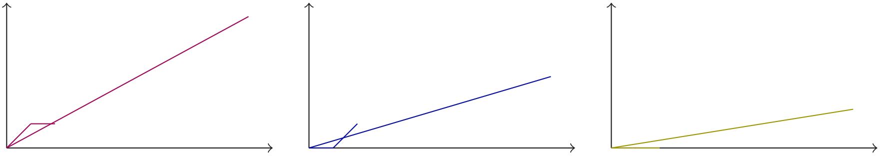

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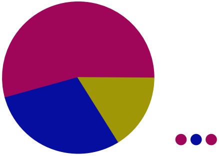

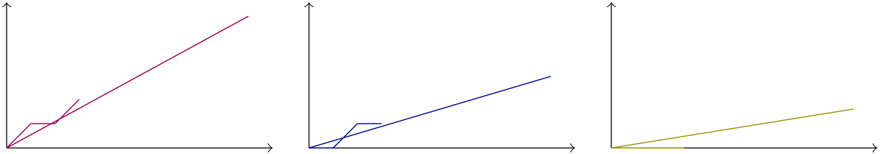

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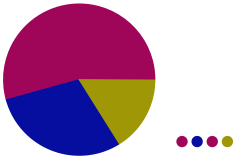

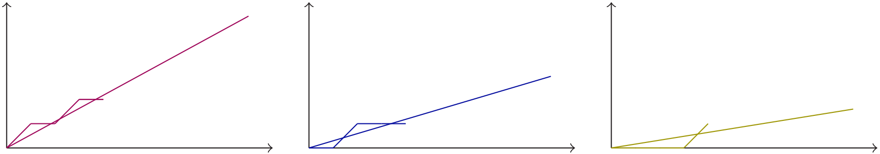

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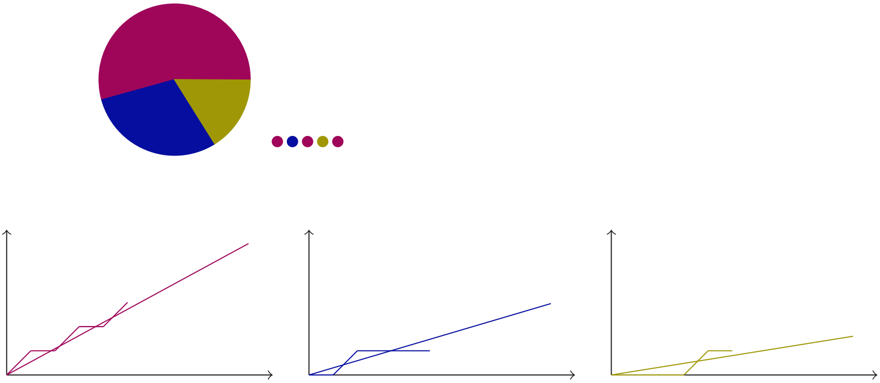

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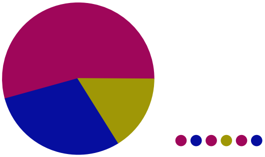

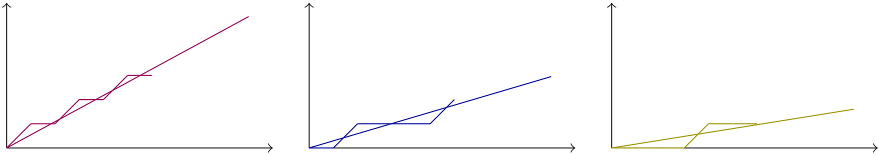

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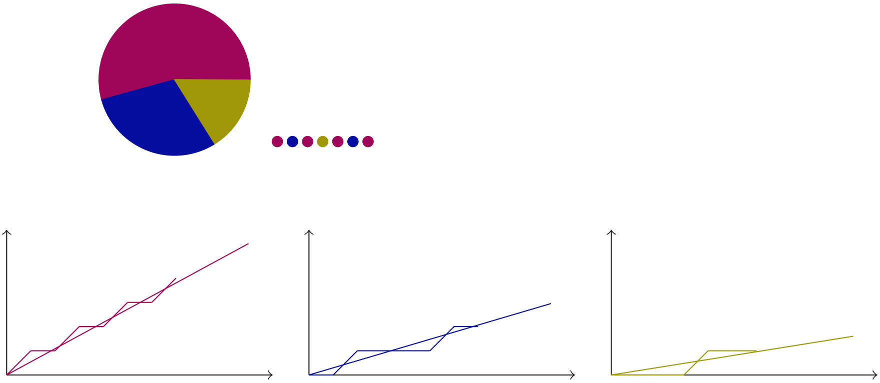

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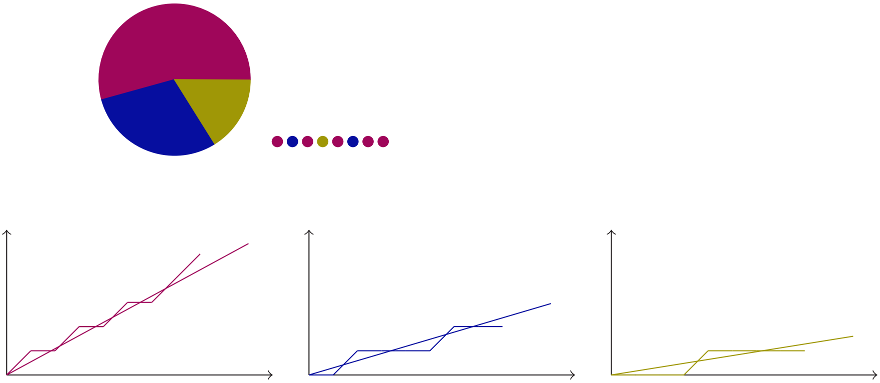

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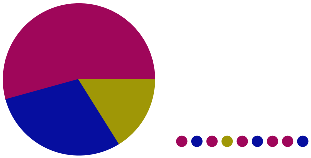

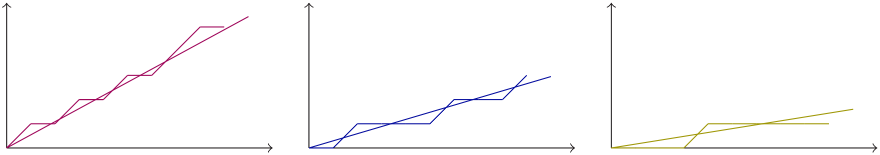

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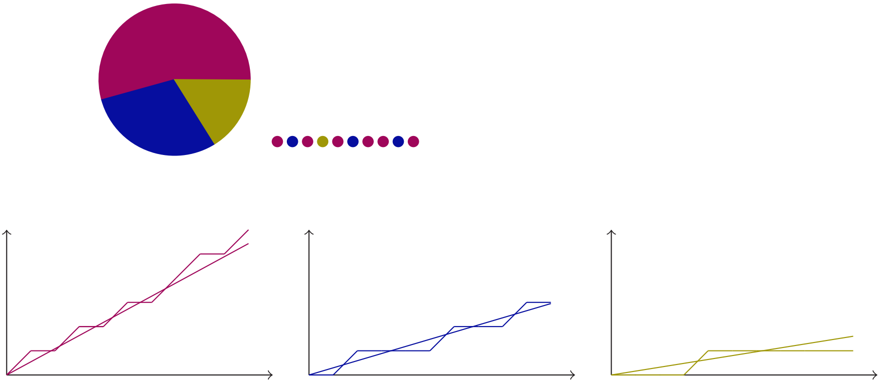

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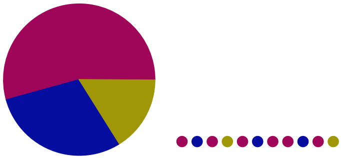

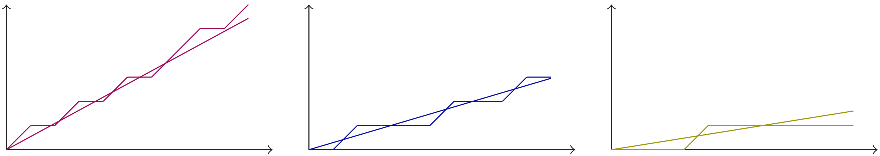

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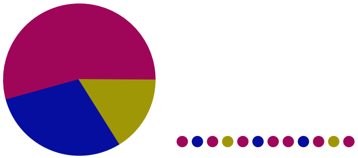

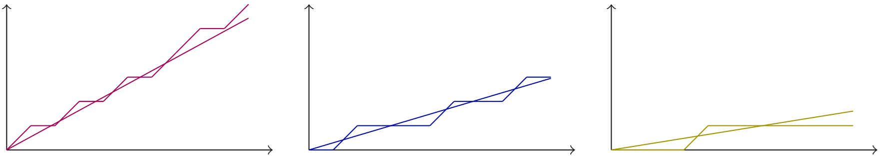

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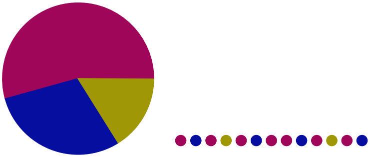

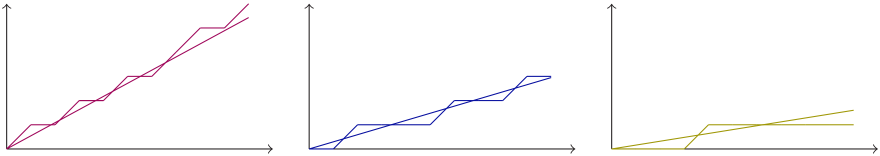

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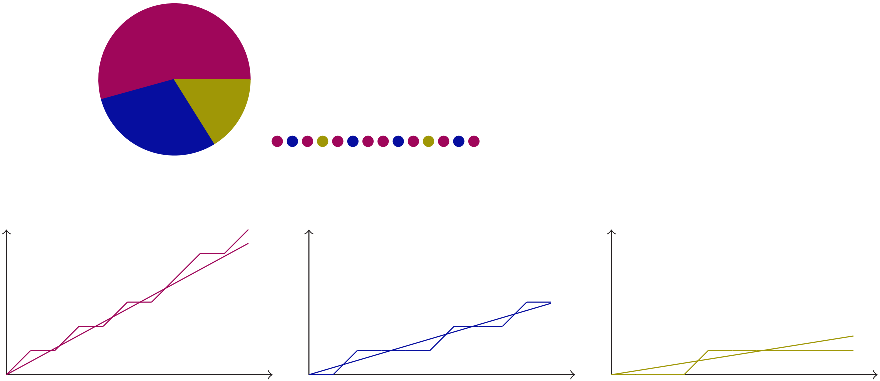

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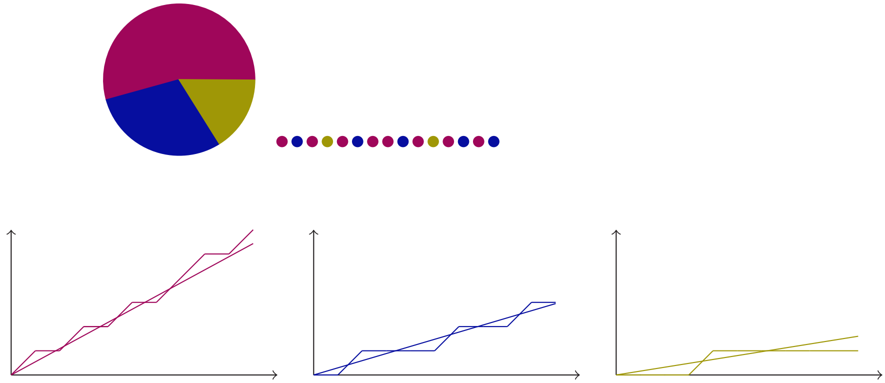

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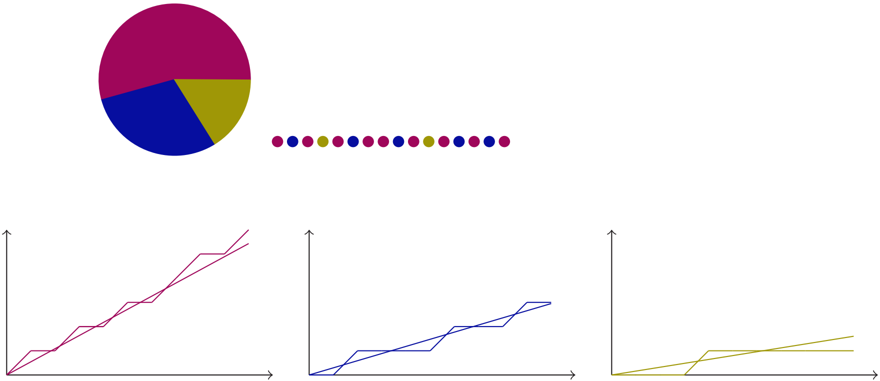

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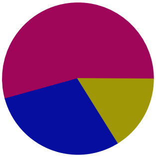

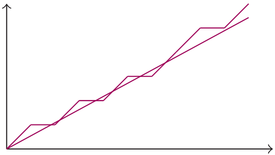

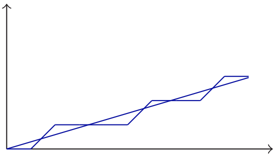

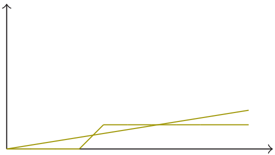

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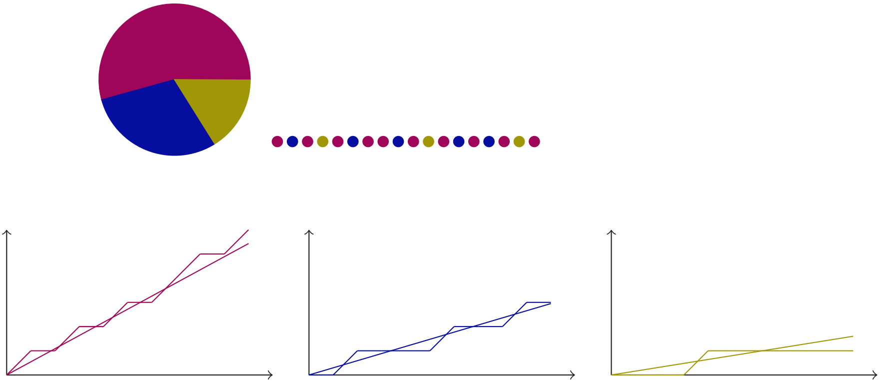

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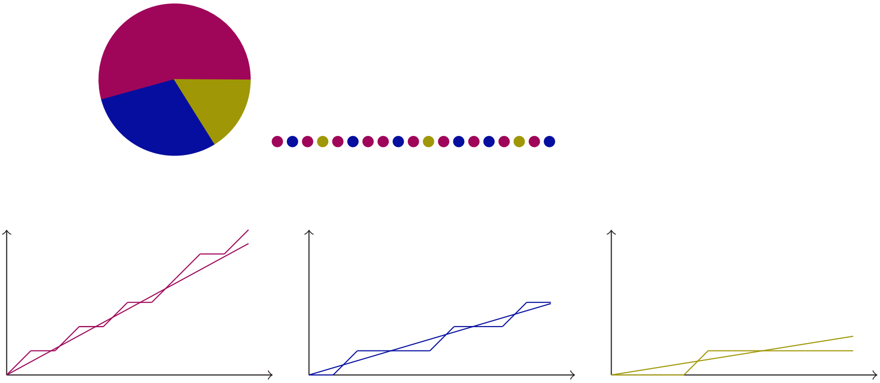

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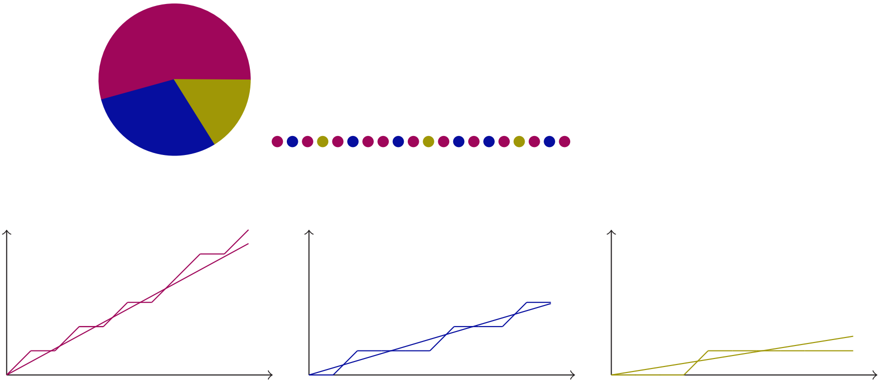

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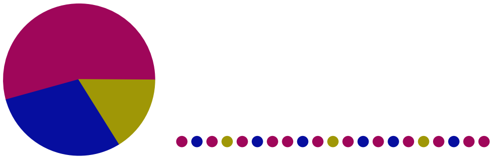

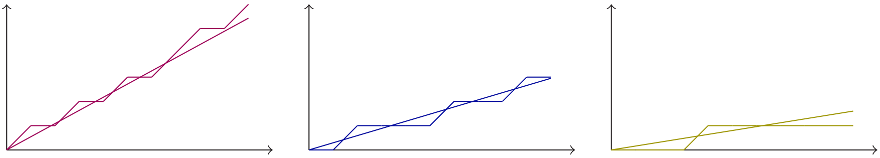

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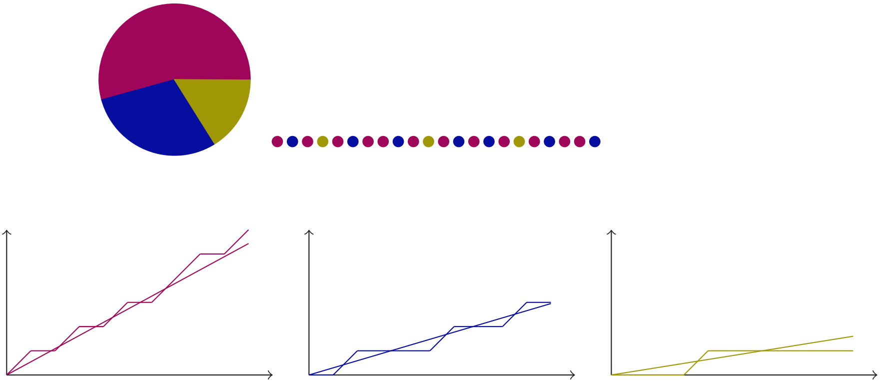

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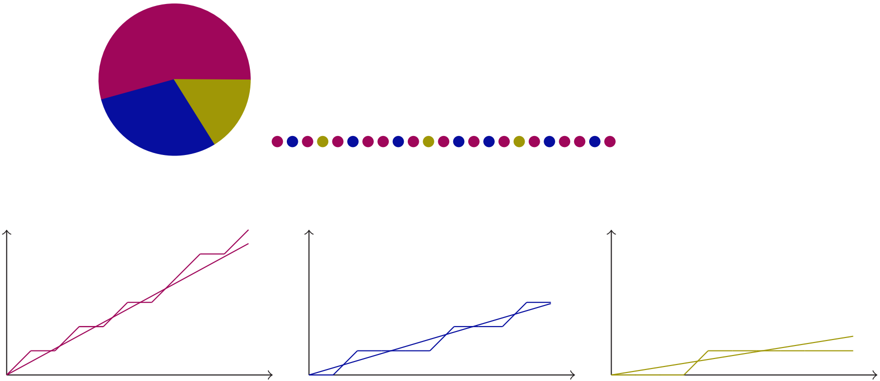

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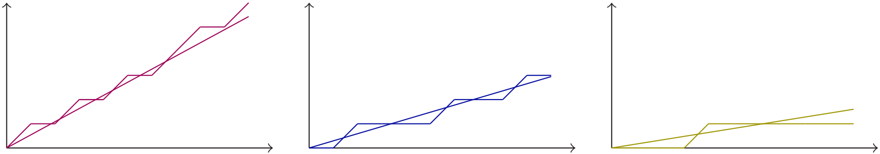

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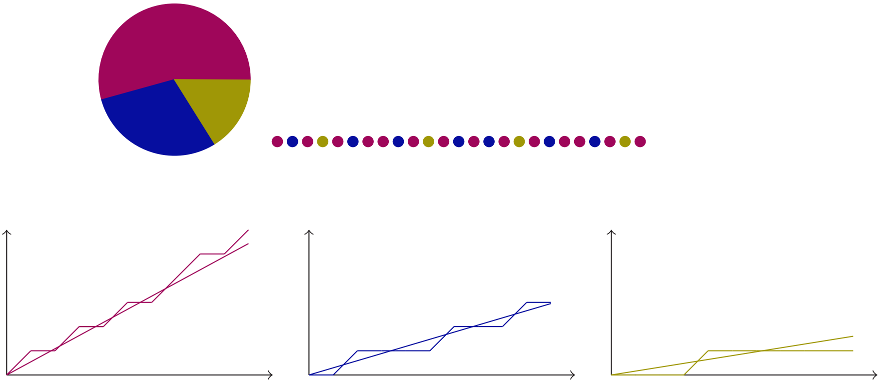

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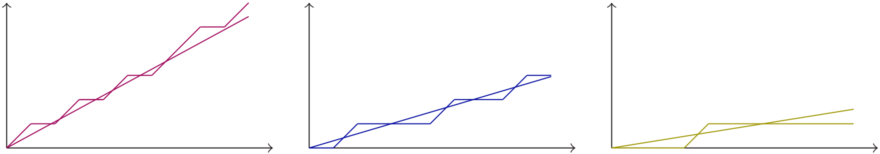

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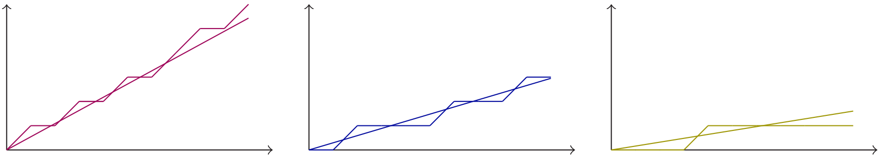

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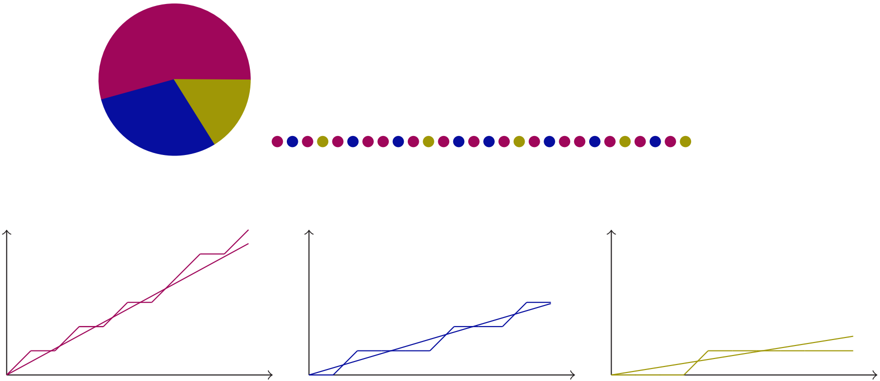

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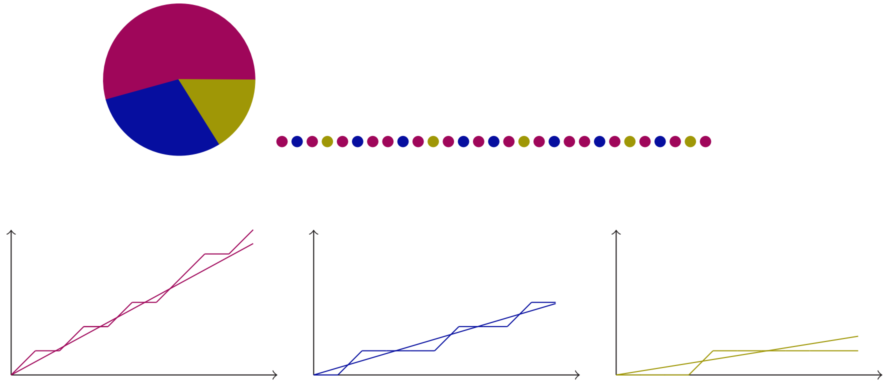

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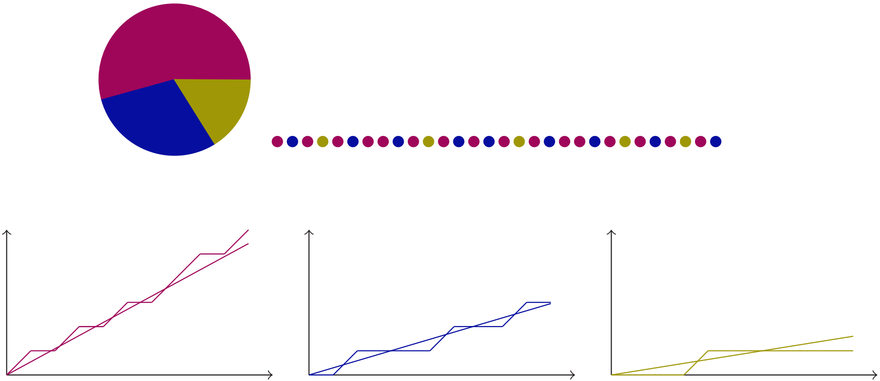

Rauzy fractal

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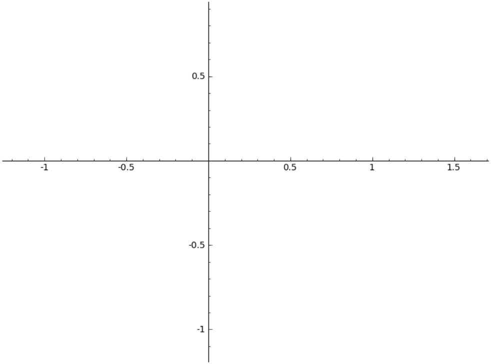
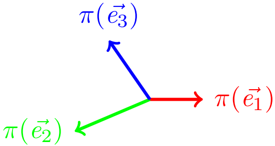


π projection along the expanding eigenline onto the contracting plane of the incidence matrix of M_σ



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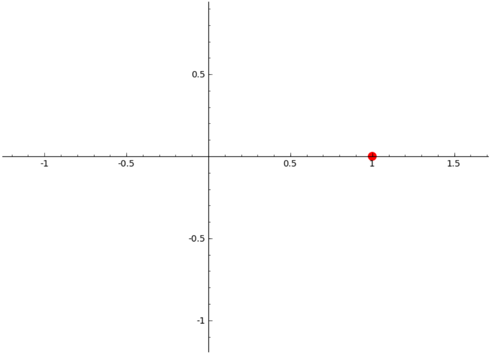

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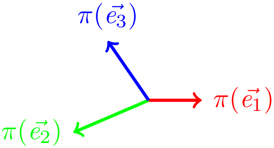

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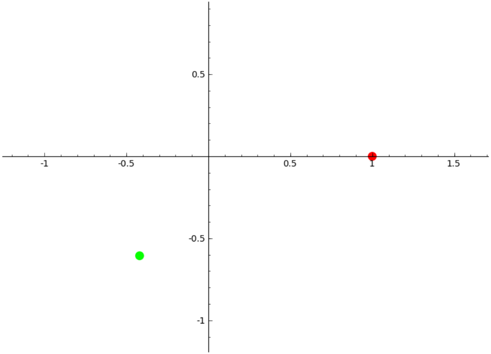

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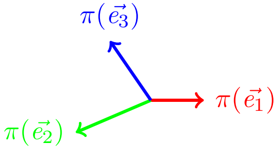

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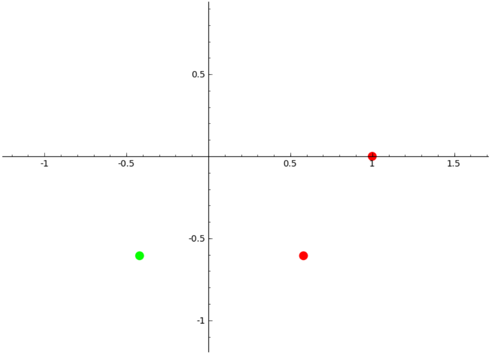

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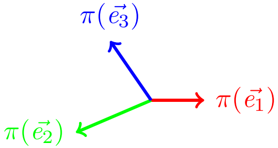

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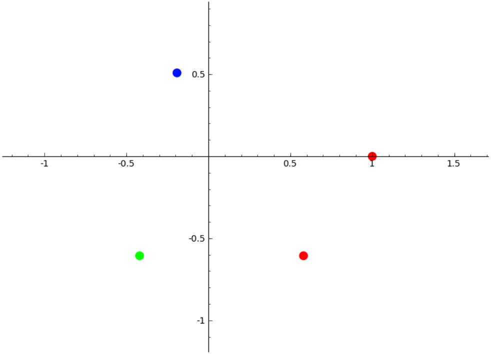

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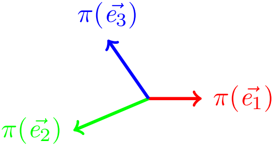

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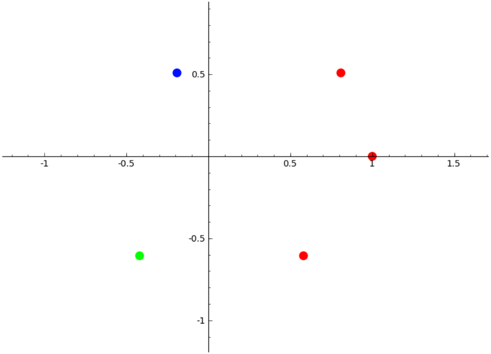

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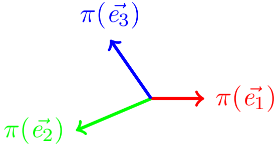

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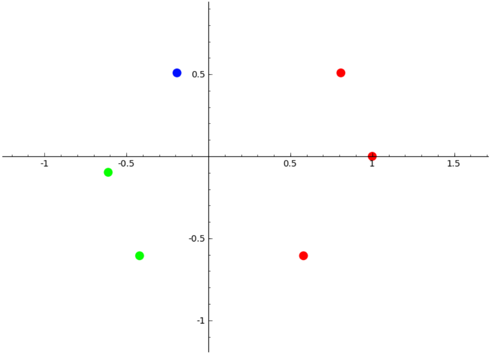

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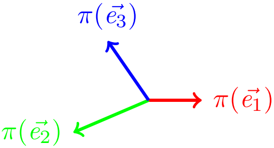

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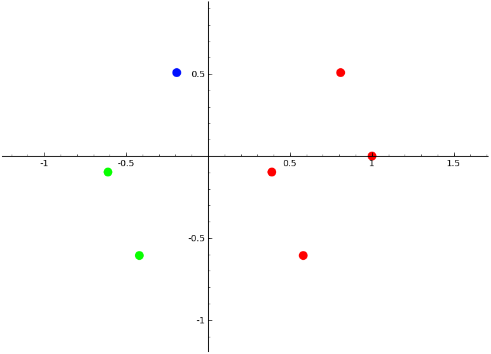

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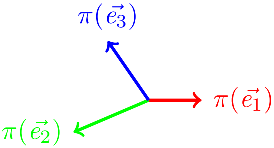

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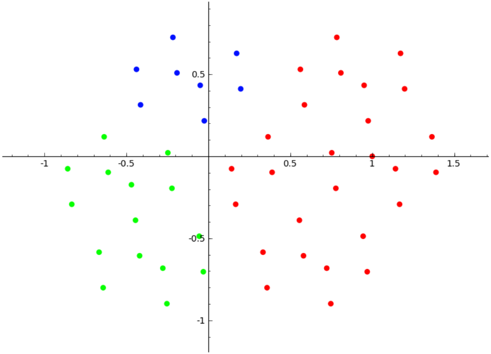

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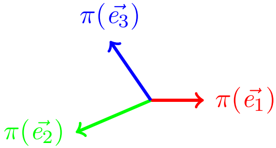

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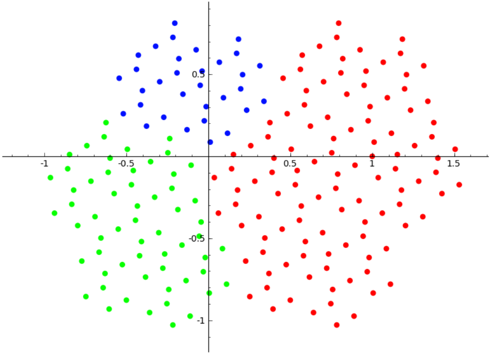

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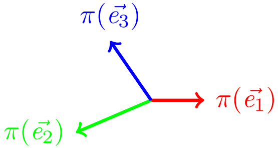

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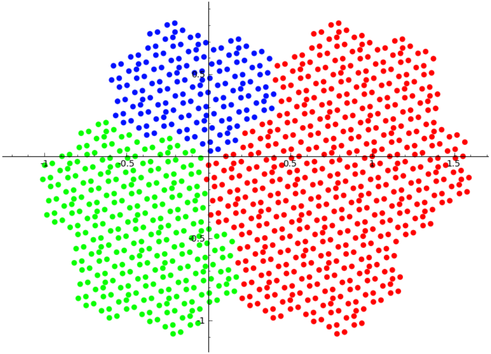

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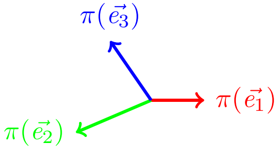

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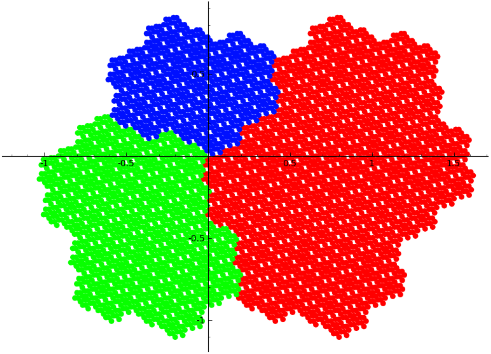

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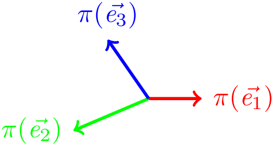

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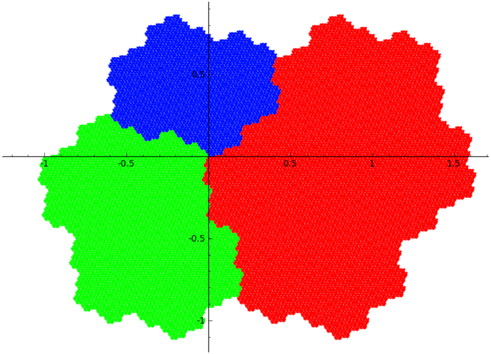

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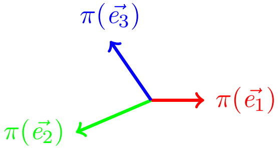


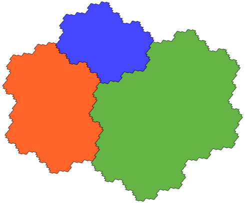
Pisot numbers, codings and fractals

$$X^3 = X^2 + X + 1$$

$$\sigma : 1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1$$

Theorem [Rauzy'82] The symbolic dynamical system (X_σ, S) is measure-theoretically isomorphic to the translation R_β on the two-dimensional torus $\mathbb{T}^2$

$$R_\beta : \mathbb{T}^2 \to \mathbb{T}^2,\ x \mapsto x + (1/\beta, 1/\beta^2)$$



How to produce symbolic codings for translations

How to produce fair assignments for a given vector of letter frequencies $\boldsymbol{\alpha}$.

- We apply a multidimensional continued fraction algorithm that generates nonnegative matrices

$$\boldsymbol{\alpha} \mapsto (M_n)_n \text{ with } \quad \boldsymbol{\alpha} \in \cap_n M_1 \cdots M_n \mathbb{R}_+^d$$

- that generates in turn a sequence of substitutions

- and thus infinite words $u = \lim \sigma_0 \cdots \sigma_n(a)$.

Beyond the Pisot conjecture

Classical exponentially convergent multidimensional continued fraction algorithms generate faithful symbolic codings for translations on the torus.

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Classical exponentially convergent multidimensional continued fraction algorithms generate faithful symbolic codings for translations on the torus.

Theorem [B.-Steiner-Thuswaldner]
For almost every $\alpha \in [0,1]^d$, there exists a faithful symbolic coding for the translation $R_\alpha : x \mapsto x + \alpha$ modulo 1 having bounded discrepancy.

- with also bounded discrepancy for all factors (multiscale)

Exponential convergence of continued fraction algorithms

According to recent numerical experiments, classical multidimensional continued fraction algorithms seem to cease to be exponentially convergent when the dimension increases. [B.-Steiner-Thuswaldner]

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Brun algorithm

d	$\lambda_2(A_B)$	d	$\lambda_2(A_B)$
2	-0.11216	7	-0.01210
3	-0.07189	8	-0.00647
4	-0.04651	9	-0.00218
5	-0.03051	10	$+0.00115$
6	-0.01974	11	$+0.00381$

To be confirmed theoretically and numerically

How to design efficient continued fraction algorithms?

What can infinite words represent?

Infinite words arise as codings of trajectories but there is more.
A word can represent

- A predicate in some logic
- A characteristic function for a subset of integers
- The sequence of digits of a real number in some numeration system
- A quasicrystal
- A tiling
- The trace of the execution of an algorithm

Reachability vs. statistical properties of orbits

- Ergodicity and long-term behavior: will a trajectory visit infinitely often a subregion and how long will it stay in this subregion?

- Model checking and reachability problems for linear dynamical systems. Will an orbit enter a given subregion of the space or even reach a given point?

  [B.,Fijalkow,Karimov,Nosan,Ohlmann,Ouaknine,Pouly, Schmitz,Shirmohammadi,Vahanwala,Worrell,...]