A denotational semantics for non-wellfounded proofs in linear logic

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Denotational Semantics

The denotational semantic is a way of assigning suitable mathematical entities to the objects of a given language.

Denotational semantics

Let's consider the numerals and numbers:

- The numerals are expression in a familiar language such as binary, octal, or decimal numerals.
- So, there are different languages to convey same concepts.
- Even in one language, there are different expression for a same concepts ($3 + 3 = 2 + 2 + 2 = 6 = ...$).

The problem of explaining these equivalences of expressions is one of the tasks of semantics and is much too important to be left to syntax alone.

Denotational semantics

A way of expressing the meaning of types and programs independent from their syntactic, operational, specification.

Main principles, since Scott:

- Formulas $\rightsquigarrow$ complete partial orders. $u \leq v$ means "u less defined than v".
- Proofs $\rightsquigarrow$ Continuous function (a finite amount of information at the input is enough to produce finite amount of information).

What we consider in this talk

Language: Non-wellfounded linear logic (μLL_∞).

Model: Category **REL** of sets and relations and non-uniform totality spaces (NUTS).

$\mu\mathsf{LL}_\infty$

$$A, B, \ldots := 1 \mid 0 \mid \bot \mid \top \mid A \oplus B \mid A \otimes B \mid A \,\&\, B \mid A \parr B \mid ?A \mid !B \mid X \mid \mu X.F \mid \nu X.F$$

$$\frac{\vdash \Gamma, F[\nu\zeta.F/\zeta]}{\vdash \Gamma, \nu\zeta.F}\ (\nu - \text{fold}) \qquad \frac{\vdash \Gamma, F[\mu\zeta.F/\zeta]}{\vdash \Gamma, \mu\zeta.F}\ (\mu - \text{fold})$$

Interpretation of formulas in **REL**

$$A(\zeta_1, \cdots, \zeta_k) \mapsto k\text{-ary CPO functor } [\![A]\!]_{\vec{\zeta}}$$

Fact (M. Wand)

If $\mathbb{F} : \mathbf{REL} \to \mathbf{REL}$ *is a CPO functor, then* $\mathbb{F}$ *has a final coalgebra which is also an initial algebra,* $\mu\mathbb{F} = \nu\mathbb{F}$*: the "least fixpoint" of* $\mathbb{F}$*.*

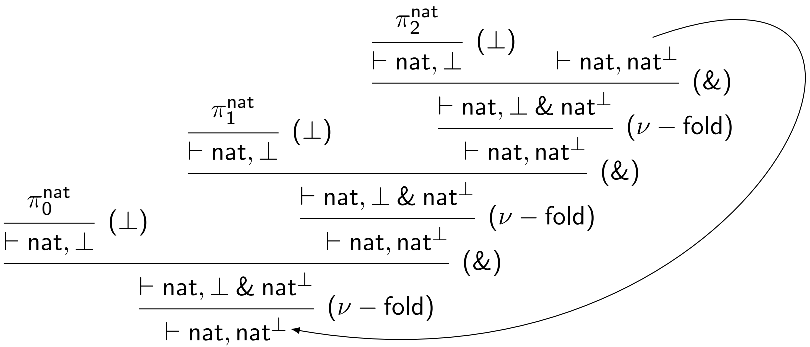
Interpretation of proofs in **REL**

$$\left[\!\!\left[\dfrac{\dfrac{\delta}{\vdash \Gamma, F[\sigma\zeta.F/\zeta]}}{\vdash \Gamma, \sigma\zeta.F}\ (\sigma - \mathsf{fold}) \right]\!\!\right] = [\![\delta]\!]$$

$$[\![\pi]\!]_{\mathbf{REL}} = \bigcup_{\rho \in \mathsf{fin}(\pi)} [\![\rho]\!]_{\mathbf{REL}}$$

Example

Consider the following circular proof $\pi_{\equiv_3}$:



$$[\![\sigma]\!]_{\mathbf{REL}} \simeq \{(\underline{2}, \underline{2}), (\underline{1}, \underline{1}), (\underline{0}, \underline{0})\}$$

$$[\![\pi_{\equiv_3}]\!]_{\mathbf{REL}} = \{(\underline{n}, \underline{m}) \mid \underline{n} = \underline{m} \mod 3\}$$

On the relation between the interpretation of finite proofs and their circular correspondent

Fact

Let π be a μLL proof. Then we have $[\![\pi]\!] = [\![\mathsf{Trans}(\pi)]\!]$ where the interpretation is given in a model $(\mathcal{L}, \overrightarrow{\mathcal{L}})$ of μLL.

Two properties of the semantics

Soundness: If π and π' are proofs of $\vdash \Gamma$ and π reduces to π' by the cut-elimination rules of $\mu\mathsf{LL}_\infty$, then $[\![\pi]\!]_{\mathbf{REL}} = [\![\pi']\!]_{\mathbf{REL}}$.

Validity implies totality: If π is a valid proof of the sequent $\vdash \Gamma$, then $[\![\pi]\!]$ is a "total element of $[\![\Gamma]\!]$".

Totality candidates on a set E

Given $\mathcal{T} \subseteq \mathcal{P}(E)$ we set

$$\mathcal{T}^{\perp} = \left\{ u' \subseteq E \;\middle|\; \forall u \in \mathcal{T} \;\; u \cap u' \neq \varnothing \right\}$$

Definition (Totality candidates)

$\mathcal{T}$ is a *totality candidate* for E if $\mathcal{T} = \mathcal{T}^{\perp\perp}$.

(Equivalently $\mathcal{T}^{\perp\perp} \subseteq \mathcal{T}$, equivalently $\mathcal{T} = \mathcal{S}^{\perp}$ for some $\mathcal{S} \subseteq \mathcal{P}(E)$.)

Fact

- *$\mathcal{T}$ is a totality candidate on E iff $\mathcal{T} \subseteq \mathcal{P}(E)$ and $\mathcal{T} = \uparrow\mathcal{T}$.*
- $\mathrm{Tot}(X)$ *(The set of all totality candidates on E), ordered with $\subseteq$, is a complete lattice (it is closed under arbitrary intersections).*

Non-uniform totality spaces (NUTS)

A NUTS is a pair $X = (|X|, \mathcal{T}X)$ where

- $|X|$ is a set
- $\mathcal{T}X$ is a totality candidate on $|X|$, that is, a $\uparrow$-closed subset of $\mathcal{P}(|X|)$.

$t \in \mathbf{NUTS}(X, Y)$ if $t \in \mathbf{REL}(|X|, |Y|)$ and

$$\forall u \in \mathcal{T}X \quad t \cdot u \in \mathcal{T}Y$$

Fact

NUTS *is a model of LL where the proofs are interpreted exactly as in* **REL**.

Interpretation of $\mu X.F$ in **NUTS**

$$
\begin{array}{ccc}
\mathbf{NUTS} & \xrightarrow{\overline{F}} & \mathbf{NUTS} \\
\downarrow & & \downarrow \\
\mathbf{REL} & \xrightarrow{F} & \mathbf{REL}
\end{array}
$$

Interpretation of $\mu X.F$ in **NUTS**

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$\overline{F} : (X, U) \mapsto (FX, \Phi U)$ where $\Phi U \in \mathcal{T}(FX)$.

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Assume μF exists.

$$g : \mathrm{Tot}(\mu F) \to \mathrm{Tot}(\mu F)$$

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By Tarski theorem, μg exists.

$$\mu\overline{F} = (\mu F, \mu g).$$

Interpretation of proofs

The interpretation of proofs in **NUTS** is same as their interpretation in **REL**.

Validity implies totality

Theorem: If π is a valid proof of the sequent $\vdash \Gamma$, then $[\![\pi]\!] \in \mathcal{T}[\![\Gamma]\!]$.

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The proof is similar to the proof of soundness of LKID^ω in [1].

We needed to adapt the proof in two aspects:

- considering $\mu\mathrm{LL}_\infty$ instead of $LKID^\omega$,
- and deal with the denotational semantics instead of Tarskian semantics.

Adapation for $\mu\mathrm{LL}_\infty$: somehow done in [2]

So, basically, the main point of this proof is adapting a Tarskian soundness theorem to a denotational semantic soundness.

[1] James Brotherston.Sequent Calculus Proof Systems for Inductive Def-initions. PhD thesis, University of Edinburgh, November 2006.

[2] Amina Doumane. On the infinitary proof theory of logics with fixedpoints. PhD thesis, Paris Diderot University, 2017.

An example

A syntatic-free proof that any term of booleans has a defined boolean value true or false

Consider $1 \oplus 1$ (The type of booleans).
$[\![1 \oplus 1]\!] = (\{(1,\star),(2,\star)\}, \mathcal{T}[\![1 \oplus 1]\!])$ where

$$\mathcal{T}([\![1 \oplus 1]\!]) = \mathcal{P}(|[\![1 \oplus 1]\!]|) \backslash \emptyset$$

For any proof π of $1 \oplus 1$, we have $[\![\pi]\!] \in \mathcal{T}[\![1 \oplus 1]\!]$.
Hence $[\![\pi]\!] \neq \emptyset$.

Future work

- Categorical axiomitazation of models of $\mu\mathsf{LL}_\infty$.
- Try to understand what sort of information can be obtained from a total interpretation, if not syntactic validity.

Thanks!