A denotational semantics for non-wellfounded proofs in linear logic

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Denotational Semantics

The denotational semantic is a way of assigning suitable mathematical entities to the objects of a given language.
Let’s consider the numerals and numbers:

- The numerals are expressions in a familiar language such as binary, octal, or decimal numerals.
- So, there are different languages to convey the same concepts.
- Even in one language, there are different expressions for the same concepts ($3 + 3 = 2 + 2 + 2 = 6 = ...$).

The problem of explaining these equivalences of expressions is one of the tasks of semantics and is much too important to be left to syntax alone.
Denotational semantics

A way of expressing the meaning of types and programs independent from their syntactic, operational, specification.

Main principles, since Scott:

- Formulas \( \sim \) complete partial orders. \( u \leq v \) means “\( u \) less defined than \( v \)”.
- Proofs \( \sim \) Continuous function (a finite amount of information at the input is enough to produce finite amount of information).
What we consider in this talk

Language: Non-wellfounded linear logic ($\mu LL_\infty$).

Model: Category REL of sets and relations and non-uniform totality spaces (NUTS).
\[\mu \mathcal{L} \mathcal{L}_\infty\]

\[A, B, \ldots := 1 \mid 0 \mid \bot \mid \top \mid A \oplus B \mid A \otimes B \mid A \& B \mid A \otimes B \mid ?A \mid !B \mid X \mid \mu X.F \mid \nu X.F\]

\[\frac{\vdash \Gamma, F[\nu \zeta . F / \zeta]}{\vdash \Gamma, \nu \zeta . F} \quad \text{(\(\nu\) - fold)}\]

\[\frac{\vdash \Gamma, F[\mu \zeta . F / \zeta]}{\vdash \Gamma, \mu \zeta . F} \quad \text{(\(\mu\) - fold)}\]
Interpretation of formulas in REL

\[ A(\zeta_1, \cdots, \zeta_k) \mapsto k\text{-ary CPO functor } [A]_{\zeta} \]

Fact (M. Wand)

If \( \mathbb{F} : \text{REL} \to \text{REL} \) is a CPO functor, then \( \mathbb{F} \) has a final coalgebra which is also an initial algebra, \( \mu \mathbb{F} = \nu \mathbb{F} \): the “least fixpoint” of \( \mathbb{F} \).
Interpretation of proofs in \( \text{REL} \)

\[
\begin{align*}
& \frac{\delta}{\vdash \Gamma, F[\sigma \zeta. F/\zeta]} \\
& \quad \vdash \Gamma, \sigma \zeta. F \\
& \hspace{1cm} (\sigma - \text{fold}) \\
\end{align*}
\]

\[
[\pi]_{\text{REL}} = \bigcup_{\rho \in \text{fin}(\pi)} [\rho]_{\text{REL}}
\]
Consider the following circular proof $\pi_{\equiv 3}$:

$$
\frac{
\frac{
\frac{
\pi_0^{\text{nat}}
}{\vdash \text{nat}, \bot}
}{\vdash \text{nat}, \bot}
{\vdash \text{nat, nat}^\perp}
}{\vdash \text{nat, nat}^\perp}
$$

$$
\frac{
\frac{
\frac{
\pi_1^{\text{nat}}
}{\vdash \text{nat}, \bot}
}{\vdash \text{nat, \bot \& nat}^\perp}
}{\vdash \text{nat, nat}^\perp}
$$

$$
\frac{
\frac{
\frac{
\pi_2^{\text{nat}}
}{\vdash \text{nat, \bot}}
}{\vdash \text{nat, \bot \& nat}^\perp}
}{\vdash \text{nat, nat}^\perp}
$$

$$
\frac{
\vdash \text{nat, \bot \& nat}^\perp
}{\vdash \text{nat, nat}^\perp}
$$

$$
\nu - \text{fold}
$$

$$
\text{J} \sigma \ K \text{REL} \simeq \{(2, 2), (1, 1), (0, 0)\}
$$

$$
\text{J} \pi_{\equiv 3} \ K \text{REL} = \{(n, m) | n = m \mod 3\}
$$
On the relation between the interpretation of finite proofs and their circular correspondent

Fact
Let $\pi$ be a $\mu$LL proof. Then we have $\downarrow \pi = \downarrow \text{Trans}(\pi)$ where the interpretation is given in a model $(\mathcal{L}, \overset{\rightarrow}{\mathcal{L}})$ of $\mu$LL.
Two properties of the semantics

**Soundness:** If $\pi$ and $\pi'$ are proofs of $\Gamma \vdash \Gamma$ and $\pi$ reduces to $\pi'$ by the cut-elimination rules of $\mu LL_\infty$, then $[\pi]_{\text{REL}} = [\pi']_{\text{REL}}$.

**Validity implies totality:** If $\pi$ is a valid proof of the sequent $\vdash \Gamma$, then $[\pi]$ is a “total element of $[\Gamma]$”.
Totality candidates on a set $E$

Given $\mathcal{T} \subseteq \mathcal{P}(E)$ we set

$$\mathcal{T}^\perp = \{ u' \subseteq E \mid \forall u \in \mathcal{T} \ u \cap u' \neq \emptyset \}$$

**Definition (Totality candidates)**

$\mathcal{T}$ is a **totality candidate** for $E$ if $\mathcal{T} = \mathcal{T}^{\perp\perp}$.

(Equivalently $\mathcal{T}^{\perp\perp} \subseteq \mathcal{T}$, equivalently $\mathcal{T} = S^{\perp}$ for some $S \subseteq \mathcal{P}(E)$.)

**Fact**

- $\mathcal{T}$ is a totality candidate on $E$ iff $\mathcal{T} \subseteq \mathcal{P}(E)$ and $\mathcal{T} = \uparrow \mathcal{T}$.
- $\text{Tot}(X)$ (*The set of all totality candidates on } E$, ordered with $\subseteq$, is a complete lattice (it is closed under arbitrary intersections)*.
Non-uniform totality spaces (NUTS)

A NUTS is a pair $X = (|X|, TX)$ where
- $|X|$ is a set
- $TX$ is a totality candidate on $|X|$, that is, a $\uparrow$-closed subset of $\mathcal{P}(|X|)$.

$t \in \text{NUTS}(X, Y)$ if $t \in \text{REL}(|X|, |Y|)$ and

$$\forall u \in TX \quad t \cdot u \in TY$$

Fact

\text{NUTS} is a model of LL where the proofs are interpreted exactly as in REL.
Interpretation of $\mu X.F$ in NUTS
Interpretation of $\mu X. F$ in **NUTS**

![Diagram]

$\overline{F} : (X, U) \mapsto (FX, \Phi U)$ where $\Phi U \in T(FX)$. 
Interpretation of $\mu X.F$ in NUTS

$\overline{F} : (X, U) \mapsto (FX, \Phi U)$ where $\Phi U \in T(FX)$.

Assume $\mu F$ exists.

$g : \text{Tot}(\mu F) \rightarrow \text{Tot}(\mu F)$

$R \mapsto \Phi R$
Interpretation of $\mu X. F$ in NUTS

$\overline{F} : (X, U) \mapsto (FX, \Phi U)$ where $\Phi U \in T(FX)$.

Assume $\mu F$ exists.

$g : \text{Tot}(\mu F) \to \text{Tot}(\mu F)$

$R \mapsto \Phi R$

By Tarski theorem, $\mu g$ exists.

$\overline{\mu F} = (\mu F, \mu g)$. 
The interpretation of proofs in \textbf{NUTS} is same as their interpretation in \textbf{REL}. 
Validity implies totality

Theorem: If $\pi$ is a valid proof of the sequent $\vdash \Gamma$, then $\llbracket \pi \rrbracket \in T[\Gamma]$. 
Validity implies totality

Theorem: If $\pi$ is a valid proof of the sequent $\Gamma$, then $[\pi] \in T[\Gamma]$.

The proof is similar to the proof of soundness of LKID$^\omega$ in $^1$.

We needed to adapt the proof in two aspects:

- considering $\mu LL_{\infty}$ instead of LKID$^\omega$,
- and deal with the denotational semantics instead of Tarskian semantics.

Adapation for $\mu LL_{\infty}$: somehow done in $^2$

So, basically, the main point of this proof is adapting a Tarskian soundness theorem to a denotational semantic soundness.

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An example

A syntactic-free proof that any term of booleans has a defined boolean value true or false

Consider $1 \oplus 1$ (The type of booleans).
\[
\llbracket 1 \oplus 1 \rrbracket = (\{ (1, \star), (2, \star) \}, \mathcal{T}[1 \oplus 1]) \text{ where }
\]
\[
\mathcal{T}(\llbracket 1 \oplus 1 \rrbracket) = \mathcal{P}(\llbracket 1 \oplus 1 \rrbracket) \setminus \emptyset
\]

For any proof $\pi$ of $1 \oplus 1$, we have $\llbracket \pi \rrbracket \in \mathcal{T}[1 \oplus 1]$. Hence $\llbracket \pi \rrbracket \neq \emptyset$. 
Future work

- Categorical axiomitazation of models of $\mu\mathsf{LL}_\infty$.
- Try to understand what sort of information can be obtained from a total interpretation, if not syntactic validity.
Thanks!