Cut-elimination for non-wellfounded proofs beyong μMALL

RECIPROG workshop

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μLL^{∞}: circular and non-wellfounded proofs for linear logic with least and greatest fixed-points

Non-Wellfounded Sequent Calculus

Consider your favourite logic $\mathscr{L}$ & add fixed points as in the μ-calculus

Pre-proofs are the trees **coinductively** generated by:

- $\mathscr{L}$ inference rules
- inference for μ, ν:

$$\frac{\Gamma, F[\mu X.F/X] \vdash \Delta}{\Gamma, \mu X.F \vdash \Delta}\,[\mu_l] \qquad \frac{\Gamma, F[\nu X.F/X] \vdash \Delta}{\Gamma, \nu X.F \vdash \Delta}\,[\nu_l]$$

$$\frac{\Gamma \vdash F[\mu X.F/X], \Delta}{\Gamma \vdash \mu X.F, \Delta}\,[\mu_r] \qquad \frac{\Gamma \vdash F[\nu X.F/X], \Delta}{\Gamma \vdash \nu X.F, \Delta}\,[\nu_r]$$

Circular (pre-)proofs: the regular fragment of infinite (pre-)proofs, ie finitely many sub-(pre)proofs.

Pre-proofs are unsound!!
Need for a validity condition

$$\dfrac{\dfrac{\dfrac{\vdots}{\vdash \mu X.X}\,[\mu]}{\vdash \mu X.X}\,[\mu] \qquad \dfrac{\dfrac{\vdots}{\vdash \nu X.X, F}\,[\nu]}{\vdash \nu X.X, F}\,[\nu]}{\vdash F}\,[\mathrm{Cut}]$$

μLL$^\infty$ Non-Wellfounded Sequent Calculus

Consider your favourite logic LL & add fixed points as in the μ-calculus

μLL$^\infty$ **Pre-proofs** are the trees **coinductively** generated by:

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Circular (pre-)proofs: the regular fragment of infinite (pre-)proofs, ie finitely many sub-(pre)proofs. μLL$^\omega$

Pre-proofs are unsound!!
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$$\dfrac{\dfrac{\dfrac{\vdots}{\vdash \mu X.X}\ [\mu]}{\vdash \mu X.X}\ [\mu] \qquad \dfrac{\dfrac{\vdots}{\vdash \nu X.X, F}\ [\nu]}{\vdash \nu X.X, F}\ [\nu]}{\vdash F}\ [\mathrm{Cut}]$$

Involutive negation, $(\ )^\perp$: operator on formula, not a connective.
One-sided sequents as lists: $\vdash A_1, \dots, A_n$. *($\Gamma \vdash \Delta$ is a short for $\vdash \Gamma^\perp, \Delta$)*
μ **and** ν **are dual binders**. Ex: $(\nu X.X \otimes X)^\perp = \mu X.X \parr X$.

μLK$^\infty$ Inferences

Inference Rules

$$\frac{}{\vdash F, F^\perp}\ (ax) \qquad \frac{\vdash \Gamma, F \qquad \vdash F^\perp, \Delta}{\vdash \Gamma, \Delta}\ (cut) \qquad \frac{\vdash \Gamma, G, F, \Delta}{\vdash \Gamma, F, G, \Delta}\ (ex)$$

$$\frac{\vdash F, F, \Gamma}{\vdash F, \Gamma}\ (c) \qquad \frac{\vdash \Gamma}{\vdash F, \Gamma}\ (w)$$

$$\frac{}{\vdash \top, \Gamma}\ (\top) \qquad \frac{\vdash F, \Gamma \qquad \vdash G, \Gamma}{\vdash F \wedge G, \Gamma}\ (\wedge) \qquad \frac{\vdash A_i, \Gamma}{\vdash A_1 \vee A_2, \Gamma}\ (\vee_i)$$

$$\frac{\vdash G[\nu X.G/X], \Gamma}{\vdash \nu X.G, \Gamma}\ (\nu) \qquad \frac{\vdash F[\mu X.F/X], \Gamma}{\vdash \mu X.F, \Gamma}\ (\mu)$$

μLL$^\infty$ Inferences

μLL formulas

F	$::=$	$a \mid \top \mid \bot \mid F \parr F \mid F \,\&\, F \mid ?F$	**negative *LL* formulas**
		$\mid a^\perp \mid 0 \mid 1 \mid F \otimes F \mid F \oplus F \mid !F$	**positive *LL* formulas**
		$\mid X \mid \mu X.F \mid \nu X.F$	**lfp & gfp**

μLL$^\infty$ Inference Rules

$$\frac{}{\vdash F, F^\perp}\ (ax) \qquad \frac{\vdash \Gamma, F \qquad \vdash F^\perp, \Delta}{\vdash \Gamma, \Delta}\ (cut) \qquad \frac{\vdash \Gamma, G, F, \Delta}{\vdash \Gamma, F, G, \Delta}\ (ex)$$

$$\frac{\vdash F, \Gamma}{\vdash ?F, \Gamma}\ (?) \qquad \frac{\vdash F, ?\Gamma}{\vdash !F, ?\Gamma}\ (!) \qquad \frac{\vdash ?F, ?F, \Gamma}{\vdash ?F, \Gamma}\ (c) \qquad \frac{\vdash \Gamma}{\vdash ?F, \Gamma}\ (w)$$

$$\frac{}{\vdash \top, \Gamma}\ (\top) \qquad \frac{\vdash F, \Gamma \qquad \vdash G, \Gamma}{\vdash F \,\&\, G, \Gamma}\ (\&) \qquad \frac{\vdash A_i, \Gamma}{\vdash A_1 \oplus A_2, \Gamma}\ (\oplus_i) \qquad \text{(no rule for 0)}$$

$$\frac{\vdash \Gamma}{\vdash \bot, \Gamma}\ (\bot) \qquad \frac{\vdash F, G, \Gamma}{\vdash F \parr G, \Gamma}\ (\parr) \qquad \frac{\vdash F, \Gamma \qquad \vdash G, \Delta}{\vdash F \otimes G, \Gamma, \Delta}\ (\otimes) \qquad \frac{}{\vdash \mathbf{1}}\ (\mathbf{1})$$

$$\frac{\vdash G[\nu X.G/X], \Gamma}{\vdash \nu X.G, \Gamma}\ (\nu) \qquad \frac{\vdash F[\mu X.F/X], \Gamma}{\vdash \mu X.F, \Gamma}\ (\mu)$$

How to distinguish valid nwf proofs from invalid ones?

μLL$^\infty$ Inferences

μLL formulas

$$\begin{array}{rcll} F & ::= & a \mid \top \mid \bot \mid F \parr F \mid F \mathbin{\&} F \mid ?F & \textbf{negative } \textit{LL} \textbf{ formulas} \\ & & \mid a^\perp \mid 0 \mid 1 \mid F \otimes F \mid F \oplus F \mid !F & \textbf{positive } \textit{LL} \textbf{ formulas} \\ & & \mid X \mid \mu X.F \mid \nu X.F & \textbf{lfp \& gfp} \end{array}$$

μLL$^\infty$ Inference Rules (with ancestor relation)

$$\dfrac{}{\vdash F, F^\perp}\ (ax) \qquad \dfrac{\vdash \Gamma, F \qquad \vdash F^\perp, \Delta}{\vdash \Gamma, \Delta}\ (cut) \qquad \dfrac{\vdash \Gamma, G, F, \Delta}{\vdash \Gamma, F, G, \Delta}\ (ex)$$

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How to distinguish valid nwf proofs from invalid ones?

Fischer-Ladner subformulas

$FL(F)$ is the least set of formula occurrences such that:

- $F \in FL(F)$;
- $G_1 \star G_2 \in FL(F) \Rightarrow G_1, G_2 \in FL(F)$ for $\star \in \{\oplus, \&, \invamp, \otimes\}$;
- $\sigma X.B \in FL(F) \Rightarrow B[\sigma X.B/X] \in FL(F)$ for $\sigma \in \{\mu, \nu\}$;
- $mG \in FL(F) \Rightarrow G \in FL(F)$ for $m \in \{!, ?\}$.

Fact

$FL(F)$ is a finite set for any formula F.

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Example: $F = \nu X.((a ⅋ a^{\perp}) \otimes (!X \otimes \mu Y.X))$

$$FL(F) = \{F\ , \qquad \}$$

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$$FL(F) = \{F\ ,\ (a \parr a^\perp) \otimes (!F \otimes \mu Y.F), \qquad \}$$

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$$FL(F) = \{F,\ (a \parr a^\perp) \otimes (!F \otimes \mu Y.F),\ a \parr a^\perp,\ a,\ a^\perp,\ !F \otimes \mu Y.F,\ \mu Y.F,\ !F\}$$

Fischer-Ladner subformulas

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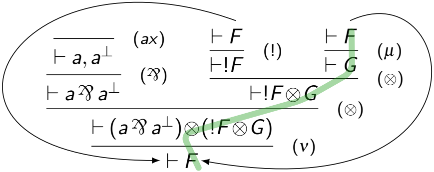
$$
\begin{tikzcd}
 & & a \parr a^\perp \arrow[r] \arrow[dr] & a \\
FL(F) = F \arrow[r] & (a \parr a^\perp) \otimes (!F \otimes \mu Y.F) \arrow[ur] \arrow[dr] & & a^\perp \\
 & & !F \otimes \mu Y.F \arrow[r] \arrow[d] & \mu Y.F \arrow[ulll, bend left] \\
 & & !F \arrow[uull, bend left] &
\end{tikzcd}
$$

Infinite threads, validity

$F = \nu X.((a \parr a^{\perp}) \otimes (!X \otimes \mu Y.X))$.
$G = \mu Y.F$



A ***thread*** on an infinite branch $(\Gamma_i)_{i \in \omega}$ is an infinite sequence of formula occurrences $(F_i)_{i \geq k}$ such that for any $i \geq k$, $F_i \in \Gamma_i$ and F_{i+1} is an immediate ancestor of F_i.

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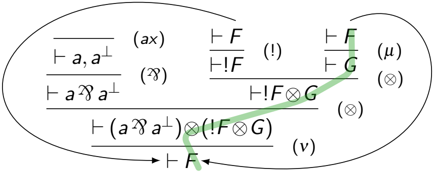

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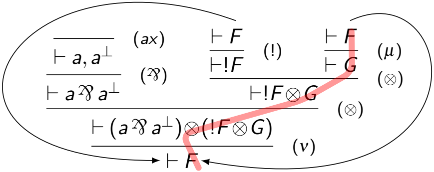
A thread is valid if it unfolds infinitely many ν. More precisely, if the minimal ***recurring*** principal formula of the thread is a ν-formula.

A proof is valid if every infinite branch contains a valid thread.

Infinite threads, validity

$$F = \nu X.((a \parr a^{\perp}) \otimes (!X \otimes \color{red}{G}))$$
$$G = \mu Y.\nu X.((a \parr a^{\perp}) \otimes (!X \otimes Y))$$



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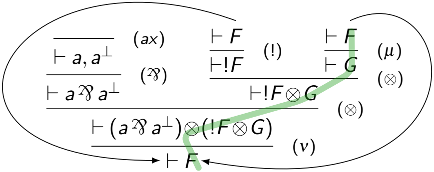

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Fixed-point encoding the exponentials

Consider the following encoding of LL exponentials:

$$
\begin{array}{lcl}
?^{\bullet} F & \triangleq & \mu X. F \oplus (\bot \oplus (X \mathbin{⅋} X)) \\
!^{\bullet} F & \triangleq & \nu X. F \mathbin{\&} (\mathbf{1} \mathbin{\&} (X \otimes X))
\end{array}
$$

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The exponential inferences can be derived:

Dereliction $(?\mathrm{d}^\bullet)$:

$$\dfrac{\dfrac{\vdash F, \Delta}{\vdash F \oplus (\bot \oplus (?^\bullet F \parr ?^\bullet F)), \Delta}\ (\oplus_1)}{\vdash ?^\bullet F, \Delta}\ (\mu)$$

Contraction $(?\mathrm{c}^\bullet)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash ?^\bullet F, ?^\bullet F, \Delta}{\vdash ?^\bullet F \parr ?^\bullet F, \Delta}\ (\parr)}{\vdash \bot \oplus (?^\bullet F \parr ?^\bullet F), \Delta}\ (\oplus_2)}{\vdash F \oplus (\bot \oplus (?^\bullet F \parr ?^\bullet F)), \Delta}\ (\oplus_2)}{\vdash ?^\bullet F, \Delta}\ (\mu)$$

Weakening $(?\mathrm{w}^\bullet)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash \Delta}{\vdash \bot, \Delta}\ (\bot)}{\vdash \bot \oplus (?^\bullet F \parr ?^\bullet F), \Delta}\ (\oplus_1)}{\vdash F \oplus (\bot \oplus (?^\bullet F \parr ?^\bullet F)), \Delta}\ (\oplus_2)}{\vdash ?^\bullet F, \Delta}\ (\mu)$$

Promotion $(!\mathrm{p}^\bullet)$:

$$\dfrac{\vdash F, ?^\bullet\Delta \qquad \dfrac{\dfrac{}{\vdash \mathbf{1}}\ (\mathbf{1})}{\vdash \mathbf{1}, ?^\bullet\Delta}\ (?\mathrm{w}^\bullet) \qquad \dfrac{\dfrac{\vdash !^\bullet F, ?^\bullet\Delta \qquad \vdash !^\bullet F, ?^\bullet\Delta}{\vdash !^\bullet F \otimes !^\bullet F, ?^\bullet\Delta, ?^\bullet\Delta}\ (\otimes)}{\vdash !^\bullet F \otimes !^\bullet F, ?^\bullet\Delta}\ (?\mathrm{c}^\bullet)}{\vdash !^\bullet F, ?^\bullet\Delta}\ (\nu)\ ,\ (\&)\ ,\ (\&)$$

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\end{aligned}
$$

Preservation of validity

π is a valid μMLL$^\infty$ pre-proof of $\vdash \Gamma$ iff
$\pi^\bullet$ is a valid μMALL$^\infty$ pre-proof of $\vdash \Gamma^\bullet$.

Preservation of provability

If $\vdash \Gamma$ is provable in μMLL$^\infty$ (resp. μMLL$^\omega$),
then $\vdash \Gamma^\bullet$ is provable in μMALL$^\infty$ (resp. μMALL$^\omega$).

Fixed-point encoding the exponentials

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If $\vdash \Gamma$ is provable in μMLL$^{\infty}$ (resp. μMLL$^{\omega}$),
then $\vdash \Gamma^{\bullet}$ is provable in μMALL$^{\infty}$ (resp. μMALL$^{\omega}$).

Shortcomings of this encoding

No soundness result for the encoding: converse result for the preservation of provability. Loss of Seely isomorphisms, etc.

Circular & finitary proofs

From finitary to circular proofs

Theorem

Finitary proofs can be transformed to (valid) circular proofs.

The key translation step is the following:

$$\dfrac{\dfrac{\pi_1}{\vdash \Gamma, S} \qquad \dfrac{\pi_2}{\vdash S^\perp, F[S]}}{\vdash \Gamma, \nu X.F}\;(\nu) \quad \longmapsto \quad \dfrac{\dfrac{[\pi_1]}{\vdash \Gamma, S} \qquad \dfrac{\dfrac{\dfrac{[\pi_2]}{\vdash S^\perp, F[S]} \qquad \dfrac{\vdash S^\perp, \nu X.F}{\vdash F[S]^\perp, F[\nu X.F]}\;(r_F)}{\vdash S^\perp, F[\nu X.F]}\;(cut)}{\vdash S^\perp, \nu X.F}\;(\nu)}{\vdash \Gamma, \nu X.F}\;(cut)$$

From circular to finitary proofs

Open problem for μLL^ω.

Cut-elimination for μLL^{∞}

Examples of circular proofs

$$\mathsf{N} = \mu X.1 \oplus X$$

$$\pi_0 = \dfrac{\dfrac{\dfrac{}{\vdash 1}\,(\mathbf{1})}{\vdash 1 \oplus N}\,(\oplus_1)}{\vdash N}\,(\mu) \qquad \pi_{\mathrm{k+1}} = \dfrac{\dfrac{\pi_k}{\vdash 1 \oplus N}\,(\oplus_2)}{\vdash N}\,(\mu) \qquad \pi_{\mathrm{succ}} = \dfrac{\dfrac{\dfrac{}{N \vdash N}\,(ax)}{N \vdash 1 \oplus N}\,(\oplus_2)}{N \vdash N}\,(\mu)$$

$$\pi_{double} = \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash 1}\,(\mathbf{1})}{\vdash 1 \oplus N}\,(\oplus_1)}{\vdash N}\,(\mu)}{1 \vdash N}\,(\bot) \qquad \dfrac{\dfrac{\dfrac{\dfrac{N \vdash N}{N \vdash 1 \oplus N}\,(\oplus_2)}{N \vdash N}\,(\mu)}{N \vdash 1 \oplus N}\,(\oplus_2)}{N \vdash N}\,(\mu)}{\dfrac{1 \oplus N \vdash N}{N \vdash N}\,(\nu)}\,(\&)$$

Examples of circular proofs

$\mathsf{N} = \mu X.1 \oplus X$

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$$\cfrac{\pi_k \qquad \pi_{succ}}{\vdash N}\ (cut) \longrightarrow^\star \pi_{k+1} \qquad\qquad \cfrac{\pi_k \qquad \pi_{double}}{\vdash N}\ (cut) \longrightarrow^\star \pi_{2k}$$

Examples of circular proofs

$\mathsf{N} = \mu X.1 \oplus X$

$$\pi_{\mathrm{dup}} \quad = \quad \dfrac{\dfrac{\dfrac{}{\vdash N_1}\,(\mu),(\oplus_1),(\mathbf{1}) \qquad \dfrac{}{\vdash N_2}\,(\mu),(\oplus_1),(\mathbf{1})}{\mathbf{1} \vdash N_1 \otimes N_2}\,(\bot),(\otimes) \qquad \dfrac{N' \vdash N_1' \otimes N_2' \qquad \dfrac{\pi_{\mathrm{succ}} \qquad \pi_{\mathrm{succ}}}{N_1' \otimes N_2' \vdash N_1 \otimes N_2}\,(\parr),(\otimes)}{N' \vdash N_1 \otimes N_2}\,(\mathrm{Cut})}{N \vdash N_1 \otimes N_2}\,(\nu),(\&)$$

$$\mathsf{WNat}(\pi) = \dfrac{\dfrac{\dfrac{\pi}{\vdash \Gamma}}{1 \vdash \Gamma}\,(\bot) \qquad N \vdash \Gamma}{\dfrac{1 \oplus N \vdash \Gamma}{N \vdash \Gamma}\,(\nu)}\,(\&)$$

Examples of circular proofs

$\mathsf{N} = \mu X.1 \oplus X$

$$\pi_{\mathrm{dup}} \quad = \quad \cfrac{\cfrac{\cfrac{}{\vdash N_1}\,(\mu),(\oplus_1),(\mathbf{1}) \qquad \cfrac{}{\vdash N_2}\,(\mu),(\oplus_1),(\mathbf{1})}{\mathbf{1} \vdash N_1 \otimes N_2}\,(\bot),(\otimes) \qquad \cfrac{N' \vdash N_1' \otimes N_2' \qquad \cfrac{\pi_{\mathrm{succ}} \qquad \pi_{\mathrm{succ}}}{N_1' \otimes N_2' \vdash N_1 \otimes N_2}\,(\parr),(\otimes)}{N' \vdash N_1 \otimes N_2}\,(\mathrm{Cut})}{N \vdash N_1 \otimes N_2}\,(\nu),(\&)$$

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$$\cfrac{\pi_k \qquad \pi_{\mathrm{dup}}}{\vdash N \otimes N}\,(cut) \quad \longrightarrow^{\star} \quad \cfrac{\pi_k \qquad \pi_k}{\vdash N \otimes N}\,(\otimes) \qquad\qquad \cfrac{\pi_k \qquad \mathrm{WNat}(\pi)}{\vdash \Gamma}\,(cut) \quad \longrightarrow^{\star} \quad \cfrac{\pi}{\vdash \Gamma}$$

Examples of circular proofs

$$S = \nu X.(1 \,\&\, (N \otimes X))$$

$$\begin{array}{lcl} enum & : & Nat \to Stream \\ enum(n) & = & n :: enum(succ(n)) \end{array}$$

$$\pi_{enum} = \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash 1}\,(\mathbf{1})}{!N \vdash 1}\,(w) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{N \vdash N}\,(ax)}{!N \vdash N}\,(?) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\pi_{\mathrm{succ}}}{N \vdash N}}{!N \vdash N}\,(?)}{!N \vdash !N}\,(!) \qquad !N \vdash S}{!N \vdash S}\,(cut)}{!N, !N \vdash N \otimes S}\,(\otimes)}{!N \vdash N \otimes S}\,(c)}{!N \vdash 1 \,\&\, (N \otimes S)}\,(\&)}{!N \vdash S}\,(\nu)$$

Examples of circular proofs

$$\color{red}{S}\color{black}{} = \nu X.(1 \,\&\, (N \otimes X))$$

$$\begin{array}{lcl} enum & : & Nat \to Stream \\ enum(n) & = & n :: enum(succ(n)) \end{array}$$

$$\pi_{enum} = \dfrac{\dfrac{\dfrac{\overline{\vdash 1}\;(\mathbf{1})}{!N \vdash 1}\;(w) \qquad \dfrac{\dfrac{\dfrac{\overline{N \vdash N}\;(ax)}{!N \vdash N}\;(?) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\pi_{succ}}{N \vdash N}}{!N \vdash N}\;(?)}{!N \vdash !N}\;(!) \qquad !N \vdash S}{!N \vdash S}\;(cut)}{!N, !N \vdash N \otimes S}\;(\otimes)}{!N \vdash N \otimes S}\;(c)}{!N \vdash 1 \,\&\, (N \otimes S)}\;(\&)}{!N \vdash S}\;(\nu)$$

$$\dfrac{\dfrac{\pi_k}{\vdash ?N}\;(?) \qquad \pi_{enum}}{\vdash S}\;(cut) \quad \longrightarrow^{\omega} \quad \dfrac{\dfrac{\overline{\vdash 1}\;(\mathbf{1}) \qquad \dfrac{\pi_k \qquad \dfrac{\overline{\vdash 1}\;(\mathbf{1}) \qquad \dfrac{\pi_{k+1} \qquad \dfrac{\overline{\vdash 1}\;(\mathbf{1}) \qquad \dfrac{\pi_{k+2} \qquad \dfrac{\cdot^{\cdot^{\cdot}}}{\vdash S}}{\vdash N \otimes S}\;(\otimes)}{\vdash S}\;(\nu)\;,\;(\&)}{\vdash N \otimes S}\;(\otimes)}{\vdash S}\;(\nu)\;,\;(\&)}{\vdash N \otimes S}\;(\otimes)}{\vdash 1 \,\&\, (N \otimes S)}\;(\&)}{\vdash S}\;(\nu)$$

Cut-elimination for μLL^∞

Theorem (Baelde, Doumane & S, 2016)

Fair μMALL$^\infty$ cut-reduction sequences converge to cut-free μMALL$^\infty$ proofs.

Cut-elimination for μLL$^\infty$

Theorem

Fair μLL$^\infty$ mcut-reduction sequences converge to cut-free μLL$^\infty$ proofs.

Idea

The proof goes by:

- considering the following encoding of LL exponential modalities:

$$\begin{array}{rcl} ?^\bullet F & = & \mu X.F \oplus (\bot \oplus (X \parr X)) \\ !^\bullet F & = & \nu X.F \,\&\, (\mathbf{1} \,\&\, (X \otimes X)) \end{array}$$

- translating μLL$^\infty$ sequents and proofs in μMALL$^\infty$,
- simulating μLL$^\infty$ cut-reduction sequences in μMALL$^\infty$ and
- applying μMALL$^\infty$ cut-elimination theorem.

Encoding μLL$^\infty$ in μMALL$^\infty$

$?^\bullet F = \mu X.F \oplus (\bot \oplus (X \parr X)) \qquad !^\bullet F = \nu X.F \,\&\, (\mathbf{1} \,\&\, (X \otimes X))$

μMALL$^\infty$ derivability of the exponential rules ($?\mathrm{d}^\bullet$, $?\mathrm{c}^\bullet$, $?\mathrm{w}^\bullet$, $!\mathrm{p}^\bullet$):

Dereliction :

$$\dfrac{\dfrac{\vdash F, \Delta}{\vdash F \oplus (\bot \oplus (?^\bullet F \parr ?^\bullet F)), \Delta}\,(\oplus_1)}{\vdash ?^\bullet F, \Delta}\,(\mu)$$

Contraction :

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash ?^\bullet F, ?^\bullet F, \Delta}{\vdash ?^\bullet F \parr ?^\bullet F, \Delta}\,(\parr)}{\vdash \bot \oplus (?^\bullet F \parr ?^\bullet F), \Delta}\,(\oplus_2)}{\vdash F \oplus (\bot \oplus (?^\bullet F \parr ?^\bullet F)), \Delta}\,(\oplus_2)}{\vdash ?^\bullet F, \Delta}\,(\mu)$$

Weakening :

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash \Delta}{\vdash \bot, \Delta}\,(\bot)}{\vdash \bot \oplus (?^\bullet F \parr ?^\bullet F), \Delta}\,(\oplus_1)}{\vdash F \oplus (\bot \oplus (?^\bullet F \parr ?^\bullet F)), \Delta}\,(\oplus_2)}{\vdash ?^\bullet F, \Delta}\,(\mu)$$

Promotion:

$$\dfrac{\vdash F, ?^\bullet \Delta \qquad \dfrac{\dfrac{}{\vdash \mathbf{1}}\,(\mathbf{1})}{\vdash \mathbf{1}, ?^\bullet \Delta}\,(?\mathrm{w}^\bullet) \qquad \dfrac{\dfrac{\vdash !^\bullet F, ?^\bullet \Delta \qquad \vdash !^\bullet F, ?^\bullet \Delta}{\vdash !^\bullet F \otimes !^\bullet F, ?^\bullet \Delta, ?^\bullet \Delta}\,(\otimes)}{\vdash !^\bullet F \otimes !^\bullet F, ?^\bullet \Delta}\,(?\mathrm{c}^\bullet)}{\vdash !^\bullet F, ?^\bullet \Delta}\,(\nu)\ ,\ (\&)\ ,\ (\&)$$

Preservation of validity

π is a valid μMLL$^\infty$ pre-proof of $\vdash \Gamma$ iff
$\pi^\bullet$ is a valid μMALL$^\infty$ pre-proof of $\vdash \Gamma^\bullet$.

Simulation of $\mu\mathsf{LL}^\infty$ cut-elimination steps

$\mu\mathsf{LL}^\infty$ cut-elimination steps can be simulated by the previous encoding.

For instance, the following reduction can be simulated by applying the external reduction rule $(\mu)/(cut)$ followed by the external reduction rule $(\oplus)/(cut)$.

$$\dfrac{\dfrac{\vdash F, G, \Gamma}{\vdash ?^\bullet F, G, \Gamma}\ (?\mathrm{d}^\bullet) \qquad \vdash G^\perp, \Delta}{\vdash ?^\bullet F, \Gamma, \Delta}\ (cut) \quad \longrightarrow^2 \quad \dfrac{\dfrac{\vdash F, G, \Gamma \qquad \vdash G^\perp, \Delta}{\vdash F, \Gamma, \Delta}\ (cut)}{\vdash ?^\bullet F, \Gamma, \Delta}\ (?\mathrm{d}^\bullet)$$

Challenge: to show that the simulation of derivation also holds

(i) for the reductions involving $[!p]$ as well as
(ii) for reductions occurring ***above*** a promotion rule (aka. in a box) since the encoding of $[!p]$ uses an infinite, circular derivation.

Simulation of μLL$^\infty$ cut-elimination steps

Cut-commutation rules

$$
\dfrac{\dfrac{\vdash F, G, \Gamma}{\vdash ?^\bullet F, G, \Gamma}\,(?\mathrm{d}^\bullet) \qquad \vdash G^\perp, \Delta}{\vdash ?^\bullet F, \Gamma, \Delta}\,(cut)
\quad \longrightarrow^2 \quad
\dfrac{\dfrac{\vdash F, G, \Gamma \qquad \vdash G^\perp, \Delta}{\vdash F, \Gamma, \Delta}\,(cut)}{\vdash ?^\bullet F, \Gamma, \Delta}\,(?\mathrm{d}^\bullet)
$$

$$
\dfrac{\dfrac{\vdash ?^\bullet F, ?^\bullet F, G, \Gamma}{\vdash ?^\bullet F, G, \Gamma}\,(?\mathrm{c}^\bullet) \qquad \vdash G^\perp, \Delta}{\vdash ?^\bullet F, \Gamma, \Delta}\,(cut)
\quad \longrightarrow^3 \quad
\dfrac{\dfrac{\vdash ?^\bullet F, ?^\bullet F, G, \Gamma \qquad \vdash G^\perp, \Delta}{\vdash ?^\bullet F, ?^\bullet F, \Gamma, \Delta}\,(cut)}{\vdash ?^\bullet F, \Gamma, \Delta}\,(?\mathrm{c}^\bullet)
$$

$$
\dfrac{\dfrac{\vdash G, \Gamma}{\vdash ?^\bullet F, G, \Gamma}\,(?\mathrm{w}^\bullet) \qquad \vdash G^\perp, \Delta}{\vdash ?^\bullet F, \Gamma, \Delta}\,(cut)
\quad \longrightarrow^3 \quad
\dfrac{\dfrac{\vdash G, \Gamma \qquad \vdash G^\perp, \Delta}{\vdash \Gamma, \Delta}\,(cut)}{\vdash ?^\bullet F, \Gamma, \Delta}\,(?\mathrm{w}^\bullet)
$$

$$
\dfrac{\dfrac{\vdash F, ?^\bullet G, ?^\bullet \Gamma}{\vdash !^\bullet F, ?^\bullet G, ?^\bullet \Gamma}\,(!\mathrm{p}^\bullet) \qquad \dfrac{\vdash G, ?^\bullet \Delta}{\vdash !^\bullet G^\perp, ?^\bullet \Delta}\,(!\mathrm{p}^\bullet)}{\vdash !^\bullet F, ?^\bullet \Gamma, ?^\bullet \Delta}\,(cut)
\quad \longrightarrow^\omega \quad
\dfrac{\dfrac{\vdash F, ?^\bullet G, ?^\bullet \Gamma \qquad \dfrac{\vdash G, ?^\bullet \Delta}{\vdash !^\bullet G^\perp, ?^\bullet \Delta}\,(!\mathrm{p}^\bullet)}{\vdash F, ?^\bullet \Gamma, ?^\bullet \Delta}\,(cut)}{\vdash !^\bullet F, ?^\bullet \Gamma, ?^\bullet \Delta}\,(!\mathrm{p}^\bullet)
$$

Simulation of μLL$^\infty$ cut-elimination steps

Key-cut rules

$$\dfrac{\dfrac{\dfrac{\pi}{\vdash F, \Gamma}}{\vdash ?^\bullet F, \Gamma}\,(?\mathrm{d}^\bullet) \qquad \dfrac{\dfrac{\pi'}{\vdash F^\perp, ?^\bullet \Delta}}{\vdash !^\bullet F^\perp, ?^\bullet \Delta}\,(!\mathrm{p}^\bullet)}{\vdash \Gamma, ?^\bullet \Delta}\,(cut) \longrightarrow^2 \dfrac{\dfrac{\pi}{\vdash F, \Gamma} \qquad \dfrac{\pi'}{\vdash F^\perp, ?^\bullet \Delta}}{\vdash \Gamma, ?^\bullet \Delta}\,(cut)$$

$$\dfrac{\dfrac{\dfrac{\pi}{\vdash ?^\bullet F, ?^\bullet F, \Gamma}}{\vdash ?^\bullet F, \Gamma}\,(?\mathrm{c}^\bullet) \qquad \dfrac{\dfrac{\pi'}{\vdash F^\perp, ?^\bullet \Delta}}{\vdash !^\bullet F^\perp, ?^\bullet \Delta}\,(!\mathrm{p}^\bullet)}{\vdash \Gamma, ?^\bullet \Delta}\,(cut) \longrightarrow^{4int, 4\times\#\Delta ext}$$

$$\dfrac{\dfrac{\dfrac{\pi}{\vdash ?^\bullet F, ?^\bullet F, \Gamma} \qquad \dfrac{\pi'}{\vdash !^\bullet F^\perp, ?^\bullet \Delta} \qquad \dfrac{\pi'}{\vdash !^\bullet F^\perp, ?^\bullet \Delta}}{\vdash \Gamma, ?^\bullet \Delta, ?^\bullet \Delta}\,(mcut)}{\vdash \Gamma, ?^\bullet \Delta}\,(?\mathrm{c}^\bullet)^\star$$

$$\dfrac{\dfrac{\dfrac{\pi}{\vdash \Gamma}}{\vdash ?^\bullet F, \Gamma}\,(?\mathrm{w}^\bullet) \qquad \dfrac{\dfrac{\pi'}{\vdash F^\perp, ?^\bullet \Delta}}{\vdash !^\bullet F^\perp, ?^\bullet \Delta}\,(!\mathrm{p}^\bullet)}{\vdash \Gamma, ?^\bullet \Delta}\,(cut) \longrightarrow^{3int, 3\times\#\Delta ext} \dfrac{\dfrac{\pi}{\vdash \Gamma}}{\vdash \Gamma, ?^\bullet \Delta}\,(?\mathrm{w}^\bullet)^\star$$

Cut-elimination for $\mu\mathsf{LL}^\infty$

1. Consider a fair cut-reduction sequence $\sigma = (\pi_i)_{i\in\omega}$ in $\mu\mathsf{LL}^\infty$ from π.
2. σ converges to a cut-free $\mu\mathsf{LL}^\infty$ pre-proof. By contradiction: Otherwise, a suffix τ of σ would contain only key-cut steps. The encoding of τ in $\mu\mathsf{MALL}^\infty$, $\tau^\bullet$ would either be unproductive or would produce an infinite tree of encodings of $?\mathsf{w}, ?\mathsf{c}$ containing no ν inference. This would contradict $\mu\mathsf{MALL}^\infty$ cut-elimination theorem.

Cut-elimination for μLL^∞

1. Consider a fair cut-reduction sequence $\sigma = (\pi_i)_{i \in \omega}$ in μLL^∞ from π.
2. σ converges to a cut-free μLL^∞ pre-proof.
3. As σ is productive and since reduction only occurs above cuts, it strongly converges to some μLL^∞ cut-free pre-proof π'.
4. $\sigma^\bullet$ is a transfinite sequence from $\pi^\bullet$ strongly converging to $\pi'^\bullet$: because $\pi'^\bullet$ – the encoding of π' – is cut-free and because only ! commutations and reductions above a promotion create infinite reductions: boxes are simulated by strongly converging sequences.

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4. $\sigma^\bullet$ is a transfinite sequence from $\pi^\bullet$ strongly converging to $\pi'^\bullet$.
5. The compression lemma applies: there exists ρ an ω-indexed μMALL$^\infty$ cut-reduction sequence converging to $\pi'^\bullet$.
6. Fairness of σ transfers to ρ: ρ is a fair μMALL$^\infty$ cut-red sequence.

Cut-elimination for μLL^∞

1. Consider a fair cut-reduction sequence $\sigma = (\pi_i)_{i\in\omega}$ in μLL^∞ from π.
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3. As σ is productive and since reduction only occurs above cuts, it strongly converges to some μLL^∞ cut-free pre-proof π'.
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5. The compression lemma applies: there exists ρ an ω-indexed $\mu MALL^\infty$ cut-reduction sequence converging to $\pi'^\bullet$.
6. Fairness of σ transfers to ρ: ρ is a fair $\mu MALL^\infty$ cut-red sequence.
7. Therefore, by $\mu MALL^\infty$ cut-elimination thm, ρ has a limit, $\pi'^\bullet$, which is a valid cut-free $\mu MALL^\infty$ proof.
8. Using preservation of validity, π' is a valid cut-free μLL^∞-proof. $\square$

About Seely isomorphisms

Two conjunctions and two disjunctions in LL: additives and multiplicatives.
In LK, they are interderivable thanks to structural rules. One has:

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,(ax)}{A, B \vdash A}\,(W_l) \qquad \dfrac{\dfrac{}{B \vdash B}\,(ax)}{A, B \vdash B}\,(W_l)}{A, B \vdash A \wedge^a B}\,(\wedge_r^a)}{A \wedge^m B \vdash A \wedge^a B}\,(\wedge_l^m)
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,(ax)}{A \wedge^a B \vdash A}\,(\wedge_l^{a1}) \qquad \dfrac{\dfrac{}{B \vdash B}\,(ax)}{A \wedge^a B \vdash B}\,(\wedge_l^{a2})}{A \wedge^a B, A \wedge^a B \vdash A \wedge^m B}\,(\wedge_r^m)}{A \wedge^a B \vdash A \wedge^m B}\,(C_l)
$$

A, B are weakened on the left, $A \wedge^a B$ is contracted on the left.

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A, B are weakened on the left, $A\wedge^a B$ is contracted on the left.

In LL, we do not have free structural rules, but only thanks to exponentials, so we need to mark formulas with exponentials where structural rules are needed, leading to: $!A\otimes !B \dashv\vdash !(A\& B)$.

$$\pi_S = \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A\vdash A}\,(ax)}{!A\vdash A}\,(?)}{!A,!B\vdash A}\,(w) \qquad \dfrac{\dfrac{\dfrac{}{B\vdash B}\,(ax)}{!B\vdash B}\,(?)}{!A,!B\vdash B}\,(w)}{!A,!B\vdash A\& B}\,(\&)}{!A,!B\vdash !(A\& B)}\,(!)}{!A\otimes !B\vdash !(A\& B)}\,(\parr) \qquad \pi'_S = \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A\vdash A}\,(ax)}{A\& B\vdash A}\,(\oplus_1)}{!(A\& B)\vdash A}\,(?)}{!(A\& B)\vdash !A}\,(!) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{B\vdash B}\,(ax)}{A\& B\vdash B}\,(\oplus_2)}{!(A\& B)\vdash B}\,(?)}{!(A\& B)\vdash !B}\,(!)}{!(A\& B),!(A\& B)\vdash !A\otimes !B}\,(\otimes)}{!(A\& B)\vdash !A\otimes !B}\,(c)$$

About Seely isomorphisms

$!A \otimes !B \dashv\vdash\, !(A \,\&\, B)$

$$\pi_S = \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,(ax)}{!A \vdash A}\,(?)}{!A, !B \vdash A}\,(w) \qquad \dfrac{\dfrac{\dfrac{}{B \vdash B}\,(ax)}{!B \vdash B}\,(?)}{!A, !B \vdash B}\,(w)}{!A, !B \vdash A \,\&\, B}\,(\&)}{\dfrac{!A, !B \vdash\, !(A \,\&\, B)}{!A \otimes !B \vdash\, !(A \,\&\, B)}\,(⅋)}\,(!)$$

$$\pi'_S = \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,(ax)}{A \,\&\, B \vdash A}\,(\oplus_1)}{!(A \,\&\, B) \vdash A}\,(?)}{!(A \,\&\, B) \vdash\, !A}\,(!) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{B \vdash B}\,(ax)}{A \,\&\, B \vdash B}\,(\oplus_2)}{!(A \,\&\, B) \vdash B}\,(?)}{!(A \,\&\, B) \vdash\, !B}\,(!)}{!(A \,\&\, B), !(A \,\&\, B) \vdash\, !A \otimes !B}\,(\otimes)}{!(A \,\&\, B) \vdash\, !A \otimes !B}\,(c)$$

$$\dfrac{\pi_S \qquad \pi'_S}{!A \otimes !B \vdash\, !A \otimes !B}\,(cut)$$

About Seely isomorphisms

$!A \otimes !B \dashv\vdash !(A \& B)$

$$\pi_S = \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,(ax)}{!A \vdash A}\,(?)}{!A, !B \vdash A}\,(w) \qquad \dfrac{\dfrac{\dfrac{}{B \vdash B}\,(ax)}{!B \vdash B}\,(?)}{!A, !B \vdash B}\,(w)}{\dfrac{!A, !B \vdash A \& B}{\dfrac{!A, !B \vdash !(A \& B)}{!A \otimes !B \vdash !(A \& B)}\,(\parr)}\,(!)}\,(\&)$$

$$\pi'_S = \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,(ax)}{A \& B \vdash A}\,(\oplus_1)}{!(A \& B) \vdash A}\,(?)}{!(A \& B) \vdash !A}\,(!) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{B \vdash B}\,(ax)}{A \& B \vdash B}\,(\oplus_2)}{!(A \& B) \vdash B}\,(?)}{!(A \& B) \vdash !B}\,(!)}{\dfrac{!(A \& B), !(A \& B) \vdash !A \otimes !B}{!(A \& B) \vdash !A \otimes !B}\,(c)}\,(\otimes)$$

$$\dfrac{\pi_S \qquad \pi'_S}{!A \otimes !B \vdash !A \otimes !B}\,(cut) \quad \rightarrow^{\star}_{cut} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,(ax)}{!A \vdash A}\,(?)}{!A, !B \vdash A}\,(w)}{!A, !B \vdash !A}\,(!) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{B \vdash B}\,(ax)}{!B \vdash B}\,(?)}{!A, !B \vdash B}\,(w)}{!A, !B \vdash !B}\,(!)}{\dfrac{!A, !B, !A, !B \vdash !A \otimes !B}{\dfrac{!A, !B \vdash !A \otimes !B}{!A \otimes !B \vdash !A \otimes !B}\,(\parr)}\,(c)^2}\,(\otimes)$$

About Seely isomorphisms

What about the fixed-point encoding?

$$\dfrac{(\pi_S)^\bullet \qquad (\pi'_S)^\bullet}{!^\bullet A \otimes !^\bullet B \vdash !^\bullet A \otimes !^\bullet B}\ (cut)$$

The left occurrences of A, B require two unfolding of the fixed-point, while the right occurrences of A, B require only one unfolding of the fixed-point.

The fixed-point unfolding structure tracks te history of the structural rules.

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What about the fixed-point encoding?

$$\dfrac{(\pi_S)^\bullet \qquad (\pi'_S)^\bullet}{!^\bullet A \otimes !^\bullet B \vdash !^\bullet A \otimes !^\bullet B}\ (cut) \qquad \to^{\omega}_{cut} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\ (ax)}{!^\bullet A \vdash A}\ (?)}{!^\bullet A, !^\bullet B \vdash A}\ (w)}{!^\bullet A, ! B \vdash ! A}\ (!) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{B \vdash B}\ (ax)}{!^\bullet B \vdash B}\ (?)}{!^\bullet A, !^\bullet B \vdash B}\ (w)}{!^\bullet A, !^\bullet B \vdash !^\bullet B}\ (!)}{!^\bullet A, !^\bullet B, !^\bullet A, !^\bullet B \vdash !^\bullet A \otimes !^\bullet B}\ (\otimes)}{!^\bullet A, !^\bullet B \vdash !^\bullet A \otimes !^\bullet B}\ (c)^2}{!^\bullet A \otimes !^\bullet B \vdash !^\bullet A \otimes !^\bullet B}\ (\parr)$$

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The fixed-point unfolding structure tracks te history of the structural rules.

Cut-elimination for μLK$^\infty$, μLJ$^\infty$

The usual call-by-value embedding of LJ in ILL (intuitionnistic LL) can be lifted to μLJ$^\infty$: indeed, the translation of proofs does not introduce cuts. For μLK$^\infty$, it is slightly trickier as the well-known T/Q-translations introduce cuts breaking validity. An alternative translation which does not introduce cuts can be used.

Moreover, one gets the skeleton of a μLL$^\infty$ (resp. μILL$^\infty$) proof which is a μLK$^\infty$ (resp. μLJ$^\infty$) proof, simply by erasing the exponentials (connectives and inferences), preserving validity.
The skeleton of a μLL$^\infty$ (resp. μILL$^\infty$) cut-reduction sequence is a μLK$^\infty$ (resp. μLJ$^\infty$) cut-reduction sequence. As a result, one has:

Theorem

If π is an μLK$^\infty$ (resp. μLJ$^\infty$) proof of $\vdash \Gamma$ (resp. $\Gamma \vdash F$), there exists a μLL$^\infty$ (resp. μILL$^\infty$) proof of the translated sequents.

Theorem

There are productive cut-reduction strategies producing cut-free μLK$^\infty$ (resp. μLJ$^\infty$) proofs.

Conclusion

- To sum up:
  - Fixed-point logics extending LL/LK/LJ with finite circular or non-wellfounded proofs;
  - A parity condition to discriminate valid/invalid proofs;
  - Syntactic cut elimination for various nwf sequent calculi: μMALL$^\infty$, μLL$^\infty$, μLJ$^\infty$, μLK$^\infty$.

Thanks!

μMALL$^\infty$ **Cut elimination**

μMALL$^\infty$ Cut Elimination Theorem

Theorem (Baelde, Doumane & S, 2016)

Fair μMALL$^\infty$ cut-reduction sequences converge to cut-free μMALL$^\infty$ proofs.

Previous result by Santocanale and Fortier
for the purely additive fragment of μMALL$^\infty$.
Proof uses a locative treatment of occurrences.

- **Strategy:** "push" the cuts away from the root.

- **Cut-Cut:**

$$\dfrac{\dfrac{\vdash \Gamma, F \qquad \vdash F^\perp, \Delta, G}{\vdash \Gamma, \Delta, G}\,(cut) \qquad \vdash G^\perp, \Sigma}{\vdash \Gamma, \Delta, \Sigma}\,(cut) \longleftrightarrow \dfrac{\vdash \Gamma, F \qquad \dfrac{\vdash F^\perp, \Delta, G \qquad \vdash G^\perp, \Sigma}{\vdash F^\perp, \Delta, \Sigma}\,(cut)}{\vdash \Gamma, \Delta, \Sigma}\,(cut)$$

μMALL$^\infty$ Cut Elimination Theorem

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$$\dfrac{\dfrac{\vdash \Gamma, F \qquad \vdash F^\perp, \Delta, G}{\vdash \Gamma, \Delta, G}\ (cut) \qquad \vdash G^\perp, \Sigma}{\vdash \Gamma, \Delta, \Sigma}\ (cut) \quad \longrightarrow \quad \dfrac{\vdash \Gamma, F \qquad \vdash F^\perp, \Delta, G \qquad \vdash G^\perp, \Sigma}{\vdash \Gamma, \Delta, \Sigma}\ (mcut)$$

Cut elimination procedure

External phase: Cut-commutation cases

$$\dfrac{\dfrac{\vdash \Delta, F, G}{\vdash \Delta, F \parr G}(\parr) \qquad \dots}{\vdash \Sigma, F \parr G}(mcut) \quad \Rightarrow \quad \dfrac{\dfrac{\vdash \Delta, F, G \qquad \dots}{\vdash \Sigma, F, G}(mcut)}{\vdash \Sigma, F \parr G}(\parr)$$

$$\dfrac{\dfrac{\vdash \Delta, F \qquad \vdash \Delta, G}{\vdash \Delta, F \mathbin{\&} G}(\&) \qquad \dots}{\vdash \Sigma, F \mathbin{\&} G}(mcut) \quad \Rightarrow \quad \dfrac{\dfrac{\vdash \Delta, F \qquad \dots}{\vdash \Sigma, F}(mcut) \qquad \dfrac{\vdash \Delta, G \qquad \dots}{\vdash \Sigma, G}(mcut)}{\vdash \Sigma, F \mathbin{\&} G}(\&)$$

$$\dfrac{\dfrac{\vdash \Delta, F[\mu X.F/X]}{\vdash \Delta, \mu X.F}(\mu) \qquad \dots}{\vdash \Sigma, \mu X.F}(mcut) \quad \Rightarrow \quad \dfrac{\dfrac{\vdash \Delta, F[\mu X.F/X] \qquad \dots}{\vdash \Sigma, F[\mu X.F/X]}(mcut)}{\vdash \Sigma, \mu X.F}(\mu)$$

+ additional cases

Cut-commutation steps are productive

Cut elimination procedure

Internal Phase: Key cases

$$\dfrac{\dots \qquad \dfrac{\vdash \Delta, F_2 \qquad \vdash \Delta, F_1}{\vdash \Delta, F_2 \,\&\, F_1}\ (\&) \qquad \dfrac{\vdash \Gamma, F_i^{\perp}}{\vdash \Gamma, F_1^{\perp} \oplus F_2^{\perp}}\ (\oplus_i)}{\vdash \Sigma}\ (mcut)$$

$$\Rightarrow \quad \dfrac{\dots \qquad \vdash \Delta, F_i \qquad \vdash \Gamma, F_i^{\perp}}{\vdash \Sigma}\ (mcut)$$

$$\dfrac{\dots \qquad \dfrac{\vdash \Delta, F[\mu X.F/X]}{\vdash \Delta, \mu X.F}\ (\mu) \qquad \dfrac{\vdash \Gamma, F^{\perp}[\nu X.F^{\perp}/X]}{\vdash \Gamma, \nu X.F^{\perp}}\ (\nu)}{\vdash \Sigma}\ (mcut)$$

$$\Rightarrow \quad \dfrac{\dots \qquad \vdash \Delta, F[\mu X.F/X] \qquad \vdash \Gamma, F^{\perp}[\nu X.F^{\perp}/X]}{\vdash \Sigma}\ (mcut)$$

+ additional cases

Key cases are not productive

Cut elimination algorithm

- Internal phase: Perform key case reductions as long as you cannot do anything else.
- External phase: Build a part of the output tree by applying cut-commutation steps as soon possible, being fair.
- Repeat.

Cut elimination algorithm

- Internal phase: Perform key case reductions as long as you cannot do anything else.
- External phase: Build a part of the output tree by applying cut-commutation steps as soon possible, being fair.
- Repeat.

Remark: We consider a **fair** strategy ie. every reduction which is available at some point will be performed eventually.

Theorem

Internal phases always halt. Cut-elimination produces a pre-proof.

Theorem

The pre-proof obtained by the cut elimination algorithm is valid.

μLL$^\omega$ is not stable by cut-elimination

Eliminating cuts from a μLL$^\omega$ proof (circular) may result in a μLL$^\infty$, non circular, proof.

Cut elimination is productive

Theorem

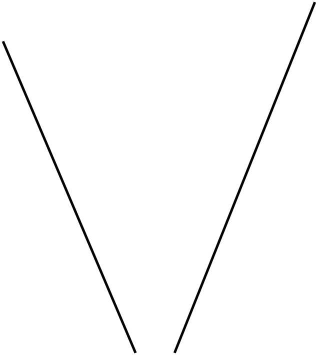
Internal phase always halts.

Cut elimination is productive

Theorem

Internal phase always halts.

Proof by contradiction: Suppose that there is a proof of F for which the internal phase does not halt.



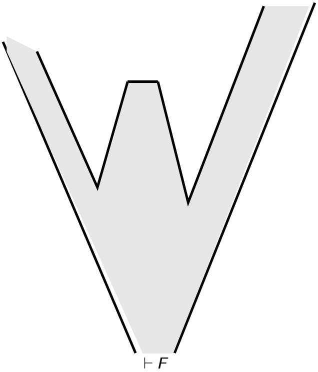
$\vdash F$

Cut elimination is productive

Theorem

Internal phase always halts.

Proof by contradiction: Consider the trace of this divergent reduction.

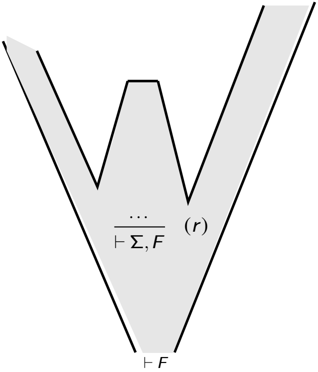


Cut elimination is productive

Theorem

Internal phase always halts.

Proof by contradiction: No rule on F is applied in the trace, otherwise the internal phase would halt.

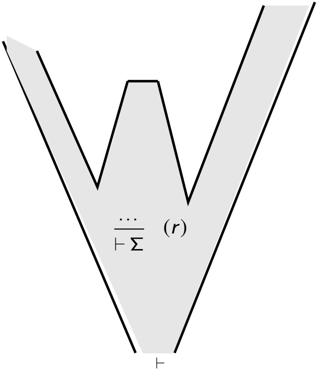


Cut elimination is productive

Theorem

Internal phase always halts.

Proof by contradiction: We can eliminate the occurrences of F from the trace. This yields a "proof" of $\vdash$.

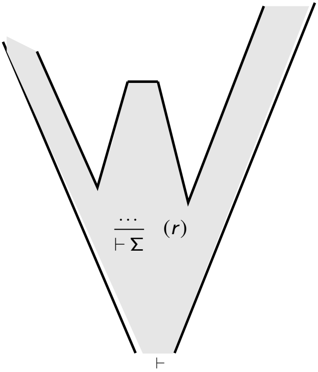


Cut elimination is productive

Theorem

Internal phase always halts.

Proof by contradiction: We show that the proof system is sound. Contradiction.



Cut elimination produces a proof

Theorem

The pre-proof obtained by the cut elimination algorithm is valid.

Cut elimination produces a proof

Theorem

The pre-proof obtained by the cut elimination algorithm is valid.

Proof: Let $\pi^\star$ be the pre-proof obtained from $\pi \vdash \Delta$ by cut elimination. Suppose that a branch b of $\pi^\star$ is not valid.

- Let θ be the sub-derivation of π explored by the reduction that produces b.
- **Fact:** Threads of θ are the threads of b, together with threads starting from cut formulas.
- The validity of θ cannot rely on the threads of b.
- θ^μ is θ where we replace in Δ any ν by a μ and any $1, \top$ by $\bot, 0$.
- Show that formulas containing only $\mu, \bot, 0$ and $MALL$ connectives are false.
- θ^μ proves a false sequent which contradicts soundness.