Infinitary normal forms for non-deterministic λ-terms

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Böhm trees

- In the pure λ-calculus, we perform β-reduction

$$(\lambda x\, M)\ N \rightarrow_\beta M\,[N/x]$$

  contextually.
- By standardization, the leftmost outermost strategy reaches normal forms when they exist.
- Böhm trees = coinductive left normal forms

$$\mathcal{BT}\,(M) = \begin{cases} \lambda \overrightarrow{x}\ (y)\ \mathcal{BT}\,(N_1) \cdots \mathcal{BT}\,(N_k) & \text{if } M \simeq_\beta \lambda \overrightarrow{x}\ (y)\ N_1 \cdots N_k \\ \bot & \text{if } M \text{ has no hnf} \end{cases}$$

  e.g. $\mathcal{BT}\,(Y) = \lambda x\ (x)\ (x)\ (x)\ \cdots$
- Böhm trees form a syntactic denotational model, that is sensible

Note: here we do not consider extensionality issues

Infinitary normal forms in a non deterministic setting

Proposals for non deterministic Böhm trees exist:

- de'Liguoro and Piperno's Böhm trees for erratic choice (1995)
- recent work by Thomas Leventis on probabilistic Böhm trees

Alternative semantic approaches:

- in domain theory: powerdomains and the like (around 1980)
- Girard's quantitative semantics (around 1980)

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Here:

The "Böhm tree" of a non-deterministic λ-term is the normal form of its Taylor expansion

Quantitative semantics

A prime aged idea (Girard, '80s, before LL)

λ-terms = analytic functions = power series
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Finiteness spaces (Ehrhard, early 2000’s)

Reformulate q.s. in linear logic using standard algebra:

- types $\rightsquigarrow$ particular topological vector spaces (or semimodules):
  $[\![A]\!] \subseteq \mathbf{S}^{|A|}$ + some additional structure
- function terms $\rightsquigarrow$ power series

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Differentiation of λ-terms (Ehrhard-Regnier 2003-2004)

So we can *differentiate* λ-terms, and compute their Taylor expansion!
And one can mimick that in the syntax:

- differential λ-calculus
- a finitary fragment: resource λ-calculus
  = the target of Taylor expansion

Resource λ-calculus

Resource terms

$$\begin{array}{rcccl} \Delta & \ni & s, t, \dots & ::= & x \mid \lambda x\, s \mid \langle s \rangle\ \bar{t} \\ !\Delta & \ni & \bar{s}, \bar{t}, \dots & ::= & [s_1, \dots, s_n] \end{array}$$

Meaning: $\langle s \rangle\ [s_1, \dots, s_n] = (Ds)_0 \cdot (s_1, \dots, s_n)$

Resource reduction

$$\langle \lambda x\, s \rangle\ \bar{t} \rightarrow_\partial \partial_x s \cdot \bar{t} \quad \text{(anywhere)}$$

$$\partial_x s \cdot \bar{t} = \begin{cases} \sum_{f \in \mathfrak{S}_n} s \left[t_{f(1)}, \dots, t_{f(n)} / x_1, \dots, x_n \right] & \text{if } \mathbf{n}_x(s) = \# \bar{t} = n \\ 0 & \text{otherwise} \end{cases}$$

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+ linearity: $\lambda x\, 0 = 0$, $\langle s \rangle\ [t_1 + t_2, u] = \langle s \rangle\ [t_1, u] + \langle s \rangle\ [t_2, u]$, ...

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- Resource reduction preserves free variables, is size-decreasing, strongly confluent and normalizing.

Taylor expansion of λ-terms

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Taylor expansion: $\mathcal{T}(M) \in \mathbf{S}^{\Delta}$

$$\mathcal{T}((M)\ N) = \sum_{n\in\mathbf{N}} \frac{1}{n!} \langle \mathcal{T}(M) \rangle\ \mathcal{T}(N)^n$$

$$\mathcal{T}(x) = x \qquad \mathcal{T}(\lambda x\, M) = \lambda x\, \mathcal{T}(M)$$

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Theorem (Ehrhard-Regnier, CiE 2006)

If $M \in \Lambda$, then $\mathcal{T}(M)$ normalizes to $\mathcal{T}(\mathcal{BT}(M))$.

Moral

In the ordinary λ-calculus $\mathcal{BT}(M) \simeq \mathsf{NF}(\mathcal{T}(M))$.

Normalizing Taylor expansions

But how can $\mathcal{T}(M)$ even normalize?

We want to set

$$\mathsf{NF}(\mathcal{T}(M)) = \sum_{s \in \Delta} \mathcal{T}(M)_s . \mathsf{NF}(s)$$

$\rightsquigarrow$ infinite sums *(and in general we might consider all kinds of coefficients)*
$\rightsquigarrow$ convergence?

Normalizing Taylor expansions: *uniformity to the rescue!*

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Theorem (Ehrhard-Regnier 2004, published in TCS in 2008)

Write $|\mathcal{T}(M)| = |\mathcal{T}(M)|$. Then for all $t \in \Delta$, there is at most one $s \in |\mathcal{T}(M)|$ such that $\mathsf{NF}(s)_t \neq 0$.

Proof.

λ-terms are uniform: their finitary approximants are pairwise coherent. $\square$

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This fails in general

$\mathsf{NF}\left(\sum_{n \in \mathbf{N}} \langle \lambda x\, x \rangle^n [y]\right) = ?$ $\qquad \langle \lambda x\, x \rangle^n [y] = \langle \lambda x\, x \rangle \, [\langle \lambda x\, x \rangle \, [\cdots [y] \cdots]]$

Taylor expansion in a non uniform setting

A generic quantitative non-uniform calculus

$$\mathbf{S}\,[\Lambda_{\mathbf{S}}] \ni M, N, \ldots ::= x \mid \lambda x\, M \mid (M)\; N \mid M + N$$

$$(\lambda x\, M)\; N \to_\beta M\,[N/x] \qquad (M+N)\; P = (M)\; P + (N)\; P$$

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Taylor expansion of a sum

$$\mathcal{T}\left(\sum_{i=1}^{n} a_i\, M_i\right) = \sum_{i=1}^{n} a_i\, \mathcal{T}\,(M_i)$$

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Then $\mathsf{NF}\,(\mathcal{T}\,(\infty_M)) = ?$
Worse: $\mathsf{NF}\,(\mathcal{T}\,(\infty_M - (\lambda x\, x)\ \infty_M)) = ?$

Finiteness structures to the rescue!

The main artifact of finiteness spaces:

Definition

- *If* $a, a' \subseteq A$, *write* $a \perp a'$ *iff* $a \cap a'$ *is finite.*
- *If* $\mathfrak{S} \subseteq \mathfrak{P}(A)$, *let* $\mathfrak{S}^{\perp} := \{a' \subseteq A \,;\, \forall a \in \mathfrak{S},\ a \perp a'\}$.
- *A finiteness structure is any* $\mathfrak{F} = \mathfrak{S}^{\perp}$.

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When is $\mathcal{T}(M)$ normalizable?

- Write $s \geq_{\partial} t$ if $s \to_{\partial}{}^{*} t + \cdots$.
- Let $\uparrow t = \{s \in \Delta \,;\, s \geq_{\partial} t\}$.
- $\mathcal{T}(M)$ is normalizable iff for all normal $t \in \Delta$, $|\mathcal{T}(M)| \perp \uparrow t$.
- $\{\uparrow t \,;\, t \text{ normal } \in \Delta\}^{\perp}$ is the finiteness structure of (supports of) normalizable vectors.

Typed terms have a finitary Taylor expansion

Let system F_+ be system F plus $\dfrac{\Gamma \vdash M : A \qquad \Gamma \vdash N : A}{\Gamma \vdash M + N : A}$.

Theorem (Ehrhard, LICS 2010)

If $M \in \mathbf{S}[\Lambda_{\mathbf{S}}]$ *is typable in system* F_+, *then* $|\mathcal{T}(M)| \in \{\uparrow t \,;\, t \in \Delta\}^{\perp}$.

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Theorem (Pagani–Tasson–V., FoSSaCS 2016)

The same holds for all strongly normalizing terms, and we even have:
$M \in \mathsf{SN}$ *iff* $|\mathcal{T}(M)| \in \{\uparrow a \,;\, a \in \mathfrak{B}\}^{\perp}$ *for a well chosen* $\mathfrak{B}$.

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Theorem (Pagani–Tasson–V., *WIP*)

The above results generalize to (weak or head) normalizability.

In particular if M is normalizable then $|\mathcal{T}(M)| \in \{\uparrow s\,;\, s \text{ normal}\}^{\perp}$.

TODO

We know:

Normalizable terms have a normalizable Taylor expansion.

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Theorem?

If $M \to_\beta N$ then $\mathcal{T}\left(M\right) \rightsquigarrow \mathcal{T}\left(N\right)$.

- And then, what about non normalizing terms?

β-reduction through Taylor expansion

Recall that:

$$\mathcal{T}\left((M)\ N\right) = \sum_{n \in \mathbf{N}} \frac{1}{n!} \left\langle \mathcal{T}\left(M\right)\right\rangle \ \mathcal{T}\left(N\right)^n$$

In quantitative semantics:

$$[\![(\lambda x\, M)\ N]\!] = [\![M\,[N/x]]\!] = \sum_{n \in \mathbf{N}} \frac{1}{n\,!} \left(\frac{\partial^n\, [\![M]\!]}{\partial x^n} \right)_{x=0} \cdot [\![N]\!]^n.$$

β-reduction through Taylor expansion: key steps

Promotion

$$\sigma^! := \sum_{n\in\mathbf{N}} \frac{1}{n\,!}\sigma^n \in \mathbf{S}^{!\Delta}$$

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- Combinatorics everywhere

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- Combinatorics everywhere
- We actually need parallel reduction: ensuring confluence requires a bit of work

Characterizing normalizability: the uniform case

Standardization: normalizability = left-reduction to a normal form.

Inductive characterization

$$\frac{(M\,[N_0/x])\;N_1\cdots N_n\Downarrow}{(\lambda x\,M)\;N_0\,N_1\cdots N_n\Downarrow} \qquad \frac{N_1\Downarrow \qquad \cdots \qquad N_n\Downarrow}{\lambda\overrightarrow{x}\;(y)\;N_1\cdots N_n\Downarrow}$$

Böhm trees: interpret the second rule coinductively.

Characterizing normalizability: the non-uniform case

Add the following rule:

$$\frac{M_1{\Downarrow} \qquad \cdots \qquad M_n{\Downarrow}}{\sum_{i=1}^{n} a_i\, M_i{\Downarrow}}$$

In general this is stronger than the naive notion of normalizability (consider $\infty_M - (I)\ \infty_M$).

Recursively head-normalizable terms

Definition (k-normalizable terms)

$$\frac{}{M{\Downarrow}_0} \qquad \frac{N_1{\Downarrow}_k \qquad \cdots \qquad N_n{\Downarrow}_k}{\lambda \overrightarrow{x}\ (y)\ N_1 \cdots N_n \Big\Downarrow_{k+1}}$$

$$\frac{(M\,[N_0/x])\ N_1 \cdots N_n \Big\Downarrow_k}{(\lambda x\, M)\ N_0\, N_1 \cdots N_n \Big\Downarrow_k} \qquad \frac{M_1{\Downarrow}_k \qquad \cdots \qquad M_n{\Downarrow}_k}{\sum_{i=1}^{n} a_i\, M_i \Big\Downarrow_k}$$

- Write $M{\Downarrow}_\infty$ if $M{\Downarrow}_k$ for all k.

Normalizing Taylor expansion, level by level

If $M{\Downarrow_k}$ then $M \to_\beta^* \mathsf{NF}_k(M)$.

Lemma

If $M{\Downarrow_k}$ *then*

- $|\mathcal{T}(M)| \in \{\uparrow s \text{ ; } s \textit{ normal and of height } \leq k\}^\perp$
- *if* s *is in normal form and of height* $\leq k$, *then* $\mathsf{NF}(\mathcal{T}(M))_s = \mathcal{T}(\mathsf{NF}_k(M))_s$.

Theorem

If $M{\Downarrow_\infty}$ *then* $|\mathcal{T}(M)| \in \{\uparrow s \text{ ; } s \textit{ normal}\}^\perp$ *and* $\mathsf{NF}(\mathcal{T}(M)) = \mathcal{T}(\mathcal{BT}(M))$.

Conclusion

Normalization and Taylor expansion commute

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Further work

- give a precise account of standardization in a generic non-uniform setting (based on work by Thomas Leventis)
- adapt those results to proof nets (WIP with Jules Chouquet)
- we claim $\mathcal{BT}(M) = \mathsf{NF}(\mathcal{T}(M))$ when it is defined:
  does this coincide with existing notions of (non extensional) Böm trees?
- when is Taylor expansion injective on normal forms?
  $\rightsquigarrow$ might lead to injectivity results for a class of quantitative denotational models

The end

Questions?