Infinitary normal forms for non-deterministic \( \lambda \)-terms

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Böhm trees

- In the pure $\lambda$-calculus, we perform $\beta$-reduction

$$\)M\ N\ \rightarrow_\beta\ M\ [N/x]\$$

cонтextually.

- By standardization, the leftmost outermost strategy reaches normal forms when they exist.

- Böhm trees = coinductive left normal forms

$$\mathcal{B}\mathcal{T}\ (M) = \begin{cases} \lambda x (y) \mathcal{B}\mathcal{T}\ (N_1) \cdots \mathcal{B}\mathcal{T}\ (N_k) & \text{if } M \simeq_\beta \lambda x (y) N_1 \cdots N_k \\ \bot & \text{if } M \text{ has no hnf} \end{cases}$$

Example: \(\mathcal{B}\mathcal{T}\ (Y) = \lambda x (x) (x) (x) \cdots\)

- Böhm trees form a syntactic denotational model, that is sensible

\textit{Note: here we do not consider extensionality issues}
Proposals for non deterministic Böhm trees exist:

- de’Liguoro and Piperno’s Böhm trees for erratic choice (1995)
- recent work by Thomas Leventis on probabilistic Böhm trees

Alternative semantic approaches:

- in domain theory: powerdomains and the like (around 1980)
- Girard’s quantitative semantics (around 1980)
Infinitary normal forms in a non deterministic setting

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Here:

*The “Böhm tree” of a non-deterministic \( \lambda \)-term is the normal form of its Taylor expansion*
Quantitative semantics

A prime aged idea (Girard, ’80s, before LL)

$\lambda$-terms = analytic functions = power series

Originally: for the $\lambda$-calculus, in an abstract categorical setting
(coefficients are sets)
Quantitative semantics

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Finiteness spaces (Ehrhard, early 2000’s)

Reformulate q.s. in linear logic using standard algebra:

- types \sim\rightarrow\text{particular topological vector spaces} (or semimodules):
  \[ [A] \subseteq S^{|A|} + \text{some additional structure} \]
- function terms \sim\rightarrow\text{power series}

Dieren tiation of λ-terms (Ehrhard-Regnier 2003-2004)

So we can dieren tiate λ-terms, and compute their Taylor expansion!

And one can mimic that in the syntax:

- dieren tial λ-calculus
- a nitary fragment: resource λ-calculus = the target of Taylor expansion
Quantitative semantics

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Finiteness spaces (Ehrhard, early 2000’s)

Reformulate q.s. in linear logic using standard algebra:

- types $\rightsquigarrow$ particular topological vector spaces (or semimodules):
  
  $[A] \subseteq \mathcal{S}^{A}$ + some additional structure

- function terms $\rightsquigarrow$ power series

Differentiation of $\lambda$-terms (Ehrhard-Regnier 2003-2004)

So we can differentiate $\lambda$-terms, and compute their Taylor expansion!
And one can mimic that in the syntax:

- differential $\lambda$-calculus

- a finitary fragment: resource $\lambda$-calculus
  
  $= \text{the target of Taylor expansion}$
Resource λ-calculus

Resource terms

\[ \Delta \ni s, t, \ldots ::= x | \lambda x s | \langle s \rangle \overline{t} \]

\[ !\Delta \ni \overline{s}, \overline{t}, \ldots ::= [s_1, \ldots, s_n] \]

Meaning: \( \langle s \rangle [s_1, \ldots, s_n] = (Ds)_0 \cdot (s_1, \ldots, s_n) \)

Resource reduction

\[ \langle \lambda x s \rangle \overline{t} \rightarrow_\partial \partial_x s \cdot \overline{t} \text{ (anywhere)} \]

\[ \partial_x s \cdot \overline{t} = \begin{cases} \sum_{f \in \mathfrak{S}_n} s[t_f(1), \ldots, t_f(n)/x_1, \ldots, x_n] & \text{if } n_x(s) = \#\overline{t} = n \\ 0 & \text{otherwise} \end{cases} \]
Resource $\lambda$-calculus

Resource terms

$$\Delta \ni s, t, \ldots ::= x \mid \lambda x s \mid \langle s \rangle \bar{t}$$

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+ linearity: $\lambda x 0 = 0$, $\langle s \rangle [t_1 + t_2, u] = \langle s \rangle [t_1, u] + \langle s \rangle [t_2, u]$, ...
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- Resource reduction preserves free variables, is size-decreasing, strongly confluent and normalizing.
Taylor expansion of $\lambda$-terms

Semantically, $(M)\ N = \sum_{n \in \mathbb{N}} \frac{1}{n!} \langle M \rangle \ N^n$ where $N^n = [N, \ldots, N]$. 
Taylor expansion of $\lambda$-terms

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Taylor expansion: $\mathcal{T} (M) \in S^\Delta$

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\mathcal{T} ((M) \, N) = \sum_{n \in \mathbb{N}} \frac{1}{n!} \langle \mathcal{T} (M) \rangle \, \mathcal{T} (N)^n
$$

$\mathcal{T} (x) = x \quad \mathcal{T} (\lambda x \, M) = \lambda x \, \mathcal{T} (M)$
Taylor expansion of $\lambda$-terms

Semantically, $(M)\ N = \sum_{n \in \mathbb{N}} \frac{1}{n!} \langle M \rangle\ N^n$ where $N^n = [N, \ldots, N]$.

Taylor expansion: $\mathcal{T}(M) \in S^\Delta$

$$\mathcal{T}((M)\ N) = \sum_{n \in \mathbb{N}} \frac{1}{n!} \langle \mathcal{T}(M) \rangle\ \mathcal{T}(N)^n$$

$$\mathcal{T}(x) = x \quad \mathcal{T}(\lambda x\ M) = \lambda x\ \mathcal{T}(M)$$

Theorem (Ehrhard-Regnier, CiE 2006)

*If* $M \in \Lambda$, *then* $\mathcal{T}(M)$ *normalizes to* $\mathcal{T}(\mathcal{B}\mathcal{T}(M))$.

Moral

In the ordinary $\lambda$-calculus $\mathcal{B}\mathcal{T}(M) \simeq \text{NF}(\mathcal{T}(M))$. 
Normalizing Taylor expansions

But how can $\mathcal{T}(M)$ even normalize?

We want to set

$$\text{NF} (\mathcal{T}(M)) = \sum_{s \in \Delta} \mathcal{T}(M)_s \cdot \text{NF}(s)$$

$\leadsto$ infinite sums \textit{(and in general we might consider all kinds of coefficients)}

$\leadsto$ convergence?
Normalizing Taylor expansions: *uniformity to the rescue!*

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**Theorem (Ehrhard-Regnier 2004, published in TCS in 2008)**

Write \(|\mathcal{T}(M)| = |\mathcal{T}(M)|. Then for all \( t \in \Delta \), there is at most one \( s \in |\mathcal{T}(M)| \) such that \( \text{NF}(s)_t \neq 0 \).

**Proof.**

\( \lambda \)-terms are uniform: their finitary approximants are pairwise coherent. \( \square \)
Normalizing Taylor expansions: *uniformity to the rescue!*

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Write $|\mathcal{T}(M)| = |\mathcal{T}(M)|$. Then for all $t \in \Delta$, there is at most one $s \in |\mathcal{T}(M)|$ such that $\text{NF}(s)_t \neq 0$.

**Proof.**

$\lambda$-terms are uniform: their finitary approximants are pairwise coherent. □

This fails in general

$$\text{NF}(\sum_{n \in \mathbb{N}} \langle \lambda x \, x \rangle^n \, [y]) = ?$$

$$\langle \lambda x \, x \rangle^n \, [y] = \langle \lambda x \, x \rangle \, [[\lambda x \, x] \, [\, \cdots [y] \cdots]]$$
Taylor expansion in a non uniform setting

A generic quantitative non-uniform calculus

\[ S[\Lambda s] \ni M, N, \ldots ::= x | \lambda x \; M \; | \; (M) \; N \; | \; M + N \]

\[
(\lambda x \; M) \; N \rightarrow_{\beta} M [N/x] \quad (M + N) \; P = (M) \; P + (N) \; P
\]
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\[ S[\Lambda_s] \ni M, N, \ldots ::= x \mid \lambda x M \mid (M) N \mid M + N \mid 0 \]

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\[(0) P = 0\]
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Taylor expansion of a sum

\[ \mathcal{T} \left( \sum_{i=1}^{n} a_i M_i \right) = \sum_{i=1}^{n} a_i \mathcal{T} (M_i) \]
Taylor expansion in a non uniform setting

A generic quantitative non-uniform calculus

\[ S \mathrel{\subseteq} M, N, \ldots ::= x \mid \lambda x \, M \mid (M) \, N \mid M + N \mid 0 \mid a \, M \]

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(\lambda x \, M) \, N \rightarrow_{\beta} M \left[ N / x \right] \quad \quad (M + N) \, P = (M) \, P + (N) \, P
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\mathcal{T} \left( \sum_{i=1}^{n} a_i \, M_i \right) = \sum_{i=1}^{n} a_i \, \mathcal{T} \left( M_i \right)
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Example

Let \( \delta_M = \lambda x \, (M + (x) \, x) \) and \( \infty_M = (\delta_M) \, \delta_M \colon \infty_M \rightarrow_{\beta^*} M + \infty_M \).
Taylor expansion in a non uniform setting

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\[ S[\Lambda_s] \ni M, N, \ldots ::= x \mid \lambda x M \mid (M) N \mid M + N \mid 0 \mid a M \]

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Taylor expansion of a sum

\[ T \left( \sum_{i=1}^{n} a_i M_i \right) = \sum_{i=1}^{n} a_i T (M_i) \]

Example

Let \( \delta_M = \lambda x (M + (x) x) \) and \( \infty_M = (\delta_M) \delta_M: \infty_M \rightarrow_\beta^* M + \infty_M \).

Then \( NF (T (\infty_M)) = ? \)
Taylor expansion in a non uniform setting

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\[ S [\Lambda s] \ni M, N, \ldots ::= x | \lambda x M | (M) N | M + N | 0 | a M \]

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(\lambda M) N \to_M M[N/x] \quad (M + N) P = (M) P + (N) P
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Example

Let \( \delta_M = \lambda x (M + (x)x) \) and \( \infty_M = (\delta_M) \delta_M \): \( \infty_M \to_M^* M + \infty_M \).

Then \( \text{NF} (T (\infty_M)) = ? \)

Worse: \( \text{NF} (T (\infty_M - (\lambda x x) \infty_M)) = ? \)
Finiteness structures to the rescue!

The main artifact of finiteness spaces:

Definition

- If $a, a' \subseteq A$, write $a \perp a'$ iff $a \cap a'$ is finite.
- If $\mathcal{G} \subseteq \mathcal{P}(A)$, let $\mathcal{G}^\perp := \{a' \subseteq A ; \forall a \in \mathcal{G}, a \perp a'\}$.
- A finiteness structure is any $\mathcal{F} = \mathcal{G}^\perp$.
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When is $\mathcal{T}(M)$ normalizable?

- Write $s \geq_\partial t$ if $s \to_{\partial^*} t + \cdots$.
- Let $\uparrow t = \{s \in \Delta ; s \geq_\partial t\}$.
- $\mathcal{T}(M)$ is normalizable iff for all normal $t \in \Delta$, $|\mathcal{T}(M)| \perp \uparrow t$.
- $\{\uparrow t ; t \text{ normal } \in \Delta\}^\perp$ is the finiteness structure of (supports of) normalizable vectors.
Typed terms have a finitary Taylor expansion

Let system $F_+$ be system $F$ plus

$$\frac{\Gamma \vdash M : A}{\Gamma \vdash M + N : A} \quad \frac{\Gamma \vdash N : A}{\Gamma \vdash M + N : A}.$$

**Theorem (Ehrhard, LICS 2010)**

*If $M \in S[\Lambda_S]$ is typable in system $F_+$, then $|T(M)| \in \{↑t ; t \in \Delta\}^\perp$.*
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Theorem (Pagani–Tasson–V., FoSSaCS 2016)

The same holds for all strongly normalizing terms, and we even have:

$M \in SN$ iff $|T(M)| \in \{ \uparrow a ; a \in \mathcal{B} \}^\perp$ for a well chosen $\mathcal{B}$.
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**Theorem (Ehrhard, LICS 2010)**

If $M \in S[\Lambda_S]$ is typable in system $F_+$, then $|\mathcal{T}(M)| \in \{\uparrow t ; t \in \Delta\} \perp$.

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The same holds for all strongly normalizing terms, and we even have: $M \in SN$ iff $|\mathcal{T}(M)| \in \{\uparrow a ; a \in \mathcal{B}\} \perp$ for a well chosen $\mathcal{B}$.

**Theorem (Pagani–Tasson–V., WIP)**

The above results generalize to (weak or head) normalizability.

In particular if $M$ is normalizable then $|\mathcal{T}(M)| \in \{\uparrow s ; s \text{ normal}\} \perp$. 
TODO

We know:

Normalizable terms have a normalizable Taylor expansion.
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\[ \text{NF}(T(M)) = T(\text{NF}(M)) \]

but the techniques of the uniform case are no longer available.
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We want:

\[ \text{NF} (\mathcal{T}(M)) = \mathcal{T}(\text{NF}(M)) \]

but the techniques of the uniform case are no longer available.

- We must follow the reduction \( M \to^* \text{NF}(M) \) through \( \mathcal{T} \).
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Normalizable terms have a normalizable Taylor expansion.

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Theorem?

If \( M \rightarrow_\beta N \) then \( \mathcal{T}(M) \rightsquigarrow \mathcal{T}(N) \).

- And then, what about non normalizing terms?
\(\beta\)-reduction through Taylor expansion

Recall that:

\[
\mathcal{T} ((M) \, N) = \sum_{n \in \mathbb{N}} \frac{1}{n!} \, \langle \mathcal{T} (M) \rangle \, \mathcal{T} (N)^n
\]

In quantitative semantics:

\[
[(\lambda x \, M) \, N] = [M \, [N/x]] = \sum_{n \in \mathbb{N}} \frac{1}{n!} \left( \frac{\partial^n [M]}{\partial x^n} \right)_{x=0} \cdot [N]^n.
\]
\[ \sigma' := \sum_{n \in \mathbb{N}} \frac{1}{n!} \sigma^n \in \mathbb{S}^{\Delta} \]
**β-reduction through Taylor expansion: key steps**

**Promotion**

\[ \sigma' := \sum_{n \in \mathbb{N}} \frac{1}{n!} \sigma^n \in S^{!\Delta} \]

**Substitution**

\[ \mathcal{T} (M [N/x]) = \partial_x \mathcal{T} (M) \cdot \mathcal{T} (N)! \]
\( \beta \)-reduction through Taylor expansion: key steps

Promotion

\[ \sigma' := \sum_{n \in \mathbb{N}} \frac{1}{n!} \sigma^n \in S^{!\Delta} \]

Substitution

\[ \mathcal{T} (M [N/x]) = \partial_x \mathcal{T} (M) \cdot \mathcal{T} (N)' \]

Reduction

\[ \langle \lambda x \sigma \rangle \tau' \leadsto \partial_x \sigma \cdot \tau' \]
β-reduction through Taylor expansion: key steps

Promotion

\[ \sigma' := \sum_{n \in \mathbb{N}} \frac{1}{n!} \sigma^n \in S^! \Delta \]

Substitution

\[ T(M[N/x]) = \partial_x T(M) \cdot T(N)' \]

Reduction

\[ \langle \lambda x \sigma \rangle \tau' \leadsto \partial_x \sigma \cdot \tau' \]

more generally:

\[ \langle \lambda x \sigma \rangle \bar{\tau} \leadsto \partial_x \sigma \cdot \bar{\tau} \] (where \( \bar{\tau} \in S^! \Delta \))
$\beta$-reduction through Taylor expansion: key steps

Promotion

$$\sigma^! := \sum_{n \in \mathbb{N}} \frac{1}{n!} \sigma^n \in S^{\Delta}$$

Substitution

$$\mathcal{T}(M[N/x]) = \partial_x \mathcal{T}(M) \cdot \mathcal{T}(N)^!$$

Reduction

$$\langle \lambda x \sigma \rangle \tau^! \leadsto \partial_x \sigma \cdot \tau^!$$

more generally:

$$\langle \lambda x \sigma \rangle \bar{\tau} \leadsto \partial_x \sigma \cdot \bar{\tau} \quad \text{(where } \bar{\tau} \in S^{\Delta})$$

- Combinatorics everywhere
β-reduction through Taylor expansion: key steps

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\[ \sigma' := \sum_{n \in \mathbb{N}} \frac{1}{n!} \sigma^n \in S^! \Delta \]

Substitution

\[ T(M[N/x]) = \partial_x T(M) \cdot T(N)! \]

Reduction

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more generally:

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(where \( \bar{\tau} \in S^! \Delta \))

- Combinatorics everywhere
- We actually need parallel reduction: ensuring confluence requires a bit of work
Characterizing normalizability: the uniform case

Standardization: normalizability = left-reduction to a normal form.

**Inductive characterization**

\[
\frac{(M \ [N_0/x]) \ N_1 \cdots N_n \downarrow}{(\lambda x \ M) \ N_0 \ N_1 \cdots N_n \downarrow} \quad \frac{\downarrow \ N_1 \cdots \downarrow \ N_n \downarrow}{\uparrow \ \lambda \ x \ (y) \ N_1 \cdots N_n \downarrow}
\]

Böhm trees: interpret the second rule coinductively.
Characterizing normalizability: the non-uniform case

Add the following rule:

\[
\frac{M_1 \Downarrow \quad \cdots \quad M_n \Downarrow}{\sum_{i=1}^{n} a_i M_i \Downarrow}
\]

In general this is stronger than the naive notion of normalizability (consider $\infty_M - (I) \infty_M$).
Recursively head-normalizable terms

**Definition (k-normalizable terms)**

\[
\begin{array}{c}
M \downarrow_0 \\
\hline
\frac{N_1 \downarrow_k \cdots N_n \downarrow_k}{\lambda x \rightarrow (y) N_1 \cdots N_n \downarrow_{k+1}} \\
\frac{(M [N_0/x]) N_1 \cdots N_n \downarrow_k}{(\lambda x M) N_0 N_1 \cdots N_n \downarrow_k} \\
\frac{M_1 \downarrow_k \cdots M_n \downarrow_k}{\sum_{i=1}^n a_i M_i \downarrow_k}
\end{array}
\]

- Write \( M \downarrow_\infty \) if \( M \downarrow_k \) for all \( k \).
Normalizing Taylor expansion, level by level

If \( M \downarrow_k \) then \( M \rightarrow^*_{\beta} \text{NF}_k(M) \).

**Lemma**

If \( M \downarrow_k \) then

- \( |\mathcal{T}(M)| \in \{\uparrow s ; s \text{ normal and of height } \leq k\} \downarrow \)
- if \( s \) is in normal form and of height \( \leq k \), then \( \text{NF}(\mathcal{T}(M))_s = \mathcal{T}(\text{NF}_k(M))_s \).

**Theorem**

If \( M \downarrow_{\infty} \) then \( |\mathcal{T}(M)| \in \{\uparrow s ; s \text{ normal}\} \downarrow \) and \( \text{NF}(\mathcal{T}(M)) = \mathcal{T}(\mathcal{B}\mathcal{T}(M)) \).
Conclusion

Normalization and Taylor expansion commute
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Further work

- give a precise account of standardization in a generic non-uniform setting (based on work by Thomas Leventis)
- adapt those results to proof nets (WIP with Jules Chouquet)
- we claim $BT(M) = NF(T(M))$ when it is defined: does this coincide with existing notions of (non extensional) Böhm trees?
- when is Taylor expansion injective on normal forms? $\approx$ might lead to injectivity results for a class of quantitative denotational models
The end

Questions?