Lex Modalities in Type Theory

Eric Finster

Rencontres PiR2
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A Lawvere-Tierney topology determines a subtopos $\mathcal{F} \subseteq \mathcal{E}$. In fact, in ordinary topos theory, *all* subtoposes are determined in this way.

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3. ... with an elimination principle:

$$(C : \bigcirc X \to \mathcal{U}) \to (\text{is-P} : (x : \bigcirc X) \to P(Cx)) \to$$
$$(s : (x : X) \to C\,[x]_P) \to (x : \bigcirc X) \to C\,x$$

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We say that a type X for which $\bigcirc_M X \simeq *$ is *M-acyclic*.

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We say such a modality is *left exact* (or just *lex* for short).

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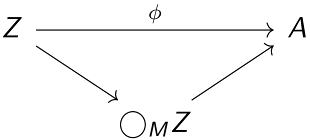

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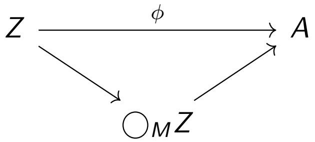

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$$\begin{array}{ccc} Z & \xrightarrow{\phi} & A \\ & \searrow \qquad \nearrow & \\ & \bigcirc_M Z & \end{array}$$

But $\bigcirc_M Z \simeq *$ by assumption.

Conclusion: every map from an M-acyclic type to an M-local type is constant.

Orthogonality

Definition

Let X and A be types. We say that X is *orthogonal* to A, written $X \perp A$ if the canonical map

$$\mathrm{cst} = \lambda a.\lambda x.a : A \to (X \to A)$$

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More generally, for a dependent type $W : X \to \mathcal{U}$, we write $W \perp A \iff (x : X) \to (W\,x \perp A)$.

Nullification

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Remark

Modulo size issues, every modality M is of this form. Take:

$$X := \Sigma_{A \in \mathcal{U}} (\bigcirc_M A \simeq *)$$

And $W :=$ fst.

Topological Localizations

Problem

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One case where we can say something is the following:

Theorem

If W takes values in hProp, that is

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But while this condition is sufficient, it is not necessary: there are localizations of higher toposes which are *not* of this form.

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The *codiagonal* ∇W of W is defined by:

$$\begin{aligned} \nabla W : X &\to \mathcal{U} \\ x &\mapsto \Sigma(W\,x) \end{aligned}$$

Cotopological Localizations

Theorem (Anel, Biedermann, Joyal, F.)

Let $W : X \to \mathcal{U}$. Suppose that

$$(x : X) \to \textit{is-contr}\ \| W\, x \|_{-1}$$

Then the following are equivalent:

- *M_W is left exact*
- *For all $x : X$*
$$\bigcirc_{\nabla W}(W\, x) \simeq *$$
- *For all $x : X$, $w_0, w_1 : W\, x$*
$$\bigcirc_W(\Delta W(x, w_0, w_1)) = \bigcirc_W(w_0 = w_1) \simeq *$$

Localizations of this form are called *cotopological*.

The Recognition Theorem

Let $W : X \to \mathcal{U}$. Define the *diagonal sequence* of W mutually recursively as follows:

$$B^n W : \mathcal{U} \qquad \Delta^n W : B^n W \to \mathcal{U}$$

$$B^0 W := X \qquad \Delta^0 W\, x := W\, x$$

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Theorem (Anel, Biedermann, Joyal, F.)

M_W is left exact if and only if for all $n : \mathbb{N}$ and $b : B^n W$

$$\bigcirc_W(\Delta^n W\, b) \simeq *$$

Generating Lex Modalities

An immediate corollary of the previous theorem is the following:

Theorem (Anel, Biedermann, Joyal, F.)

Let $W : X \to \mathcal{U}$. Then the dependent family

$$\Delta^{\infty} : \bigsqcup_{n=0}^{\infty} B^n W \xrightarrow{\Delta^n} \mathcal{U}$$

generates a lex modality.