Let $\mathcal{E}$ be a topos and $\Omega$ its subobject classifier.
Recall the following definition:

A Lawvere-Tierney topology on $\mathcal{E}$ is a map $j : \Omega \to \Omega$ satisfying:

1. $j \circ \text{true} = \text{true}$
2. $j \circ j = j$
3. $j \circ \land = \land \circ (j \circ j)$

A Lawvere-Tierney topology determines a subtopos $\mathcal{F} \subseteq \mathcal{E}$. In fact, in ordinary topos theory, all subtoposes are determined in this way.
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1. A property of types:

$$P : U \to U \quad (X : U) \to \text{is-prop } (P X)$$
Modalities

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$$P : \mathcal{U} \rightarrow \mathcal{U} \quad (X : \mathcal{U}) \rightarrow \text{is-prop } (P X)$$

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$$\bigcirc : \mathcal{U} \rightarrow \mathcal{U}$$

$$(X : \mathcal{U}) \rightarrow P (\bigcirc X)$$

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   $$ \bigcirc : \mathcal{U} \to \mathcal{U} $$
   $$ (X : \mathcal{U}) \to P (\bigcirc X) $$
   $$ [\neg]_P : X \to \bigcirc X $$

3. ... with an elimination principle:

   $$ (C : \bigcirc X \to \mathcal{U}) \to (\text{is-P} : (x : \bigcirc X) \to P (Cx)) \to $$
   $$ (s : (x : X) \to C [x]_P) \to (x : \bigcirc X) \to C x $$
Let $M$ be a modality and write $\bigcirc = \bigcirc_M$. Using the universal property of $\bigcirc_M$ it is possible to show that:

1. $\bigcirc \ast \simeq \ast$
2. $\bigcirc (X) \simeq \bigcirc X$
3. $\bigcirc (X \times Y) \simeq (\bigcirc X) \times (\bigcirc Y)$

Hence we can view a modality as a kind of Lawvere-Tierney topology on all types.

We say that a type $X$ for which $\mathcal{P}X$ is $M$-modal.

There is a universe of modal types defined by:

$U_M = \Sigma_{X \in U} \mathcal{P}X$

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Properties of Modalities

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Eric Finster
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Let $M$ be a modality and write $\bigcirc = \bigcirc_M$. Using the universal property of $\bigcirc_M$ it is possible to show that:

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Let $M$ be a modality and write $\bigcirc = \bigcirc_M$. Using the universal property of $\bigcirc_M$ it is possible to show that:

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A modality $M$ satisfies the following closure properties:

1. If $B : A \rightarrow U \mathcal{M}$ then $\Pi A B : U \mathcal{M}$
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Hence a modality almost gives an internal model of type theory. The only missing piece is the universe $U \mathcal{M}$ itself!

Theorem (Shulman, Rijke, Spitters)

The following are equivalent:

1. $U \mathcal{M}$ is $\mathcal{M}$-modal
2. If $Z$ is $\mathcal{M}$-acyclic then for all $z_0, z_1 : Z$, so is $z_0 = Z z_1$

We say such a modality is left exact (or just lex for short).
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**Lex Modalities and Subtoposes**

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But $\bigcirc_M Z \simeq \ast$ by assumption.

Conclusion: every map from an $M$-acyclic type to an $M$-local type is constant.
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Definition

Let $X$ and $A$ be types. We say that $X$ is orthogonal to $A$, written $X \perp A$ if the canonical map

$$cst = \lambda a.\lambda x.a : A \to (X \to A)$$

is an equivalence.

Example $S_{n+1} \perp A \iff A$ is $n$-truncated

More generally, for a dependent type $W : X \to U$, we write $W \perp A \iff (x : X) \to (W x \perp A)$. 
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$S^{n+1} \perp A \iff A$ is $n$-truncated

More generally, for a dependent type $W : X \to U$, we write $W \perp A \iff (x : X) \to (W x \perp A)$. 
Nullification

Let \( \mathcal{W} : X \to \mathcal{U} \) be a type family.
Nullification

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Associated to $W$ is a modality $M_W = (P_W, \bigcirc_W, \| - \|_W, \ldots)$ which we call *nullification* with respect to $W$. One defines:

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The local operator $\bigcirc_W$ can be constructed by an HIT.
Nullification

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Nullification

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\[
P_W A = W \bot A
\]

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**Example**

\[
M_{S^{n+1}} = \| - \|_n
\]

**Remark**

*Modulo size issues, every modality \( M \) is of this form. Take:*

\[
X \coloneqq \Sigma_{A \in \mathcal{U}} (\bigcirc_M A \simeq *)
\]

*And \( W \coloneqq \text{fst}.*
Problem

*Given* $W$, when is nullification with respect to $W$ left exact?

**Theorem**

If $W$ takes values in $h$Prop, that is $(x: X) \rightarrow is-prop(W x)$, then $M_W$ is left exact.

Localizations of this form are called *topological*.

The class of topological localizations are exactly those which arise from Lawvere-Tieney topologies.

But while this condition is sufficient, it is not necessary: there are localizations of higher toposes which are *not* of this form.
Topological Localizations

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One case where we can say something is the following:

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Let $W : X \to U$. 

Let $W : X \to \mathcal{U}$.

The *diagonal* $\Delta W$ of $W$ is defined by:

$$
\Delta W : \Sigma_{x : X} (W x \times W x) \to \mathcal{U}
$$

$$(x, w_0, w_1) \mapsto w_0 = w_1$$
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The *diagonal* $\Delta W$ of $W$ is defined by:

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$$(x, w_0, w_1) \mapsto w_0 = w_1$$

The *codiagonal* $\nabla W$ of $W$ is defined by:

$$\nabla W : X \to U$$

$$x \mapsto \Sigma( W x)$$
Theorem (Anel, Biedermann, Joyal, F.)

Let $W : X \to \mathcal{U}$. Suppose that

$$(x : X) \to \text{is-contr} \parallel W x \parallel_{-1}$$

Then the following are equivalent:

- $M_W$ is left exact
- For all $x : X$
  $$\bigcirc \nabla W(W x) \simeq *$$
- For all $x : X$, $w_0, w_1 : W x$
  $$\bigcirc_W(\Delta W(x, w_0, w_1)) = \bigcirc_W(w_0 = w_1) \simeq *$$

Localizations of this form are called cotopological.
Let $W : X \to \mathcal{U}$. Define the *diagonal sequence* of $W$ mutually recursively as follows:

\[
\begin{align*}
B^n W & : \mathcal{U} \\
B^0 W & := X \\
B^{n+1} W & := \Sigma_{b : B^n X}(\Delta^n b \times \Delta^n b) \\
\Delta^n W & : B^n W \to \mathcal{U} \\
\Delta^0 W \times & := W \times \\
\Delta^{n+1} W (b, p, q) & := p = q
\end{align*}
\]
The Recognition Theorem

Let $W : X \to U$. Define the diagonal sequence of $W$ mutually recursively as follows:

\[
\begin{align*}
B^n W : & \ U \\
B^0 W : & = X \\
B^{n+1} W : & = \Sigma_{b : B^n X} (\Delta^n b \times \Delta^n b) \\
\Delta^n W : & : B^n W \to U \\
\Delta^0 W \times : & = W \times \\
\Delta^{n+1} W (b, p, q) : & = p = q
\end{align*}
\]

Theorem (Anel, Biedermann, Joyal, F.)

$M_W$ is left exact if and only if for all $n : \mathbb{N}$ and $b : B^n W$

\[\bigcirc_W (\Delta^n W b) \simeq *\]
An immediate corollary of the previous theorem is the following:

**Theorem (Anel, Biedermann, Joyal, F.)**

Let $W : X \to U$. Then the dependent family

$$\Delta^\infty : \bigcup_{n=0}^\infty B^n W \xrightarrow{\Delta^n} U$$

generates a lex modality.