A few comments on interfaces for computational effects

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$\pi.r^2$ à Fontainebleau

The good old simply typed λ-calculus

Let's begin with a basic λ-calculus with products and unit.

$$M, N ::= x \mid MN \mid \lambda x.M \mid (M, N) \mid \pi_i(M) \mid \star$$

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Its semantics can be given by cartesian closed categories (CCC).

$$\boxed{\lambda^{\to}} \xrightarrow[{[\![\cdot]\!]}]{} \boxed{\mathrm{CCC}}$$

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- It means... not very useful for programming *computers*.

We want *computational effects*: input/output, exceptions, non–determinism, continuations, etc.

If we just throw effects to it, difficulties arise:

- Evaluation order must be handed carefully.
- Equational reasoning stops working so gracefully.

Moggi: λ_C-calculus

Moggi proposes a calculus based on semantics through monads.

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$M, N ::= \ldots \mid \text{let } x \Leftarrow M \text{ in } N \mid [M]$

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> *Programs should form a category.*

Two new constructions on terms:

$$M, N ::= \ldots \mid \text{let } x \Leftarrow M \text{ in } N \mid [M]$$

Based on the monad semantics.

$$\boxed{\lambda_C} \xrightarrow{[\![\cdot]\!]} \boxed{\text{CCC} + \text{monad}}$$

Wadler: internalising monads

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Abstracted as a type-class:

$$
\begin{array}{l}
\textbf{class}\ \mathit{Monad}\ m\ \textbf{where} \\
\quad \mathit{return} :: a \to m\ a \\
\quad (\gg\!=) :: m\ a \to (a \to m\ b) \to m\ b
\end{array}
$$

Effects as monads: examples

data $Exc\ a = Fail \mid Success\ a$
instance $Monad\ Exc$ **where** ...

data $Nondet\ a = Nondet\ [a]$
instance $Monad\ Nondet$ **where** ...

data $\mathrm{State}_s\ a = \mathrm{State}_s\ (s \to (a, s))$
instance $Monad\ \mathrm{State}_s$ **where** ...

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Operations

In general, a monad m is endowed with operations of the form

$$op :: a \to m\ b$$

Computations are written using *return*, $(\gg\!=)$ and these operations.

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$$or :: (m\ a, m\ a) \to m\ a \qquad \text{vs.} \qquad choose :: () \to m\ Bool$$

$$\left(\begin{array}{lcl} or\ (m_1, m_2) & = & choose\ () \gg\!= \lambda b \to \mathbf{if}\ b\ \mathbf{then}\ m_1\ \mathbf{else}\ m_2 \\ choose\ () & = & or\ (return\ True, return\ False) \end{array}\right)$$

Usage example

The constructor State_s is endowed with two *operations*:

$$\begin{aligned} &get :: () \rightarrow \mathsf{State}_s\ s \\ &put :: s \ \ \rightarrow \mathsf{State}_s\ () \end{aligned}$$

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A program that generates fresh variables `"x0"`, `"x1"`, ... by using a state with an *Int*:

$$
\begin{aligned}
&\mathit{freshVar} :: \mathsf{State}_{Int}\ \mathit{String} \\
&\mathit{freshVar} = \mathit{get}\ () \gg\!= \lambda s \to \\
&\qquad\qquad\quad \mathit{put}\ (s+1) \gg\!= \lambda() \to \\
&\qquad\qquad\quad \mathit{return}\ (\texttt{"x"} +\!\!+ \mathit{show}\ s)
\end{aligned}
$$

However...

Some effects would benefit of having static information around.

$$(\gg\!=) :: m\ a \to (a \to m\ b) \to m\ b$$

The combinator $(\gg\!=)$ is too dynamic.

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These limitations lead to new interfaces.

Arrows

Hughes introduced a new interface for computations.

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\textbf{class } \mathit{Arrow}\ a\ \textbf{where} \\
\quad \begin{array}{ll}
\mathit{arr} & :: (x \to y) \to a\ x\ y \\
(\ggg) & :: a\ x\ y \to a\ y\ z \to a\ x\ z \\
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This interface provides static information.

- Originally applied to parsers.
- Used in reactive functional programming.
- A monad and a comonad give rise to an arrow.

Applicative functors

Later, McBride and Patterson introduced yet another type-class to represent effects.

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\begin{array}{l}
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- Popular instances: monoids, zip lists, parsers.
- The constructor WR_A can be instantiated as an *Applicative*.
- $\mathit{Monad}\ m \Rightarrow \mathit{Applicative}\ m$. Not the converse.

An unified framework?

So far, we have seen three different interfaces to represent computational effects.

Applicative *Monad* *Arrow*

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It would be nice if it was possible to give an *unified* framework for these interfaces.

Looking again...

The three interfaces have a neuter lifting operation:

$$\mathit{return} :: a \to m\ a$$
$$\mathit{pure} :: a \to f\ a$$
$$\mathit{arr} :: (a \to b) \to (a \leadsto b)$$

And also sequencing of computation:

$$(\gg=) :: m\ a \to (a \to m\ b) \to m\ b$$
$$(\circledast) :: f\ (a \to b) \to f\ a \to f\ b$$
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They look a bit like monoids...

Generalised monoids

A monoidal category $(\mathcal{C}, \otimes, I)$ is a category $\mathcal{C}$ together with:

- a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$,
- an object I,
- natural isomorphisms α, λ and ρ subject to coherence.

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A monoid in $(\mathcal{C}, \otimes, I)$ is a triple (M, m, e) where:

- M is an object,
- $m : M \otimes M \to M$ is a morphism,
- $e : I \to M$ is a morphism.

Monads as monoids

Endofunctors have a substitution monoidal structure:

- $(F \circ G)X = \int^Y FY \times (Y \to GX)$ as tensor.
- The identity functor as unit.

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Arrows as monoids

Strong profunctors inherit Bénabou's monoidal structure:

- $(P \otimes Q)(X, Y) = \int^{W} P(X, W) \times Q(W, Y)$ as tensor.
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Summing up: computational effects as monoids

$\boxed{\textit{Monad}}$ Monoid in $(\mathrm{End}, \circ, \mathrm{Id})$

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Monad Monoid in $(\mathrm{End}, \circ, \mathrm{Id})$

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Not just a pretty theory...

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Not just a pretty theory...

We can put the unification to work!

Exploiting the unification for fun and profit (1)

Free monoid.

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$$\begin{tikzcd} \boxed{\text{Free monoid}} \arrow[r] & \boxed{\begin{array}{c}\text{Free monoid in}\\ \text{a monoidal cat.}\end{array}} \end{tikzcd}$$

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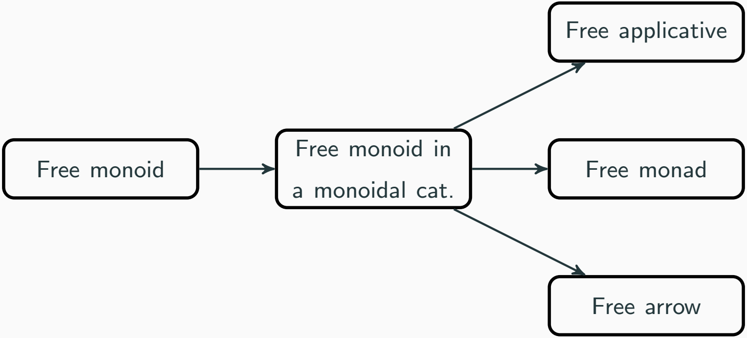

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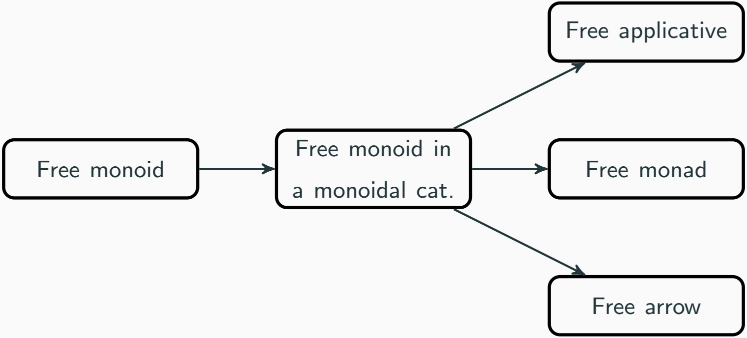


Free monoid express abstract programs, useful for DSLs.

Free notions of computation from operations

Given a set of operations

$$\{op_w : I_w \to O_w\}_{w \in W}$$

We can construct a functor:

$$F(X) = \sum_{w \in W} I_w \times X^{O_w}$$

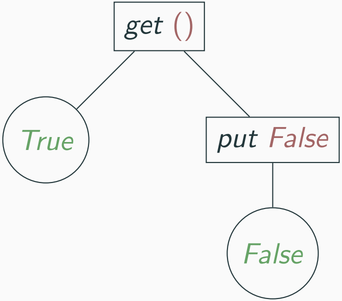
Or a strong profunctor:

$$P(X, Y) = \sum_{w \in W} {I_w}^X \times Y^{O_W \times X}$$

Free monads from operations

$$\{get : () \to Bool, put : Bool \to ()\}$$

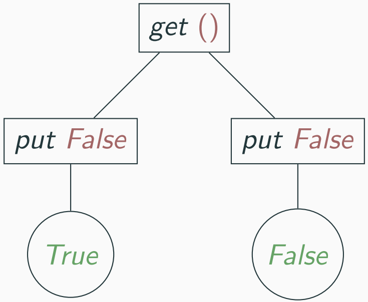
The set of trees.



Free applicative functors from operations

$$\{get : () \to Bool, put : Bool \to ()\}$$

The set of trees with static control and data flow.



Free arrows from operations

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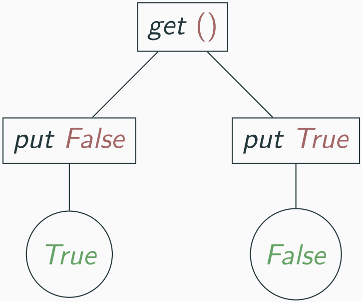
The set of trees with static control flow.



Exploiting the unification for fun and profit (2)

Exponential. In the presence of exponentials (closure), it is possible to construct a generalisation of the monoid of endofunctions $X \Rightarrow X$.

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It underlies Hughes lists and the codensity monad.

Exploiting the unification for fun and profit (3)

There are monoidal adjunctions (in a broad sense)

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(\mathrm{End}, \star, \mathrm{Id}) \arrow[r, bend left, "_{!}"] \arrow[r, phantom, "\perp"] & (\mathrm{Prof}, \otimes, \mathrm{Hom}) \arrow[l, bend left, "_^{*}"] \arrow[r, bend left, "_^{*}"] \arrow[r, phantom, "\perp"] & (\mathrm{End}, \circ, \mathrm{Id}) \arrow[l, bend left, "_{*}"]
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Monoidal functors lift to categories of monoids.

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Monoidal functors lift to categories of monoids.

The relationship between the interfaces is explained by the lifted functors.

Handling non-determinism

Non-deterministic monads extend monads with failure and choice:

$$
\begin{array}{l}
\textbf{class}\ \mathit{Monad}\ m \Rightarrow \mathit{MonadPlus}\ m\ \textbf{where} \\
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An instance provides two extra operations:

- *mzero* represents a failing computation.
- *mplus* expresses a computation that behaves non-deterministically.

Near-rigs

Analogously, *Alternative* and *ArrowPlus* extend applicative functors and arrows.

Alternative *MonadPlus* *ArrowPlus*

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Free near-rigs can be constructed, and also a represetation. However, the theory of near-rig categories is not well developed.

Near-rig categories

A near-rig category $(\mathcal{C}, \otimes, I, \oplus, J)$ is a category $\mathcal{C}$ together with:

- bifunctors $\otimes, \oplus : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$
- objects I,
- natural isomorphisms

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Coherence with skew structures?

Conclusions and further work

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- Monoidal categories work as an unification of interfaces for computational effects.
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Further work?

- Handle non-algebraic operations? Concurrency?
- Develop a calculus for computational effects as monoids.
- Formalize part of it in a proof assistant.