La légèreté markovienne des empilements infinis

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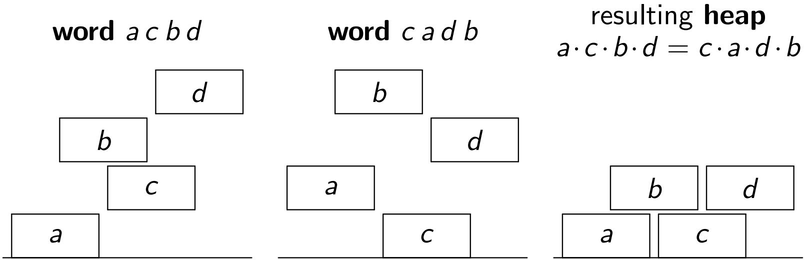
Outline

Framework

Uniform measure and applications

Framework: an example of heap monoid (dimer model)



- Alphabet $\Sigma = \{a, b, c, d\}$, Σ-labeled pieces
- Heaps obtained in a Tetris-like way by letting pieces fall
- Commutation relations between pieces:

$$a \cdot c = c \cdot a \qquad a \cdot d = d \cdot a \qquad b \cdot d = d \cdot b$$

Framework: an example of heap monoid (dimer model)

Structure

- Piling up heaps is an associative operation: free partially commutative monoid:

$$\mathcal{M} = \langle a, b, c, d \mid ac = ca,\ ad = da,\ bd = db \rangle$$

Framework: an example of heap monoid (dimer model)

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Object of study: uniform distributions

- Size $|x|$ of a heap = the number of pieces in x
- Finite uniform distribution ν_k on $\mathcal{M}_k = \{x \in \mathcal{M} : |x| = k\}$

$$x \in \mathcal{M}_k \qquad \nu_k\big(\{x\}\big) = \frac{1}{\#\mathcal{M}_k}$$

- **Question:** are there more occurrences of a or of b in a large random heap? What is the limiting ratio, if any?

Generalization

Heap monoids/trace monoids

- Σ an alphabet, I an irreflexive and symmetric relation on Σ
-

$$\mathcal{M}(\Sigma, I) = \left\langle \Sigma \;\middle|\; ab = ba \text{ for } (a, b) \in I \right\rangle$$

equivalence classes of words in Σ^* modulo commutations

Same questions

- Asymptotics for statistics associated to the uniform distributions on heaps of size n when $n \to \infty$?

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Space of infinite heaps: boundary at infinity

Compactification of the monoid $\mathcal{M}$

- We consider infinite heaps and their set $\partial\mathcal{M}$
- The countable set $\mathcal{M}$ is embedded into the compact metric space $\overline{\mathcal{M}} = \mathcal{M} \cup \partial\mathcal{M}$

Uniform distributions on $\overline{\mathcal{M}}$

- ν_k corresponds to a finite probability distribution on $\overline{\mathcal{M}}$

Convergence of spherical uniform distributions

Theorem

The sequence $(\nu_k)_k$ converges weakly toward a (continuous) probability distribution ν_∞ concentrated on $\partial\mathcal{M}$, called the **uniform measure at infinity**, and with a **Bernoulli** characterization:

$$\nu_\infty\{\xi \in \partial\mathcal{M} \;:\; x \leq \xi\} = p^{|x|}$$

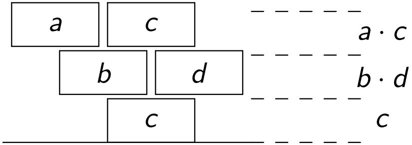
For the running example

- $p = 1/3$
- 4 blocks a, b, c, d each one with individual probability $1/3$!

Strategy

- Study the properties of the uniform measure ν_∞ on $\partial\mathcal{M}$
- Derive asymptotics for the finite uniform measures ν_k

Facts from Combinatorics: cliques and Cartier-Foata normal form



- Every heap is made of successive horizontal layers
- Corresponds to the Cartier-Foata normal form of heaps
- Not all pairs of layers can be seen successively:

  $b \cdot d$ over c is OK $\qquad$ $b \cdot d$ over b is not OK

- There is an automaton accepting correct sequences of layers

Cliques for the running example

Possible layers correspond to cliques of the commutation relation on a, b, c, d:

Size	Layers	Number
0	ϵ	1
1	$a,\ b,\ c,\ d$	4
2	$a \cdot c,\ a \cdot d,\ b \cdot d$	3

Möbius polynomial: $\mu_{\mathcal{M}}(t) = 1 - 4t + 3t^2$

Möbius polynomial and generating series

Möbius polynomial $\mu_{\mathcal{M}}(t) = \sum_{\gamma \in \mathscr{C}} (-1)^{|\gamma|} t^{|\gamma|}$

Generating series $G_{\mathcal{M}}(t) = \sum_{x \in \mathcal{M}} t^{|x|} = \sum_{k \geq 0} (\#\mathcal{M}_k) t^k$

Inversion formula $G_{\mathcal{M}}(t) = \dfrac{1}{\mu_{\mathcal{M}}(t)}$

$p =$ radius of convergence of $G_{\mathcal{M}}(t)$

$=$ inverse of the growth rate of $\mathcal{M}$

$=$ smallest root of $\mu_{\mathcal{M}}(t)$

$=$ Bernoulli exponent for the uniform measure on $\partial\mathcal{M}$

$$\nu_\infty\{\xi \in \partial\mathcal{M} \ : \ x \leq \xi\} = p^{|x|}$$

Markov chain realization

Theorem

- Under the uniform measure at infinity, the sequence $(x_i)_{i\geq 1}$ of layers (cliques) that appear in an infinite random heap is an ergodic Markov chain on non empty cliques:

$$a, \quad b, \quad c, \quad d, \quad a\cdot c, \quad a\cdot d, \quad b\cdot d$$

- There are explicit formula for the initial distribution and the transition matrix of the chain.

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Corollary

- Fix an integer $j \geq 1$ and consider the j first layers $(y_1,\ldots,y_j)$ of a random heap of size k large.
- Then the law of $(y_1,\ldots,y_j)$ converges toward the law of $(x_1,\ldots,x_j)$ when $k\to\infty$.

Application: evaluating an average cost

Theorem

Let $f : \{a, b, c, d\} \to \mathbb{R}$ be cost values, extended to heaps by additivity:

$$\langle f, x_1 \cdot \ldots \cdot x_k \rangle = f(x_1) + \ldots + f(x_k) \qquad x_i \in \{a, b, c, d\}$$

Let π be the stationary measure of the Markov chain on cliques under the uniform measure at infinity

Then:

$$\lim_{k\to\infty} \frac{1}{k} \mathbb{E}_{\nu_k} \langle f, \cdot \rangle = \frac{\displaystyle\sum_{x\in\mathscr{C}} \pi(x)\langle f, x\rangle}{\displaystyle\sum_{x\in\mathscr{C}} \pi(x)|x|}$$

Density of pieces for the running example

Stationary measure of the chain of cliques:

$$\pi(a) = \pi(d) = \frac{6}{57} \quad \pi(b) = \pi(c) = \frac{18}{57} \quad \pi(a \cdot c) = \pi(b \cdot d) = \frac{4}{57} \quad \pi(a \cdot d) = \frac{1}{57}$$

$$\lim_{k \to \infty} \frac{1}{k} \mathbb{E}_{\nu_k} \langle f, \cdot \rangle = \frac{\sum_x \pi(x) \langle f, x \rangle}{\sum_x \pi(x) |x|} = \frac{1}{6}\big(f(a) + 2f(b) + 2f(c) + f(d)\big)$$

Average number of as and bs in a heap of size k:

$$f = \delta_a \qquad\qquad g = \delta_b$$

$$\lim_{k \to \infty} \frac{1}{k} \mathbb{E}_{\nu_k} \langle f, \cdot \rangle = \frac{1}{6} \qquad\qquad \lim_{k \to \infty} \frac{1}{k} \mathbb{E}_{\nu_k} \langle g, \cdot \rangle = \frac{1}{3}$$

In the limit, there are twice more bs than as in a large random heap

Additional remarks

- Similar techniques apply to braid monoids with the Garside normal form
- This solved some recent conjecture on the asymptotics of the Garside normal form of large positive braids