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Grothendieck bifibrations and Quillen model categories

Journées PPS 2017

1. Liminaries

2. Glueing model structures

3. Applications

1

Liminaries

Grothendieck bifibrations

A Grothendieck bifibration is a functor $p : \mathcal{E} \to \mathcal{B}$ such that

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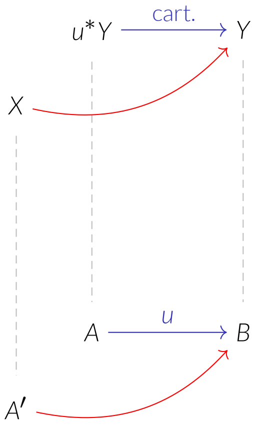

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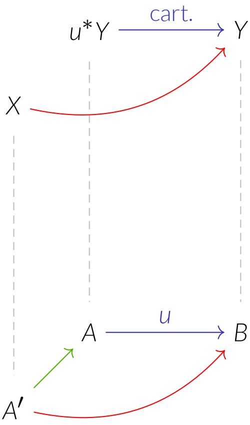

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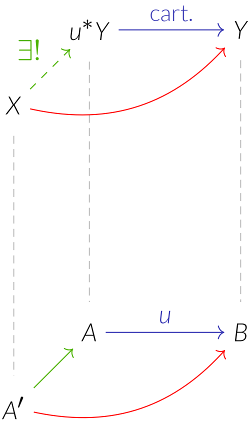

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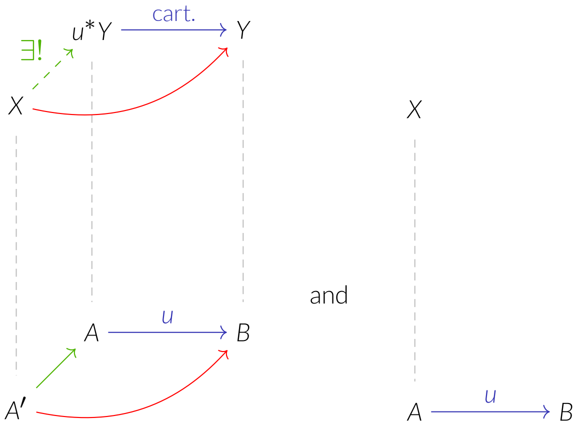

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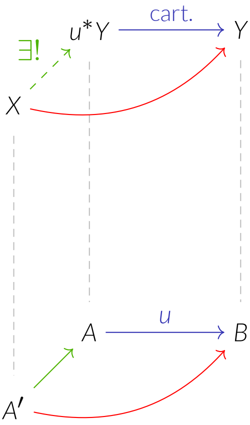

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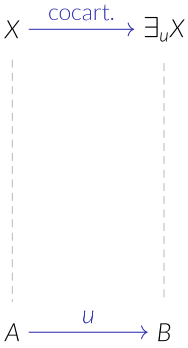

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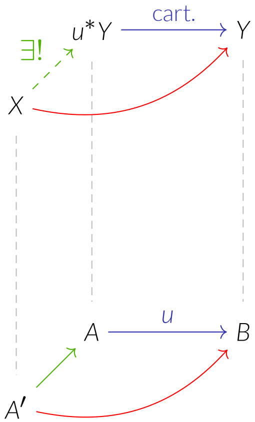

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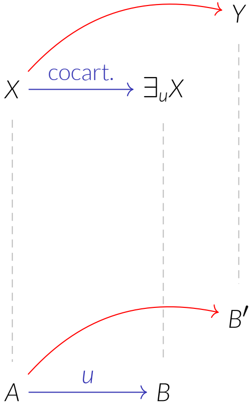

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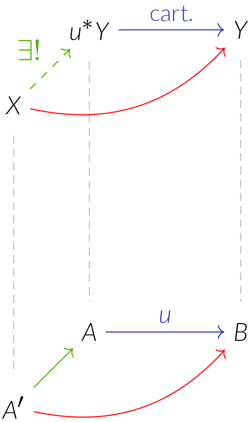

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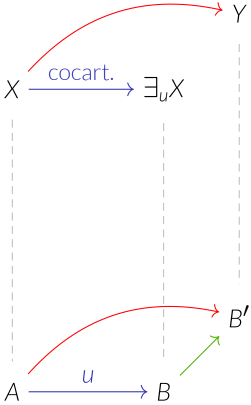

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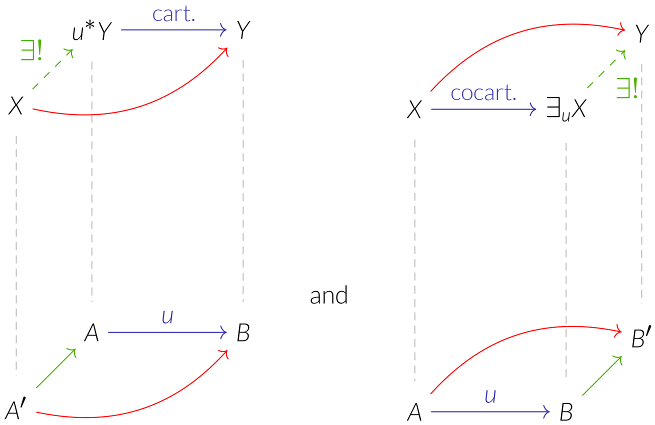
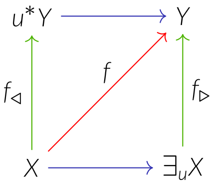


Push and pull

For any $f : X \to Y$ in $\mathcal{E}$ over $u : A \to B$,



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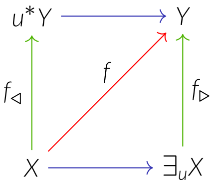

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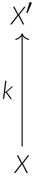

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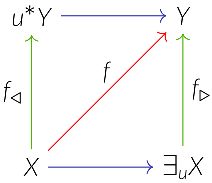

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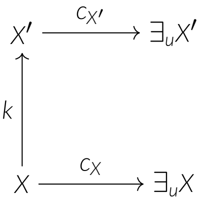

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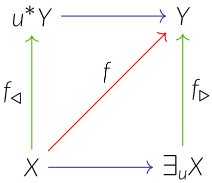

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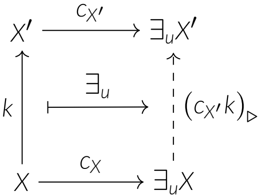

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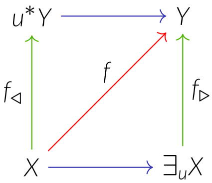

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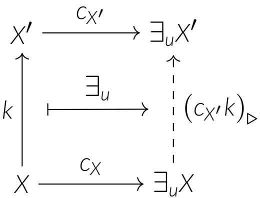

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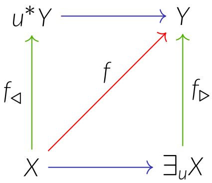

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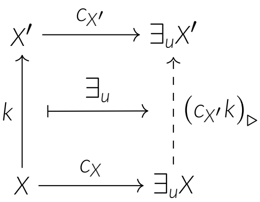

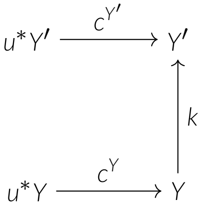

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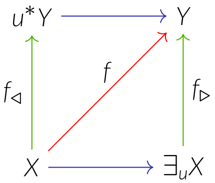

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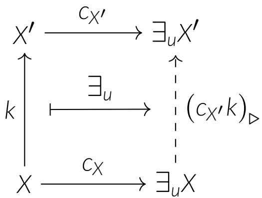

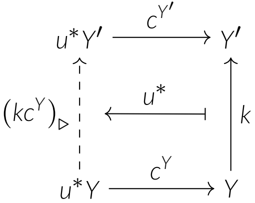

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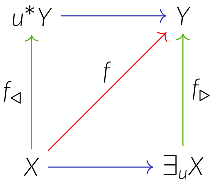

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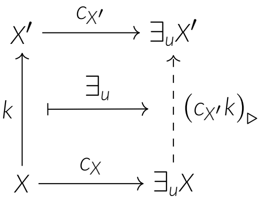

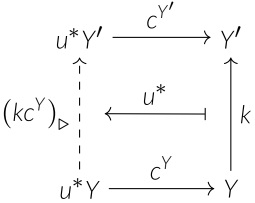

This produces an adjunction $\exists_u : \mathcal{E}_A \leftrightarrows \mathcal{E}_B : u^*$ between fibers.

Bifibrations in logic

Bifibrations in logic

- SubSet $\longrightarrow$ Set :

Bifibrations in logic

- SubSet $\longrightarrow$ Set :

objects
$(A, X \subseteq A)$

Bifibrations in logic

morphisms $(A, X) \to (B, Y)$
$u : A \to B$ s.t. $u(X) \subseteq Y$

- SubSet $\longrightarrow$ Set :

objects
$(A, X \subseteq A)$

Bifibrations in logic

- SubSet $\longrightarrow$ Set :

cocartesian

$$\begin{array}{ccc} X & \longrightarrow & u(X) \\ \vdots & & \vdots \\ A & \xrightarrow{u} & B \end{array}$$

cartesian

$$\begin{array}{ccc} u^{-1}(Y) & \longrightarrow & Y \\ \vdots & & \vdots \\ A & \xrightarrow{u} & B \end{array}$$

Bifibrations in logic

- SubSet $\longrightarrow$ Set :

cocartesian

$$\begin{array}{ccc} X & \longrightarrow & \{b : \exists a \in X,\ u(a) = b\} \\ \vdots & & \vdots \\ A & \xrightarrow{\quad u \quad} & B \end{array}$$

cartesian

$$\begin{array}{ccc} \{a : u(a) \in Y\} & \longrightarrow & Y \\ \vdots & & \vdots \\ A & \xrightarrow{\quad u \quad} & B \end{array}$$

Bifibrations in logic

- SubSet $\longrightarrow$ Set
- $\mathcal{F}(T) \to \mathrm{ctx}_{\mathcal{L}}$ for a first-order existential theory T over $\mathcal{L}$:

Bifibrations in logic

- SubSet $\longrightarrow$ Set
- $\mathcal{F}(T) \longrightarrow \mathrm{ctx}_{\mathcal{L}}$ for a first-order existential theory T over $\mathcal{L}$:

objects
formulas $\varphi(\bar{x})$

objects
integers n

Bifibrations in logic

morphisms $\varphi(\bar{x}) \to \psi(\bar{y})$
$t_1(\bar{x}), \ldots, t_m(\bar{x})$ s.t.
$T \vdash \varphi(\bar{x}) \to \psi(\bar{y}/\bar{t})$

morphisms $n \to m$
terms $t_1(\bar{x}), \ldots, t_m(\bar{x})$

- $\mathsf{SubSet} \longrightarrow \mathsf{Set}$
- $\mathcal{F}(T) \to \mathrm{ctx}_{\mathcal{L}}$ for a first-order existential theory T over $\mathcal{L}$:

objects
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Bifibrations in logic

- SubSet $\longrightarrow$ Set
- $\mathcal{F}(T) \to \mathrm{ctx}_{\mathcal{L}}$ for a first-order existential theory T over $\mathcal{L}$:

cocartesian

$$\begin{array}{ccc} \varphi(\bar{x}) & \longrightarrow & \exists \bar{x}, \varphi(\bar{x}) \wedge \bigwedge_{i=1}^{m} y_i = t_i(\bar{x}) \\ \vdots & & \vdots \\ n & \xrightarrow{\quad \bar{t} \quad} & m \end{array}$$

cartesian

$$\begin{array}{ccc} \psi(t_1(\bar{x}), \ldots, t_m(\bar{x})) & \longrightarrow & \psi(\bar{y}) \\ \vdots & & \vdots \\ n & \xrightarrow{\quad \bar{t} \quad} & m \end{array}$$

Bifibrations in logic

- $\text{SubSet} \to \text{Set}$
- $\mathcal{F}(T) \to \text{ctx}_{\mathcal{L}}$ for a first-order existential theory T over $\mathcal{L}$
- a model $\mathfrak{M}$ for T is exactly

$$\begin{array}{ccc} \mathcal{F}(T) & \longrightarrow & \text{SubSet} \\ \downarrow & & \downarrow \\ \text{ctx}_{\mathcal{L}} & \xrightarrow[\mathfrak{M}]{} & \text{Set} \end{array}$$

Bifibrations in logic

- $\text{SubSet} \longrightarrow \text{Set}$
- $\mathcal{F}(T) \longrightarrow \text{ctx}_{\mathcal{L}}$ for a first-order existential theory T over $\mathcal{L}$
- a model $\mathfrak{M}$ for T is exactly

preserves (co)cartesian morphisms
+ other structures

$$\begin{array}{ccc} \mathcal{F}(T) & \longrightarrow & \text{SubSet} \\ \downarrow & & \downarrow \\ \text{ctx}_{\mathcal{L}} & \xrightarrow[\mathfrak{M}]{} & \text{Set} \end{array}$$

product preserving

Bifibrations in CS

- Hoare $\longrightarrow$ State :

Bifibrations in CS

- Hoare $\longrightarrow$ State :

objects
predicates P

objects
states s

Bifibrations in CS

morphisms $P \to Q$
Hoare provable triples $\{P\}S\{Q\}$

morphisms $s \to t$
programs S from s to t

- Hoare $\to$ State :

objects
predicates P

objects
states s

Bifibrations in CS

- Hoare $\longrightarrow$ State :

cocartesian

$$\begin{array}{ccc} \{P\} & \longrightarrow & sp(S, \{P\}) \\ \vdots & & \vdots \\ s & \xrightarrow{S} & t \end{array}$$

cartesian

$$\begin{array}{ccc} wp(S, \{Q\}) & \longrightarrow & \{Q\} \\ \vdots & & \vdots \\ s & \xrightarrow{S} & t \end{array}$$

Bifibrations in CS

- Hoare $\longrightarrow$ State
- $\mathcal{D} \longrightarrow \mathrm{ctx}_{\mathcal{T}}$ for a dependent type theory $\mathcal{T}$

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objects
$\Gamma \vdash B$ type

objects
context Γ

Bifibrations in CS

morphisms $\Gamma \vdash B \to \Delta \vdash C$
$t_0, \dots t_n$ with a term
$\Gamma, B \vdash u : C(\overline{t})$

morphisms $\Gamma \to B_0, B_1, \dots, B_n$
terms
$t_0 : B_0, t_1 : B_1(t_0), \dots, t_n : B_n(t_0, \dots, t_{n-1})$

- Hoare $\to$ State
- $\mathcal{D} \to \mathrm{ctx}_{\mathcal{T}}$ for a dependent type theory $\mathcal{T}$:

objects
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Bifibrations in CS

- Hoare $\longrightarrow$ State
- $\mathcal{D} \longrightarrow \mathrm{ctx}_{\mathcal{T}}$ for a dependent type theory $\mathcal{T}$:

cocartesian

$$\begin{array}{ccc} B & \longrightarrow & (x_i : A_i) \vdash \sum_{y:A} B(\bar{x}, y) \\ \vdots & & \vdots \\ \Gamma, y : A & \xrightarrow{(x_i:A_i)_{A_i \in \Gamma}} & \Gamma \end{array}$$

cartesian

$$\begin{array}{ccc} C(t_0, \ldots, t_n) & \longrightarrow & C \\ \vdots & & \vdots \\ \Gamma & \xrightarrow{\bar{t}} & \Delta \end{array}$$

Weak factorization system

In a category $\mathcal{M}$, denote $j \boxslash q$ when for any

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle j}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ B & \longrightarrow & Y \end{array}$$

Weak factorization system

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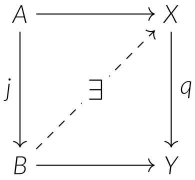

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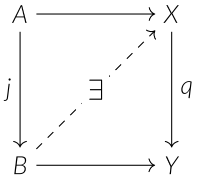

Define $\mathfrak{L}^{\boxslash}$ as the class of q such that $j \boxslash q$ for all $j \in \mathfrak{L}$.
Define ${}^{\boxslash}\mathfrak{R}$ as the class of j such that $j \boxslash q$ for all $q \in \mathfrak{R}$.

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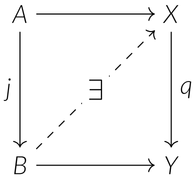

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Definition

A weak factorisation system on a category $\mathcal{M}$ is a couple $(\mathfrak{L}, \mathfrak{R})$ such that

- $\mathfrak{L} = {}^{\boxslash}\mathfrak{R}$ and $\mathfrak{L}^{\boxslash} = \mathfrak{R}$
- each morphism factors as $q \circ j$ for $j \in \mathfrak{L}$ and $q \in \mathfrak{R}$.

Model category

Definition

A model category is a complete and cocomplete category $\mathcal{M}$ together with $\mathbf{Cof}, \mathbf{W}, \mathbf{Fib}$ such that

- $\mathbf{W}$ has 2-out-of-3,
- $(\mathbf{Cof} \cap \mathbf{W}, \mathbf{Fib})$ and $(\mathbf{Cof}, \mathbf{W} \cap \mathbf{Fib})$ are weak factorisation systems.

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An adjunction $L : \mathcal{M} \leftrightarrows \mathcal{N} : R$ is Quillen when:

$$L(\mathbf{Cof}_{\mathcal{M}}) \subseteq \mathbf{Cof}_{\mathcal{N}} \qquad R(\mathbf{Fib}_{\mathcal{N}}) \subseteq \mathbf{Fib}_{\mathcal{M}}$$

2

Glueing model structures

Lifting weak factorisation systems

Define a wfs-adjunction as an adjunction $L : \mathcal{M} \leftrightarrows \mathcal{N} : R$ such that

$$L(\mathfrak{L}_{\mathcal{M}}) \subseteq \mathfrak{L}_{\mathcal{N}} \qquad R(\mathfrak{R}_{\mathcal{N}}) \subseteq \mathfrak{R}_{\mathcal{M}}$$

Lifting weak factorisation systems

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Proposition (folklore)

Given a bifibration $p : \mathcal{E} \to \mathcal{B}$ with a wfs $(\mathfrak{L}, \mathfrak{R})$ on $\mathcal{B}$ and a wfs $(\mathfrak{L}_A, \mathfrak{R}_A)$ on each fiber $\mathcal{E}_A$, if each pair $(\exists_u, u^*)$ is a wfs-adjunction, the following classes yield a wfs on $\mathcal{E}$:

$$\mathfrak{L}_{\mathcal{E}} = \{f : p(f) \in \mathfrak{L}, f_{\triangleright} \in \mathfrak{L}_B\} \qquad \mathfrak{R}_{\mathcal{E}} = \{f : p(f) \in \mathfrak{R}, f_{\triangleleft} \in \mathfrak{R}_A\}$$

Lifting weak factorisation systems

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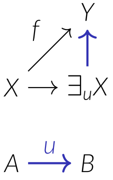

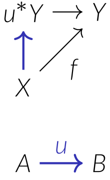

General principle

From now on, $p : \mathcal{E} \to \mathcal{B}$ is a bifibration with model structure $(\mathbf{Cof}, \mathbf{W}, \mathbf{Fib})$ on $\mathcal{B}$ and $(\mathbf{Cof}_A, \mathbf{W}_A, \mathbf{Fib}_A)$ on each fiber $\mathcal{E}_A$ such that every $(\exists_u, u^*)$ is Quillen.

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Call f in $\mathcal{E}$

- a total cofibration if $p(f) \in \mathbf{Cof}$ and $f_\triangleright \in \mathbf{Cof}_B$,
- a total fibration if $p(f) \in \mathbf{Fib}$ and $f_\triangleleft \in \mathbf{Fib}_A$.

Denote $\mathbf{Cof}_\mathcal{E}$ and $\mathbf{Fib}_\mathcal{E}$ the corresponding classes.

General principle

From now on, $p : \mathcal{E} \to \mathcal{B}$ is a bifibration with model structure $(\mathbf{Cof}, \mathbf{W}, \mathbf{Fib})$ on $\mathcal{B}$ and $(\mathbf{Cof}_A, \mathbf{W}_A, \mathbf{Fib}_A)$ on each fiber $\mathcal{E}_A$ such that every $(\exists_u, u^*)$ is Quillen.

Call f in $\mathcal{E}$

- a total acyclic cofibration if $p(f) \in \mathbf{Cof} \cap \mathbf{W}$ and $f_\triangleright \in \mathbf{Cof}_B \cap \mathbf{W}_B$,
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Key observation

If we find a class $\mathbf{W}_\mathcal{E}$ of total weak equivalences such that

- $\mathbf{W}_\mathcal{E}$ has 2-out-of-3,
- $\mathbf{WCof}_\mathcal{E} = \mathbf{Cof}_\mathcal{E} \cap \mathbf{W}_\mathcal{E}$ and $\mathbf{WFib}_\mathcal{E} = \mathbf{Fib}_\mathcal{E} \cap \mathbf{W}_\mathcal{E}$.

then $\mathbf{Cof}_\mathcal{E}, \mathbf{W}_\mathcal{E}, \mathbf{Fib}_\mathcal{E}$ is a model structure on $\mathcal{E}$.

Total weak equivalences

Main Idea

You don't have a choice for the total weak equivalences:

$$\mathrm{W}_{\mathcal{E}} \stackrel{\text{def}}{=} \mathrm{WFib}_{\mathcal{E}} \circ \mathrm{WCof}_{\mathcal{E}}$$

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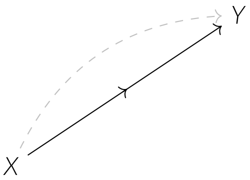

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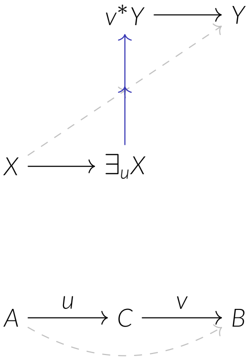

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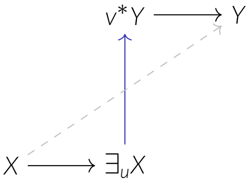
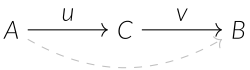


Call $f : X \to Y$ acyclic relatively to (u, v) if $p(f) = vu$ with $u \in \mathbf{Cof} \cap \mathbf{W}$ and $v \in \mathbf{Fib} \cap \mathbf{W}$, and $\exists_u X \to v^*Y \in \mathbf{W}_C$.



Define f to be a total weak equivalence if it is acyclic relatively to some pair.

Necessary conditions

H_1 $\exists_u$ preserves and reflects weak equivalences if $u \in \mathbf{Cof} \cap \mathbf{W}$.

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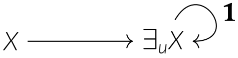

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$$\begin{array}{ccc} X' & \longrightarrow & \exists_u X' \\ {\scriptstyle k}\uparrow & & \uparrow{\scriptstyle \exists_u k} \\ X & \longrightarrow & \exists_u X \end{array}$$

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- H_1 $\exists_u$ preserves and reflects weak equivalences if $u \in \mathbf{Cof} \cap \mathbf{W}$.
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$$\begin{array}{ccc} v^*Y' & \longrightarrow & Y' \\ {\scriptstyle v^*k}\uparrow & & \uparrow{\scriptstyle k} \\ v^*Y & \longrightarrow & Y \end{array}$$

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hBC every square of the form

$$\begin{array}{ccc} A & \overset{u'}{\rightarrowtail} & C' \\ {\scriptstyle v}\downarrow\!\!\downarrow & & \downarrow\!\!\downarrow{\scriptstyle v'} \\ C & \underset{u}{\rightarrowtail} & B \end{array} \qquad \begin{array}{l} u, u' \in \mathbf{Cof} \cap \mathbf{W} \\ v, v' \in \mathbf{Fib} \cap \mathbf{W} \end{array}$$

satisfies the homotopical Beck-Chevalley condition, i.e. the mate $\exists_{u'} v^* \to v'^* \exists_u$ is pointwise in $\mathbf{W}_{C'}$.

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$$\begin{array}{ccccc} & & v'^*\exists_u Z & & \\ & & \uparrow & \searrow & \\ & & \exists_{u'} v^* Z & & \\ & \nearrow & & & \\ v^*Z & & & & \exists_u Z \\ & \searrow & & \nearrow & \\ & & Z & & \end{array}$$

Main result

Theorem

The bifibration $p : \mathcal{E} \to \mathcal{B}$ satisfies H_1, H_2 and hBC if and only if $\mathbf{Cof}_{\mathcal{E}}, \mathbf{W}_{\mathcal{E}}, \mathbf{Fib}_{\mathcal{E}}$ give a model structure on $\mathcal{E}$ where

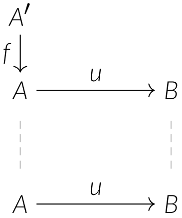
- $\mathbf{Cof}_{\mathcal{E}} = \{f : p(f) \in \mathbf{Cof},\ f_{\triangleright} \in \mathbf{Cof}_B\}$
- $\mathbf{Fib}_{\mathcal{E}} = \{f : p(f) \in \mathbf{Fib},\ f_{\triangleleft} \in \mathbf{Fib}_A\}$

3

Applications

Warm up example: the codomain bifibration

Given a category $\mathcal{C}$, the functor $\mathbf{cod} : \mathrm{Arr}(\mathcal{C}) \to \mathcal{C},\ (A' \to A) \mapsto A$ is a bifibration where pulls and pushs are given as follow:



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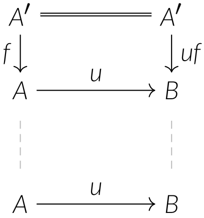

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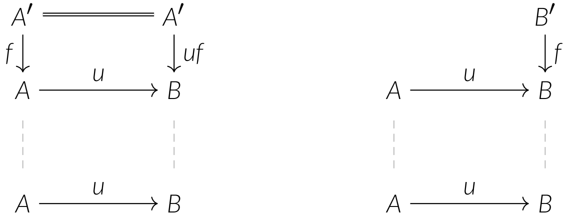

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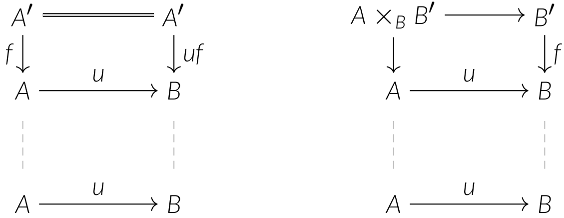

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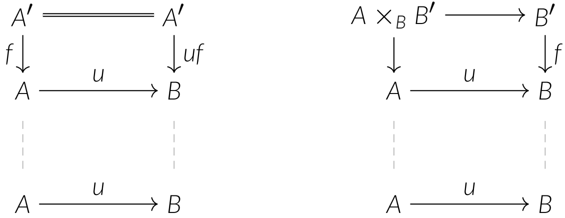

If $\mathcal{C}$ is a model category, each fiber $\mathbf{cod}_A \simeq \mathcal{C}_{/A}$ inherit a canonical model structure. The conditions H_1, H_2 and hBC are satisfied, hence we get a model structure on $\mathrm{Arr}(\mathcal{C})$...

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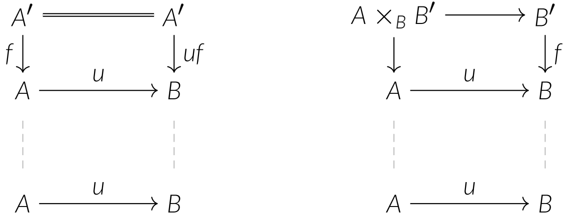

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Related works

- *Model category structures in bifibred categories* (1994) Roig
  Bifibrations and Weak Factorisation Systems (2012) Stanculescu

  Defines $\mathbf{W}_{\mathcal{E}}$ as those f such that $p(f) \in \mathbf{W}$ and $f_{\triangleleft} \in \mathbf{W}_A$.

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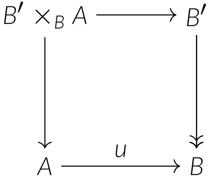

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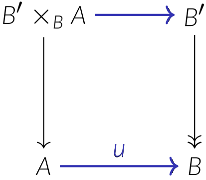
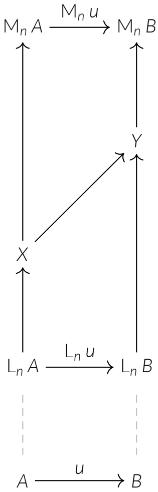


(right properness)

The Reedy construction

Consider the category of standard simplices: $\mathbf{\Delta}$.
Denote $\mathbf{\Delta}_n$ for the full subcategory with objects $[k]$ s.t. $k \leq n$.

The restriction $p_n : [\mathbf{\Delta}_{n+1}, \mathcal{M}] \to [\mathbf{\Delta}_n, \mathcal{M}]$ is a bifibration where:



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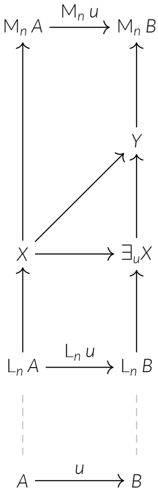

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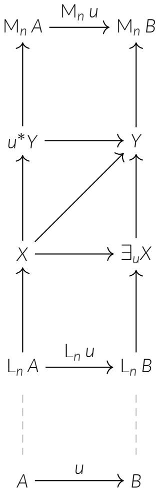

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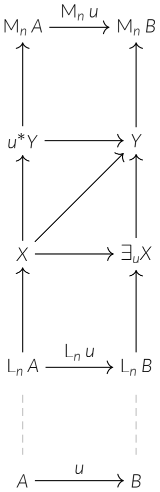

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If $[\mathbf{\Delta}_\lambda, \mathcal{M}]$ has the Reedy model structure, we then obtain a model structure on $[\mathbf{\Delta}_{\lambda+1}, \mathcal{M}]$...

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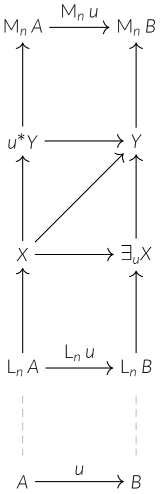

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If $[\boldsymbol{\Delta}_\lambda, \mathcal{M}]$ has the Reedy model structure, we then obtain a model structure on $[\boldsymbol{\Delta}_{\lambda+1}, \mathcal{M}]$... which is again the Reedy model structure.

Thank you.

http://www.normalesup.org/~cagne/
https://pierrecagne.github.io

... and its generalizations

In *Reedy categories and their generalizations* (2015) by Mike Shulman:

Theorem 3.11

If $\mathcal{M}$ and $\mathcal{N}$ model categories, $F, G : \mathcal{M} \to \mathcal{N}$ and $\alpha : F \to G$ st

- F cocont. and maps $\mathbf{Cof} \cap \mathbf{W}$ to couniversal weak equivalences,
- G cont. and maps $\mathbf{Fib} \cap \mathbf{W}$ to universal weak equivalences,

then the biglueing $\mathrm{Gl}(\alpha)$ is a model category.

The functor $\mathrm{Gl}(\alpha) \to \mathcal{M}$ is a bifibration and H_1, H_2 and hBC are satisfied, yielding the same model structure as in the article.

Sketch of proof: $\mathbf{WCof}_{\mathcal{E}} = \mathbf{Cof}_{\mathcal{E}} \cap \mathbf{W}_{\mathcal{E}}$

A total acyclic cofibration f is a cofibration and is acyclic relatively to $(w, \mathbf{1}_B)$. Hence:

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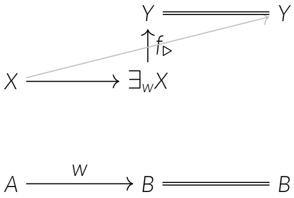

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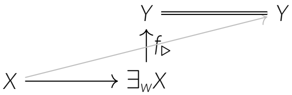

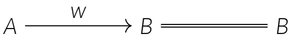

Conversely, if f is a cofibration acyclic relatively to (u, v), then it is also relatively to $(w, \mathbf{1}_B)$. Hence $f_\triangleright$ weak, making f into a total acyclic cofibration:

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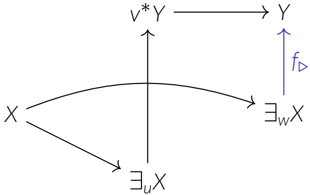

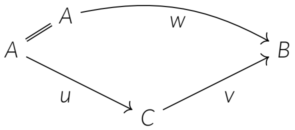

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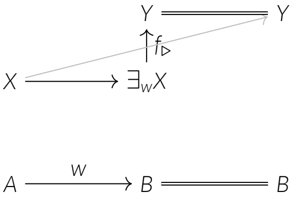

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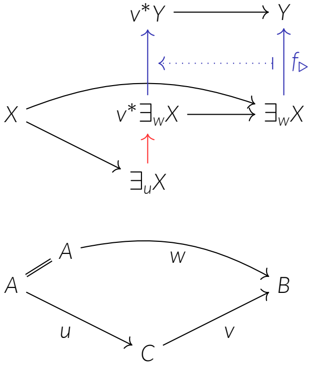
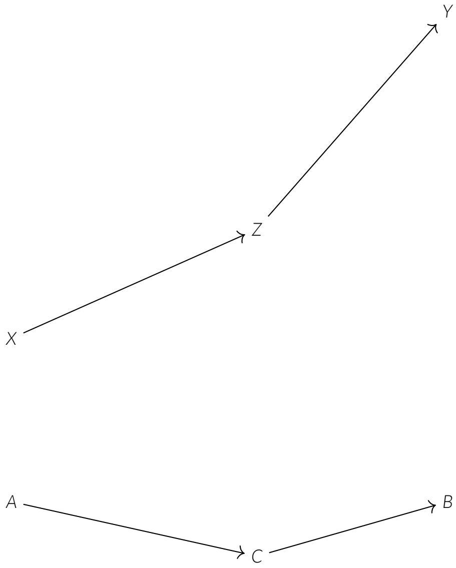
Conversely, if f is a cofibration acyclic relatively to (u, v), then it is also relatively to $(w, \mathbf{1}_B)$. Hence $f_\triangleright$ weak, making f into a total acyclic cofibration:

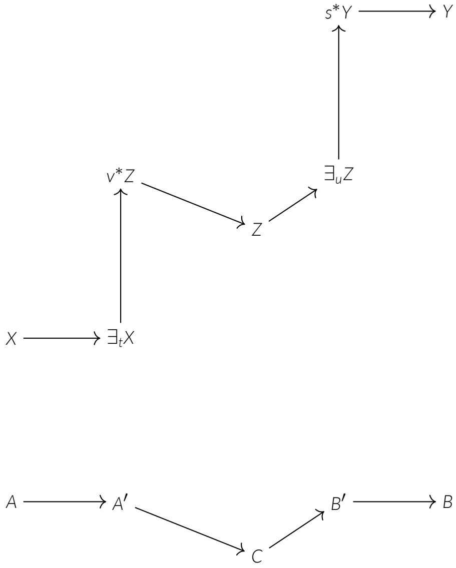


$$\mathbf{WCof}_\mathcal{E} \supset \mathbf{Cof}_\mathcal{E} \cap \mathbf{W}_\mathcal{E}$$

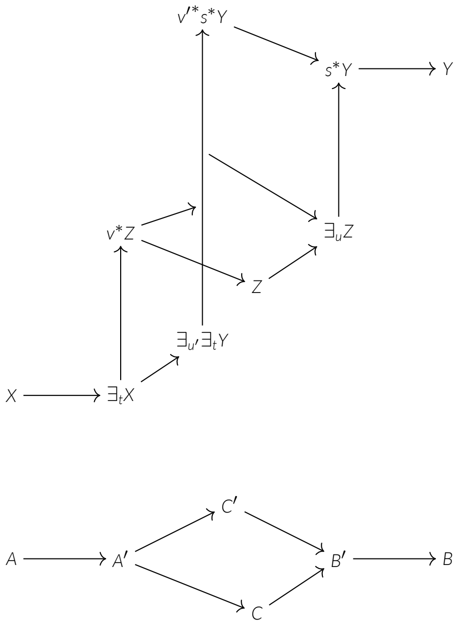
Sketch of proof: 2-out-of-3



Sketch of proof: 2-out-of-3



Sketch of proof: 2-out-of-3



Sketch of proof: 2-out-of-3