Logiques à points fixes, preuves infinies & fils rebondissants

Journées de rentrée du PPPS
Paris, 13 octobre 2017

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Based on joint works with D. Baelde, A. Doumane and G. Jaber
Logics with least and greatest fixed points

- Model *inductive and coinductive data structures and reasoning*.
- Their proof theory is under-developed and not well-understood.
- Infinite proofs in CS:
  - **Verification device.** Complete deduction system giving algorithms for checking validity (Tableaux, sequent calculi)
  - **Completeness arguments.** Intermediate objects between syntax and semantics for modal \( \mu \)-calculus (Kozen, Kaivola, Walukiewicz):

\[
\mu \text{-calculus formula} \rightarrow \text{Circular proof} \rightarrow \text{Finite axiomatization}
\]

... but rarely as proof–program objects in themselves
Structural proof theory of least and greatest fixed points

- **Sequent calculi**: MALL with fixed points
  - $\mu$MALL: well-founded proofs. Rules for induction, correctness is local, cut-elim and focalization but no subformula property.
  - $\mu$MALL$_\infty$: non-well-founded proofs. Simple inference rules for fixed points, global correctness criterion, cut-elimination with subformula property and focalization. **Circular** fragment.
Structural proof theory of least and greatest fixed points

- **Sequent calculi:** MALL *with fixed points*
  - $\mu$MALL: well-founded proofs. Rules for induction, correctness is local, cut-elim and focalization but no subformula property.
  - $\mu$MALL$^\infty$: non-well-founded proofs. Simple inference rules for fixed points, global correctness criterion, cut-elimination with subformula property and focalization. Circular fragment.

- **Aim of the talk:** Infinite proof as proof–program objects
  - Develop such a proof-theoretical study, from a Curry-Howard perspective (focalization and cut-elimination);
  - How remove constraints on the use of cuts in infinitary proofs, *ie*, allow cuts in cycles?

**Outline**

- Logiques à points fixes  $\mu$MALL
- Preuves infinies  $\mu$MALL$^\infty$
- Fils rebondissants  From straight threads to bouncing ones.
PROPOSITION 31

Any composite number is measured by some prime number.

Let $A$ be a composite number; I say that $A$ is measured by some prime number.
For, since $A$ is composite, some number will measure it.
Let a number measure it, and let it be $B$.
Now, if $B$ is prime, what was enjoined will have been done.
But if it is composite, some number will measure it.
Let a number measure it, and let it be $C$.
Then, since $C$ measures $B$, and $B$ measures $A$,
therefore $C$ also measures $A$.

And, if $C$ is prime, what was enjoined will have been done.
But if it is composite, some number will measure it.
Thus, if the investigation be continued in this way, some prime number will be found which will measure the number before it, which will also measure $A$.
For, if it is not found, an infinite series of numbers will measure the number $A$, each of which is less than the other:

which is impossible in numbers.

Therefore some prime number will be found which will measure the one before it, which will also measure $A$.
Therefore any composite number is measured by some prime number.

Q. E. D.
Back to Euclid’s *Elements* (Book VII)

**Proposition 31**

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Then, since $C$ measures $B$,

and $B$ measures $A$,

therefore $C$ also measures $A$.

And, if $C$ is prime, what was enjoined will have been done.

But if it is composite, some number will measure it.

Thus, if the investigation be continued in this way, some prime number will be found which will measure the number before it, which will also measure $A$.

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Therefore any composite number is measured by some prime number.

Q. E. D.

Root of Fermat’s infinite descent proof method.
Irrationality of $\sqrt{2}$ in sequent calculus

\[
\begin{align*}
0 < x_0, x_0^2 = 2x_1^2 & \vdash \\
0 < x_1 < x_0 & \wedge \\
\exists x_2. x_0 = 2x_2 & \\
\end{align*}
\]
Irrationality of $\sqrt{2}$ in sequent calculus

\[
\begin{align*}
0 < x_0, x_0^2 = 2x_1^2 \vdash \\
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\exists x_2. x_0 = 2x_2
\end{align*}
\]

\[
\begin{align*}
0 < x_1, x_1^2 = 2x_2^2 \vdash \\
0 < x_2 < x_1 \land \\
\exists x_3. x_1 = 2x_3
\end{align*}
\]

\[
\begin{align*}
0 < x_3, x_3^2 = 2x_4^2 \vdash \\
0 < x_4 < x_3 \land \\
\exists x_5. x_3 = 2x_4
\end{align*}
\]

\[
\begin{align*}
x_3 < x_2, 0 < x_3, 4x_4^2 = 2x_3^2 \vdash
\end{align*}
\]

A natural question:

Is infinite descent equivalent to proof by induction?
$\mu$MALL: MALL with Least and Greatest Fixed Points
**µMALL Formulas and Sequent calculus**

(Baelde & Miller 2007, Baelde 2012)

µMALL formulas

\[ F ::= \top | \bot | F \otimes F | F \& F \]

\[ | 0 | 1 | F \otimes F | F \oplus F \]

\[ | X | \mu X.F | \nu X.F \]

**MALL formulas**  
**least and greatest fixed point**

- \( \mu \) and \( \nu \) are dual.

\[ \textbf{Ex: } (\nu X.X \otimes X) \bot = \mu X.X \otimes X. \]

- Data types encodings:

  - Nat :\(=\)\(\mu X.1 \oplus X\)
  - List\((A)\) :\(=\)\(\mu X.1 \oplus (A \otimes X)\)
  - Stream\((A)\) :\(=\)\(\nu X.A \otimes X\)

µMALL Sequent Calculus  

⇒ **See following slides**

MALL inference rules together with inference rules for \( \mu \) and \( \nu \)
Knaster-Tarski fixed point theorem

Knaster-Tarski theorem

Let $F$ be a monotonic function over a complete lattice.

$$F(\mu F) \leq \mu F \quad F(X) \leq X \Rightarrow \mu F \leq X$$

$$\nu F \leq F(\nu F) \quad X \leq F(X) \Rightarrow X \leq \nu F$$
Knaster-Tarski fixed point theorem

Knaster-Tarski theorem

Let $F$ be a monotonic function over a complete lattice.

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F(\mu F) \leq \mu F \quad F(X) \leq X \Rightarrow \mu F \leq X
\]
\[
\nu F \leq F(\nu F) \quad X \leq F(X) \Rightarrow X \leq \nu F
\]

Let $F$ be a formula where $X$ occurs only positively.

This gives right and left rules for $\mu$:

\[
F[\mu X.F] \vdash \mu X.F \quad (\mu_r)
\]
\[
\frac{F[S] \vdash S}{\mu X.F \vdash S} \quad (\mu_l)
\]
Knaster-Tarski fixed point theorem

Knaster-Tarski theorem
Let $F$ be a monotonic function over a complete lattice.

\[ F(\mu F) \leq \mu F \quad F(X) \leq X \Rightarrow \mu F \leq X \]
\[ \nu F \leq F(\nu F) \quad X \leq F(X) \Rightarrow X \leq \nu F \]

*Let $F$ be a formula where $X$ occurs only positively.*

This gives right and left rules for $\mu$:

\[ H \vdash F[\mu X.F/X] \quad \frac{F[S/X] \vdash S}{\mu X.F \vdash S} \quad (\mu_l) \]

\[ H \vdash \mu X.F \quad \frac{H \vdash F[\mu X.F/X]}{(\mu_r)} \]
Knaster-Tarski fixed point theorem

Knaster-Tarski theorem
Let $F$ be a monotonic function over a complete lattice.

$F(\mu F) \leq \mu F$ \hspace{1cm} $F(X) \leq X \Rightarrow \mu F \leq X$

$\nu F \leq F(\nu F)$ \hspace{1cm} $X \leq F(X) \Rightarrow X \leq \nu F$

*Let $F$ be a formula where $X$ occurs only positively.*

This gives right and left rules for $\mu$:

\[
\Gamma \vdash \Delta, F[\mu X. F] \quad (\mu_r) \quad \frac{S, \Gamma \vdash \Delta}{\mu X. F, \Gamma \vdash \Delta} \quad (\mu_l)
\]
Knaster-Tarski fixed point theorem

Knaster-Tarski theorem
Let \( F \) be a monotonic function over a complete lattice.

\[
F(\mu F) \leq \mu F \quad F(X) \leq X \Rightarrow \mu F \leq X \\
\nu F \leq F(\nu F) \quad X \leq F(X) \Rightarrow X \leq \nu F
\]

Let \( F \) be a formula where \( X \) occurs only positively.

This gives right and left rules for \( \nu \):

\[
\frac{F[\nu X.F], \Gamma \vdash \Delta}{\nu X.F, \Gamma \vdash \Delta} (\nu_l) \quad \frac{\Gamma \vdash \Delta, S \quad S \vdash F[S]}{\Gamma \vdash \Delta, \nu X.F} (\nu_r)
\]
Knaster-Tarski fixed point theorem

Knaster-Tarski theorem

Let $F$ be a monotonic function over a complete lattice.

$F(\mu F) \leq \mu F$ \hspace{1cm} $F(X) \leq X \Rightarrow \mu F \leq X$

$\nu F \leq F(\nu F)$ \hspace{1cm} $X \leq F(X) \Rightarrow X \leq \nu F$

Let $F$ be a formula where $X$ occurs only positively.

One-sided rules for $\mu$ and $\nu$:

$$\frac{\Gamma \vdash \Delta, F[\mu X.F]}{\Gamma \vdash \Delta, \mu X.F} \quad (\mu_r)$$

$$\frac{\Gamma \vdash \Delta, S \quad S \vdash F[S]}{\Gamma \vdash \Delta, \nu X.F} \quad (\nu_r)$$
\( \mu \text{MALL}^\infty \): infinite and circular proofs
Infinitary Sequent Calculus

Consider your favourite logic $\mathcal{L}$ & add fixed points as in $\mu$MALL:

Pre-proofs are the trees coinductively generated by:

- $\mathcal{L}$ inference rules
- $\mu$, $\nu$-rules:

\[
\frac{\Gamma, F[\mu X.F/X]}{\Gamma, \mu X.F} \quad (\mu) \\
\frac{\Gamma, F[\nu X.F/X]}{\Gamma, \nu X.F} \quad (\nu)
\]

Fischer-Ladner Subformulas: induced by fixed-point unrolling:

\[F[\sigma X.F/X] \in FL(\sigma X.F), \ \sigma \in \{\mu, \nu\}.\]

Circular (pre-)proofs: the regular fragment of infinite (pre-)proofs, ie finitely many sub-(pre)proofs.
\( \mu \text{MALL}^\infty \) Infinitary Sequent Calculus

Consider your favourite logic \( \mathcal{L} \) & add fixed points as in \( \mu \text{MALL} \):

\( \mu \text{MALL}^\infty \) Pre-proofs are the trees \textbf{coinductively} generated by:

- MALL inference rules
- \( \mu, \nu \)-rules:

\[
\Gamma, F\sigma X.F / X \frac{\vdash}{\mu} \Gamma, \mu X.F \\
\frac{\vdash}{\nu} \Gamma, \nu X.F
\]

\textbf{Fischer-Ladner Subformulas:} induced by fixed-point unrolling:

\[ F[\sigma X.F / X] \in \text{FL}(\sigma X.F), \sigma \in \{\mu, \nu\}. \]

\textbf{Circular (pre-)proofs:} the regular fragment of infinite (pre-)proofs, ie finitely many sub-(pre)proofs.

\( \mu \text{MALL}^\omega \)
\(\mu\text{MALL}^\infty\) Infinitary Sequent Calculus

Consider your favourite logic \(\mathcal{L}\) & add fixed points as in \(\mu\text{MALL}^\infty\):

\(\mu\text{MALL}^\infty\) Pre-proofs are the trees **coinductively** generated by:

- MALL inference rules
- \(\mu, \nu\)-rules:
  \[
  \frac{\Gamma, F[\mu X.F/X]}{\Gamma, \mu X.F} \quad (\mu) \quad \frac{\Gamma, F[\nu X.F/X]}{\Gamma, \nu X.F} \quad (\nu)
  \]

**Fischer-Ladner Subformulas:** induced by fixed-point unrolling:

\[
F[\sigma X.F/X] \in \text{FL}(\sigma X.F), \ \sigma \in \{\mu, \nu\}.
\]

**Circular (pre-)proofs:** the regular fragment of infinite (pre-)proofs, ie finitely many sub-(pre)proofs.

\(\mu\text{MALL}^\omega\)

Pre-proofs are unsound!! **Need for a validity condition**

\[
\frac{\vdash \mu X.X}{\vdash \mu X.X} \quad (\mu) \quad \frac{\vdash \nu X.X, F}{\vdash \nu X.X, F} \quad (\nu) \quad \frac{\vdash \nu X.X, F}{\vdash \mu X.X} \quad (\nu) \quad \frac{\vdash \nu X.X, F}{\vdash \nu X.X, F} \quad (\text{Cut})
\]
Propositional classical $\mu$-calculus

Syntax

$$F ::= a \mid a^\perp \mid \top \mid \bot \mid F_1 \lor F_2 \mid F_1 \land F_2 \mid \mu X.F \mid \nu X.F \mid X$$

Truth semantics

$$[[\mu X.F]]^\mathcal{E} := \text{lfp}(b \mapsto [[F]]^\mathcal{E} + (X \mapsto b))$$
$$[[\nu X.F]]^\mathcal{E} := \text{gfp}(b \mapsto [[F]]^\mathcal{E} + (X \mapsto b))$$

Examples

$$[[\mu X.X]]^\mathcal{E} = 0 \quad [[\nu X.X]]^\mathcal{E} = 1$$
$$[[\mu X.a \land X]]^\mathcal{E} = 0 \quad [[\nu X.a \land X]]^\mathcal{E} = [[a]]^\mathcal{E}$$
Parity games

\[ S := \nu X. (\mu Y. 1 \lor Y) \land X \quad N := \mu Y. 1 \lor Y \]

**Prover wins if:**
- either the play is finite and ends in a Refuter node,
- or among formulas encountered infinitely often, the minimum wrt. subformula ordering is a greatest fixed point:

\[ \min(\inf(\rho)) = \nu \]

Here, Prover wins iff 1 is reached or \( S \) is visited infinitely often.
Validity condition (1)

Every infinite branch contains a thread $t$ such that $\text{min}(\text{inf}(t)) = \nu$. 
Validity condition (2)

\[ F \;:=\; \nu X. \; X \land (\mu Y. \; X \lor Y) \]
\[ G \;:=\; \mu Y. \; F \lor Y \]

Prover wins if, among formulas encountered infinitely often, the minimum wrt. subformula ordering is a greatest fixed point:

\[ \min(\inf(p)) = \nu \]
Validity condition (2)

\[
\begin{align*}
\therefore & \quad F, G \\
\therefore & \quad G \\
\mid & \quad F \land G, G & & \quad (\land) \\
\therefore & \quad F \land G, G \\
\therefore & \quad F, G & & \quad (\lor)
\end{align*}
\]

Validity condition

Every infinite branch contains a thread \( t \) such that \( \min(\inf(t)) = \nu \).
Validity condition (2)

\[
\begin{align*}
\frac{}{\vdash F, G} & \quad (\heartsuit) \\
\frac{\vdash F \land G, G}{\vdash G} & \quad (\wedge) \\
\frac{}{\vdash F, G} & \quad (\spadesuit) \\
\frac{\vdash F \lor G}{\vdash F} & \quad (\diamondsuit) \\
\frac{}{\vdash G} & \quad (\clubsuit) \\
\frac{}{\vdash F \land G} & \quad (\wedge) \\
\frac{}{\vdash F} & \quad (\diamondsuit) \\
\frac{}{\vdash G} & \quad (\clubsuit) \\
\end{align*}
\]

Validity condition

Every infinite branch contains a thread \( t \) such that \( \min(\inf(t)) = \nu \).
Validity condition (2)

\[
\begin{align*}
(\heartsuit) & \quad (\spadesuit) \\
\vdash F, G & \quad \vdash G \\
\vdash F \land G, G & \quad (\land) \\
(\heartsuit) \vdash F, G & \quad (\lor) \\
(\diamondsuit) & \quad (\spadesuit) \quad (\mu) \\
\vdash F & \quad (\land) \\
(\diamondsuit) \vdash G & \quad (\lor) \\
\vdash F \land G & \quad (\land) \\
(\diamondsuit) \vdash F & \quad (\lor) 
\end{align*}
\]

Validity condition

Every infinite branch contains a thread \( t \) such that \( \min(\inf(t)) = \nu \).
Validity condition (2)

Validity condition

Every infinite branch contains a thread \( t \) such that \( \min(\inf(t)) = \nu \).
Validity condition (2)

\[
\begin{align*}
(\heartsuit) & \quad (\spadesuit) \\
\vdash F, G & \quad \vdash F, G \\
\vdash F \land G, G & \quad \vdash G, G \quad \text{(weak)} \\
(\heartsuit) & \quad (\spadesuit) \\
\vdash F \land G, G & \quad \vdash F, G \\
(\heartsuit) & \quad (\spadesuit) \\
\vdash F \lor G & \quad \vdash G \\
(\spadesuit) & \quad (\mu) \\
\vdash F \land G & \quad \vdash F \\
(\diamondsuit) & \quad (\lor) \\
\vdash F & \quad \vdash F \land G \\
(\forall) & \quad (\forall) \\
\vdash F & \quad \vdash F \lor G \\
(\diamondsuit) & \quad (\forall) \\
\vdash F \\
(\forall) & \quad (\forall) \\
\vdash F \land G & \quad \vdash F \\
(\diamondsuit) & \quad (\forall) \\
\vdash F & \quad \vdash F \land G
\end{align*}
\]

Need to distinguish formula occurrences!

Validity condition

Every infinite branch contains a thread \( t \) such that \( \min(\text{inf}(t)) = \nu \).
Validity condition (3)

- A *thread* is a sequence of formulas that traces the evolution of a formula along an infinite branch.

- Cut spawns two new threads, on cut formulas:  
  \[ \vdash \Gamma, F \vdash F^{\bot}, \Delta, \Delta \vdash \Gamma, \Delta \]

- A thread is *valid* if its outermost formula is a \( \nu \)-formula.

- A pre-proof is *valid* if every branch contains a valid thread.

- A valid pre-proof is called *proof*.

- Validity of regular pre-proofs is *decidable*. 

Details
Example of a $\mu$MALL$^\omega$ proof

- Inductive and coinductive definitions

\[ N = \mu X.1 \oplus X \quad S = \nu X.N \otimes X \]

- Proofs-programs over these data types

\[
\text{\textit{double}}(n) = \begin{cases} 
0 & \text{if } n = 0 \\
\text{\textit{succ}}(\text{\textit{succ}}(\text{\textit{double}}(m))) & \text{if } n = \text{\textit{succ}}(m)
\end{cases}
\]
Example of a $\mu\text{MALL}^\omega$ proof

- Inductive and coinductive definitions

\[
N = \mu X. 1 \oplus X \quad \quad S = \nu X. N \otimes X
\]

- Proofs-programs over these data types

\[
double(n) = \begin{cases} 
0 & \text{if } n = 0 \\
\text{succ}(\text{succ}(\text{double}(m))) & \text{if } n = \text{succ}(m)
\end{cases}
\]

\[
\Pi_{\text{double}} = \frac{1 \vdash 1}{1 \vdash 1 \oplus N} \quad \quad \frac{N \vdash N}{N \vdash 1 \oplus N} \quad \quad \frac{N \vdash N}{N \vdash N} \quad \quad \frac{1 \vdash 1 \oplus N}{1 \vdash N} \quad \quad \frac{N \vdash N}{N \vdash N} \quad \quad \frac{1 \vdash 1 \oplus N}{1 \vdash N} \quad \quad \frac{N \vdash N}{N \vdash N}
\]

\[
\quad \quad \frac{N \vdash N}{N \vdash N} \quad \quad \frac{N \vdash N}{N \vdash N} \quad \quad \frac{N \vdash N}{N \vdash N} \quad \quad \frac{N \vdash N}{N \vdash N} \quad \quad \frac{N \vdash N}{N \vdash N}
\]

- (Ax) \quad \quad \frac{N \vdash N}{N \vdash N} \quad \quad \frac{N \vdash N}{N \vdash N} \quad \quad \frac{N \vdash N}{N \vdash N} \quad \quad \frac{N \vdash N}{N \vdash N} \quad \quad \frac{N \vdash N}{N \vdash N}

- (\oplus_1) \quad \quad \frac{N \vdash N}{N \vdash N} \quad \quad \frac{N \vdash N}{N \vdash N} \quad \quad \frac{N \vdash N}{N \vdash N} \quad \quad \frac{N \vdash N}{N \vdash N} \quad \quad \frac{N \vdash N}{N \vdash N}

- (\mu_r) \quad \quad \frac{N \vdash N}{N \vdash N} \quad \quad \frac{N \vdash N}{N \vdash N} \quad \quad \frac{N \vdash N}{N \vdash N} \quad \quad \frac{N \vdash N}{N \vdash N} \quad \quad \frac{N \vdash N}{N \vdash N}

- (\oplus_2) \quad \quad \frac{N \vdash N}{N \vdash N} \quad \quad \frac{N \vdash N}{N \vdash N} \quad \quad \frac{N \vdash N}{N \vdash N} \quad \quad \frac{N \vdash N}{N \vdash N} \quad \quad \frac{N \vdash N}{N \vdash N}

- (\mu_l) \quad \quad \frac{N \vdash N}{N \vdash N} \quad \quad \frac{N \vdash N}{N \vdash N} \quad \quad \frac{N \vdash N}{N \vdash N} \quad \quad \frac{N \vdash N}{N \vdash N} \quad \quad \frac{N \vdash N}{N \vdash N}
Circular & finitary proofs

From finitary to circular proofs

Theorem

Finitary proofs can be transformed to (valid) infinitary proofs.

The key translation step is the following:

\[
\begin{align*}
\pi_1 & \vdash \Gamma, S \\
\pi_2 & \vdash S^\perp, F[S] \\
\end{align*}
\]

(\nu)

\[
\begin{align*}
\pi_1 & \vdash \Gamma, S \\
& \vdash \Gamma, \nu X. F \\
\end{align*}
\]

From circular to finitary proofs

Brotherston-Simpson’s conjecture recently (dis)proved (FOSSACS 2017, LICS 2017). No general result for \( \mu \text{MALL}^\omega \).
μMALL∞ Cut elimination and Focalization

Before:
Additive Cut elimination only (Santocanale & Fortier):

Lattice logic [μ-calculus over ALL (⊕, &, 0, ⊤).]
- Sequents are of the form $F ⊢ G$.
- Infinite branches have exactly one left and one right thread.

Now:

Theorem [BDS 2016]
μMALL∞ enjoys cut-elimination and focalization.

Proofs in two steps:
- cut-elimination/focalization procedures are productive;
- the resulting pre-proof is valid.
Relaxing validity:
From straight threads
to bouncing threads
Invalid but productive proofs

Motto of the previous part:

*Cut-elimination is productive on valid pre-proofs and always produces valid pre-proofs.*
Invalid but productive proofs

Motto of the previous part:

*Cut-elimination is productive on valid pre-proofs and always produces valid pre-proofs.*

However, some productive computations are invalid, e.g. convoluted identity over $S = \nu X. X$:

$$
\frac{S \vdash S}{S \vdash S} \quad \text{(ax)}
\frac{S \vdash S}{S \vdash S} \quad \text{(\nu_R)}
\frac{S \vdash S}{S \vdash S} \quad \text{(cut)}
$$

$$
\frac{S \vdash S}{S \vdash S} \quad \text{(\nu_L)}
$$

Allow for a more relaxed validity criterion?

\[
f(\_ :: s) = g(f(s))
\]

\[
g(x) = \_ :: x
\]
Towards bouncing threads

Notations

- $s \sqsubseteq s'$ when the sequent occurrence $s'$ is a premise of $s$;
- $F \sqsubseteq F'$ when $F'$ is a sub-occurrence of $F$.

Definition

A **b-thread** is an infinite sequence $(F_i, d_i, s_i)_{i \in \omega}$ such that, for all $i$, $F_i \in s_i$ and one of the following holds:

- $d_i = d_{i+1} = \uparrow$, $s_i \sqsubseteq s_{i+1}$ and $F_i \sqsubseteq F_{i+1}$.
- $d_i = d_{i+1} = \downarrow$, $s_{i+1} \sqsubseteq s_i$ and $F_{i+1} \sqsubseteq F_i$.
- $d_i = \uparrow$, $d_{i+1} = \downarrow$, $s_i = s_{i+1} = \{F_i, F_{i+1}\}$ is an axiom.
- $d_i = \downarrow$, $d_{i+1} = \uparrow$, $s_i$ and $s_{i+1}$ premises of cut on $F_i = F_{i+1}^\perp$. 
Example: b-thread

\[
\begin{align*}
1 \vdash N & \quad (0) \\
N \vdash N & \quad (ax) \\
N \vdash N & \quad (\text{succ}) \\
1 \vdash B & \quad (tt) \\
1 \vdash B & \quad (ff) \\
N \vdash B & \quad (\text{cut}) \\
N \vdash B &
\end{align*}
\]

This thread gets erased/disconnected during cut elimination!
Example: b-thread

This thread gets erased/disconnected during cut elimination!
Persistent b-threads

Let $\Sigma = \{l, r, i, l^*, r^*, i^*\}$, and let $M$ be the set of finite or infinite words over it, plus 0, quotiented by the following equations:

$$0 \cdot x = x \cdot 0 = 0$$

$$u^* \cdot u = 1 \quad \text{and} \quad v^* \cdot u = 0 \quad \text{for all} \quad u \neq v \in \{l, r, i\}$$

**Definition**

The **weight** of a thread is $w(t) = \prod_{i \in \omega} a_i$:

- $a_i = l/r/i$ when $F_i = F_{\alpha}$ and $F_{i+1} = F_{\alpha a_i}$;
- $a_i = l^*/r^*/i^*$ symmetrically;
- $a_i = 1$ otherwise.

The thread is **persistent** if $w(t) \not\equiv M 0$.

---

*Reminiscent from Geometry of Interaction, well-bracketed threads...*
Visible part

Definition

A thread $t$ admits a **visible part** $\nu$ when:
$w(t) =_M \nu$ and $\nu$ is a (possibly infinite) word over $\{l, r, i\}$.

Definition

When the weight admits a visible part $\nu = (\nu_j)_{j \in \lambda}$ ($\lambda \leq \omega$)
we say that the thread *visibly explores* $(G_j)_{j \in \lambda}$ defined by
$G_0 = F_0$ (the first formula in the thread) and,
$G_{i+1}$ unique descendent of $G_i = F_{\alpha}$ with address $\alpha \nu_j$. 
Example: visible part

\[ \begin{align*}
1 \vdash N & \quad \text{(0)} \quad N \vdash N \\
N \vdash N & \quad \text{(ax)} \quad (\text{succ}) \quad 1 \vdash B \\
N \vdash N & \quad (tt) \quad 1 \vdash B \\
N \vdash B & \quad (\text{cut}) \quad N \vdash B
\end{align*} \]

\[ \alpha \alpha_i \alpha_{ir} \alpha_{ir} \alpha_i \alpha \alpha_i \alpha_{ir} \alpha_{iri} \alpha_{iril} \]
Example: visible part

\[ S \vdash S \]

(\(\star\))

(\(\nu_R\))

(\(\nu_L\))

(\(\text{cut}\))

(\(\nu_L\))

\[ \alpha \alpha_i (\alpha_i \alpha \alpha_i) \alpha \]
Bouncing Validity

A b-thread is **valid** if:

- its weight admits an infinite visible part,
- the minimum of the visibly explored formulas is a $\nu$ formula,
- and it never bounces on its starting formula.

An infinite branch is **valid** if:

- there is a valid b-thread starting from one of its sequents which visits the branch infinitely often.

**Definition (Proof)**

A **proof** is a pre-proof in which every infinite branch is valid.
Bouncing Validity

A b-thread is **valid** if:
- its weight admits an infinite visible part,
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An infinite branch is **valid** if:
- there is a valid b-thread starting from one of its sequents
- which visits the branch infinitely often.

Definition (Proof)
A **proof** is a pre-proof in which every infinite branch is valid.
Cut elimination with bouncing threads

Theorem

*If every infinite branch of a $\mu$MLL$^\infty$ pre-proof $\pi$ is supported by a valid bouncing thread, then the cut-elimination of $\pi$ is productive.*

**Idea:** perform just enough cut elimination to reduce **bouncing-valid** derivation into **straight-valid** one.

Theorem

*Bouncing-valid derivation converge to straight-valid ones.*

Extension to Additives (WIP): criterion for every slices of the proof.
Cut elimination with bouncing threads

Theorem

*Bouncing-valid derivation converge to straight-valid ones.*

Proof.

Consider head reductions of selected cuts:

- **bouncer** = cut on which a validating thread bounces.

By contradiction, it is *productive*:

- reducing bouncers corresponds to weight simplifications;
- eventually, the visible part of some thread must be produced.

The result is *straight-valid*:

- a resulting infinite branch induces an original infinite branch;
- valid bouncing threads result in valid straight threads.
Conclusion
Conclusion

To sum up:

- Fixed point logics extending MALL with finite or infinite proofs;
- Syntactic cut elimination;
- Focalization;
- Relaxing threads in the multiplicative case.

Future work:

- Go beyond Linear Logic and handle structural rules;
- Equivalence of circular fragment of $\mu MALL^\infty$ and $\mu MALL$: Translate infinitary proofs to finitary ones;
- Same question as above by preserving the computational content.
- Other styles: Natural deduction, circular $\lambda$-calculus
To sum up:

- Fixed point logics extending MALL with finite or infinite proofs;
- Syntactic cut elimination;
- Focalization;
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Future work:

- Go beyond Linear Logic and handle structural rules;
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- Other styles: Natural deduction, circular $\lambda$-calculus

Thank you for your attention!
Appendix
Decidability of the validity condition
Parity automata

Definition

A *parity automaton* is a finite state word automaton, whose states are ordered and given a parity bit $\nu/\mu$, which accepts runs $(q_i)_{i \in \omega}$ such that $\min(\inf((q_i)_i))$ has parity $\nu$. 
Parity automata

Definition

A parity automaton is a finite state word automaton, whose states are ordered and given a parity bit $\nu/\mu$, which accepts runs $(q_i)_{i \in \omega}$ such that $\min(\inf((q_i)_i))$ has parity $\nu$.

Remarks

- States are usually given a color in $\mathbb{N}$, equivalently.
- Only co-accessible states need to be ordered.
Parity automata

Definition
A *parity automaton* is a finite state word automaton, whose states are ordered and given a parity bit $\nu/\mu$, which accepts runs $(q_i)_{i \in \omega}$ such that $\min(\inf((q_i)_i))$ has parity $\nu$.

Remarks
- States are usually given a color in $\mathbb{N}$, equivalently.
- Only co-accessible states need to be ordered.

Closure properties
Parity automata can be determinized, complemented, and intersected, thus their inclusion is decidable.
Decidability of the validity condition

Theorem

The validity of circular pre-proofs is decidable.

Proof.

Consider a pre-proof $\Pi$ i.e. a graph with nodes $s_i = (F_j^i)_{j \in [1; n_i]}$.

- Let $A_B$ be the branch automaton with states $s_i$, transitions $s_i \to^k s_j$ when $s_j$ is the $k$-th premise of $s_i$, and which accepts all runs.

- Let $A_T$ be the thread automaton with states $F_j^i$ or $\bot_i$, transitions $F_j^i \to^k F_p^q$ when
  - $s_i \to^k s_p$ and
  - $F_j^i = \bot$ or the ancestor of $F_p^q$
  and acceptance based on the subformula ordering.

We have that $\Pi$ is valid iff $A_B \subseteq A_T$. 

To slide on threads
MALL proof system
LL Sequent Calculus

\[ F ::= a \mid F \otimes F \mid F \oplus F \mid 1 \mid 0 \mid ! F \]

\[ a \perp \mid F \otimes F \mid F \& F \mid \bot \mid \top \mid ? F \]

\[ \vdash \Gamma, A, A \perp \text{(Ax)} \]

\[ \vdash \Gamma, A \vdash \Delta, A \perp \text{(Cut)} \]

\[ \vdash 1 \text{ [1]} \quad \vdash \Gamma, A \vdash \Delta, B \quad \text{[\otimes]} \quad \vdash \Gamma, A, B \quad \text{[\otimes]} \quad \vdash \Gamma \quad \text{[\perp]} \]

\[ \vdash \Gamma, A_i \quad \text{[\oplus_i]} \quad i \in \{1, 2\} \quad \vdash \Gamma, A_1 \oplus A_2 \quad \vdash \Gamma, A \vdash \Gamma, B \quad \text{[\&]} \quad \vdash \Gamma, \top \quad \text{[\top]} \]

\[ \vdash ? \Gamma, B \quad \text{[!]} \quad \vdash ? \Gamma, B \quad \text{[?]} \quad \vdash \Gamma \quad \text{[?w]} \quad \vdash \Gamma, ? B, ? B \quad \text{[? c]} \]
MALL Sequent Calculus

\[ F ::= a \mid F \otimes F \mid F \oplus F \mid 1 \mid 0 \quad \text{positive} \]
\[ a \perp \mid F \oslash F \mid F \& F \mid \bot \mid \top \quad \text{negative} \]

\[ \vdash \Gamma, A, A \perp \quad \text{(Ax)} \quad \vdash \Gamma, A \quad \vdash \Delta, A \perp \quad \text{(Cut)} \]

\[ \vdash 1 \quad \text{[1]} \quad \vdash \Gamma, A \quad \vdash \Delta, B \quad \text{(\otimes)} \quad \vdash \Gamma, A, B \quad \text{[\oslash]} \quad \vdash \Gamma \quad \text{[\perp]} \]

\[ \vdash \Gamma, A_i \quad \text{[\oplus_i]} \quad i \in \{1, 2\} \quad \vdash \Gamma, A \quad \vdash \Gamma, B \quad \text{[\&]} \quad \vdash \Gamma, \top \quad \text{[\top]} \]
Focalization in $\mu \mathrm{MALL}^{\infty}$
\[ \mu \text{MALL}^\infty \text{ Focalization} \]

1/ Reversibility of negative sequents:

- Similar to MALL reversibility except that it cannot be treated with local rule permutations as shown by the following example.

\[
\begin{align*}
\pi & \\
\frac{\vdash F, P, Q}{\vdash F, P \otimes Q} & \quad (\otimes) \\
\vdash F, P \otimes Q & \quad (\&)
\end{align*}
\]

\[
\begin{align*}
\frac{\vdash F, P \otimes Q}{\vdash F \& F, P \otimes Q} & \quad (\otimes_1) \\
\frac{\vdash (F \& F) \oplus 0, P \otimes Q}{\vdash F, P \otimes Q} & \quad (\oplus)
\end{align*}
\]

\[
\begin{align*}
\vdash F, P \otimes Q & \quad (\&)
\end{align*}
\]


\(\mu \text{MALL}^\infty\) Focalization

1/ **Reversibility of negative sequents:**

- Similar to MALL reversibility except that it cannot be treated with local rule permutations as shown by the following example.

2/ **Focalization of positive sequents:**

- Positive trunks are finite trees (due to the polarization of fixed points formulas);
- The rest of the proof goes as for MALL.

3 & 4/ **Productivity and validity of focalization process:**

- Productivity of the focusing process is essentially direct from \(\text{MALL}\) case (guarded calls to the focalization process);
- Preservation of validity relies on an analysis of the kind of permutations involved in focalization: valid thread cannot be infinitely postponed.
Cut elimination in $\mu\text{MALL}^\infty$
The purely additive case

Lattice logic
- Consider $\mu$-calculus over additive linear logic ($\oplus$, $\&$, 0, $\top$).
- Sequents are of the form $F \vdash G$.
- Infinite branches have exactly one left and one right thread.

Results [Santocanale 2002, Santocanale & Fortier 2013]
- Primitive recursive functions encoded as proofs.
- Restricted cut-elimination through semantics.
- Syntactic cut elimination.
Cut elimination strategy

The following strategy is used in the additive and MALL cases:

1. Reduce bottom-most cuts first.

2. Perform internal chatter while necessary, e.g.
   \[ \Gamma, F[\mu X. F/\mu X] \vdash \Gamma, \mu X. F(\mu X) \]
   \[ \vdash F[\mu X. F/\mu X] \]
   \[ \vdash \nu X. F[\nu X/\mu X], \Delta \vdash \nu X. F[\nu X/\mu X], \Delta(\nu X) \vdash \Gamma, \Delta \rightarrow \text{int} \vdash \Gamma, F[\mu X. F/\mu X] \vdash F[\mu X. F/\mu X] \]

3. Perform external rules, which produce cut-free result, e.g.
   \[ \Gamma, F, H \vdash \Gamma, F \oplus G, H \oplus G \]
   \[ \vdash H[\oplus G] \]
   \[ \vdash \Gamma, F, \Delta \rightarrow \text{ext} \vdash \Gamma, F, H \vdash H[\oplus G], \Delta \vdash \Gamma, F, H \rightarrow \text{ext} \vdash \Gamma, F, H \]

4. Repeat.
Cut elimination strategy

The following strategy is used in the additive and MALL cases:

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Cut elimination strategy

The following strategy is used in the additive and MALL cases:

1. Reduce bottom-most cuts first.

2. Perform *internal* chatter while necessary, e.g.

\[
\begin{align*}
&\Gamma, \Delta \vdash \Gamma, \mu X.F[\mu X.F/X] \\
&\Gamma, \Delta \vdash \Gamma, \mu X.F \\
&\Gamma, \Delta \vdash \Gamma, \nu X.F[\nu X.F/X] \rightarrow_{\text{int}} \\
&\Gamma, \Delta \vdash \Gamma, \nu X.F[\nu X.F/X] \\
&\Gamma, \Delta \vdash \Gamma, \nu X.F \\
\end{align*}
\]
**Cut elimination strategy**

The following strategy is used in the additive and MALL cases:

1. Reduce bottom-most cuts first.

2. Perform *internal* chatter while necessary, e.g.

   \[
   \frac{\vdash \Gamma, F[\mu X. F/X]}{\vdash \Gamma, \mu X. F} \quad (\mu) \quad \frac{\vdash F^\perp[vX. F^\perp/X], \Delta}{\vdash vX. F^\perp, \Delta} \quad (\nu) \quad \frac{\vdash \Gamma, F[\mu X. F/X]}{\vdash F^\perp[vX. F^\perp/X], \Delta} \quad \rightarrow_{\text{int}} \quad \frac{\vdash \Gamma, \Delta}{\vdash \Gamma, \Delta}
   \]

3. Perform *external* rules, which produce cut-free result, e.g.

   \[
   \frac{\vdash \Gamma, F, H}{\vdash \Gamma, F \oplus G, H} \quad (\oplus) \quad \frac{\vdash \Gamma, F \oplus G, \Delta}{\vdash \Gamma, F^\perp, \Delta} \quad \rightarrow_{\text{ext}} \quad \frac{\vdash \Gamma, F, \Delta}{\vdash \Gamma, F \oplus G, \Delta} \quad (\oplus)
   \]

4. Repeat.
Lemma (Productivity)

*Cut reductions are productive, i.e. $\rightarrow_{\text{int}}$ terminates.*

Proof sketch.

Assuming the contrary, we have an infinite internal reduction. Consider the sub-tree whose sequents take part to that reduction.
Reviewing additive cut elimination \((F \vdash G)\)

**Lemma (Productivity)**

*Cut reductions are productive, i.e. \(\rightarrow_{\text{int}}\) terminates.*

**Proof sketch.**

Assuming the contrary, we have an infinite internal reduction. Consider the sub-tree whose sequents take part to that reduction.

- The rightmost branch must have a left \(\mu\)-thread.
Lemma (Productivity)

Cut reductions are productive, i.e. $\rightarrow_{\text{int}}$ terminates.

Proof sketch.

Assuming the contrary, we have an infinite internal reduction. Consider the sub-tree whose sequents take part to that reduction.

- The rightmost branch must have a left $\mu$-thread.
- It has a dual branch to the left, with a right $\mu$-thread, thus a left $\nu$-thread by validity.
- And so on...
Reviewing additive cut elimination \((F \vdash G)\)

Lemma (Productivity)

*Cut reductions are productive, i.e. \(\rightarrow_{\text{int}}\) terminates.*

Proof sketch.

Assuming the contrary, we have an infinite internal reduction. Consider the sub-tree whose sequents take part to that reduction.

- The rightmost branch must have a left \(\mu\)-thread.
- It has a dual branch to the left, with a right \(\mu\)-thread, thus a left \(\nu\)-thread by validity.
- And so on...
- The limit, leftmost branch, still has a right \(\mu\)-thread, thus a left \(\nu\)-thread, contradicting its leftmostness.
**μMALL∞ Cut elimination procedure**

- **Strategy:** “push” the cuts away from the root.

- **Cut-Cut:**

\[
\begin{align*}
\vdash \Gamma, F & \quad \vdash F^\perp, \Delta, G \\
\vdash \Gamma, \Delta, G & \quad \text{(Cut)} \\
\vdash \Gamma, \Delta, \Sigma & \quad \vdash G^\perp, \Sigma \\
\vdash \Gamma, \Delta, \Sigma & \quad \vdash \Gamma, \Delta, \Sigma
\end{align*}
\]
$\mu MALL\infty$ Cut elimination procedure

• **Strategy:** “push” the cuts away from the root.

• **Cut-Cut:**

$$
\begin{align*}
\vdash \Gamma, F & \vdash F^\perp, \Delta, G \\
& \quad \vdash \Gamma, \Delta, G \quad \text{(Cut)} \\
& \quad \vdash \Gamma, \Delta, G \\
\downarrow & \\
\vdash \Gamma, F & \vdash F^\perp, \Delta, G & \vdash G^\perp, \Sigma & \vdash \Gamma, \Delta, \Sigma \\
& \quad \text{(Cut)} \\
& \quad \text{(Cut)} & \quad \text{(mcut)}
\end{align*}
$$
\( \mu \text{MALL}\infty \) Cut elimination procedure - External operations

\[
\begin{array}{c}
\frac{\vdash \Delta, F, G}{\vdash \Delta, F \otimes G} \quad (\otimes) \\
\frac{\vdash \Delta, F \otimes G}{\vdash \Sigma, F \otimes G} \quad (\text{mcut})
\end{array}
\implies
\begin{array}{c}
\frac{\vdash \Delta, F, G}{\vdash \Sigma, F \otimes G} \quad (\otimes)
\end{array}
\]

\[
\begin{array}{c}
\frac{\vdash \Delta, F}{\vdash \Delta, G} \quad (\&) \\
\frac{\vdash \Delta, F \& G}{\vdash \Sigma, F \& G} \quad (\text{mcut})
\end{array}
\implies
\begin{array}{c}
\frac{\vdash \Delta, F}{\vdash \Sigma, F} \quad (\text{mcut})
\end{array}
\]

\[
\begin{array}{c}
\frac{\vdash \Delta, F[\mu X.F/X]}{\vdash \Delta, \mu X.F} \quad (\mu) \\
\frac{\vdash \Delta, \mu X.F}{\vdash \Sigma, \mu X.F} \quad (\text{mcut})
\end{array}
\implies
\begin{array}{c}
\frac{\vdash \Delta, F[\mu X.F/X]}{\vdash \Sigma, F[\mu X.F/X]} \quad (\mu)
\end{array}
\]

External operations are productive
\[ \mu \text{MALL} \infty \text{ Cut elimination procedure - Internal operations} \]

\[ \frac{\vdash \Delta, F_2 \quad \vdash \Delta, F_1}{\vdash \Delta, F_2 \& F_1} \quad (\&) \quad \frac{\vdash \Gamma, F_i^\bot}{\vdash \Gamma, F_1^\bot \oplus F_2^\bot} \quad (\oplus i) \quad \frac{\vdash \Gamma, F_1^\bot}{\vdash \Gamma, F_1^\bot \ominus F_2^\bot} \quad (\ominus) \]

\[ \vdash \Gamma, F_i \vdash \Gamma, F_i^\bot \]

\[ \Rightarrow \frac{\vdash \Delta, F_i}{\vdash \Sigma} \quad (\text{mcut}) \]

\[ \frac{\vdash \Delta, F[\mu X.F/X]}{\vdash \Delta, \mu X.F} \quad (\mu) \quad \frac{\vdash \Gamma, F^\bot[\nu X.F^\bot/X]}{\vdash \Gamma, \nu X.F^\bot} \quad (\nu) \]

\[ \frac{\vdash \Gamma, F^\bot[\nu X.F^\bot/X]}{\vdash \Gamma, \nu X.F^\bot} \quad (\text{mcut}) \]

\[ \vdash \Sigma \quad \text{(mcut)} \]

\[ \Rightarrow \frac{\vdash \Delta, F[\mu X.F/X]}{\vdash \Sigma} \quad \text{(mcut)} \]

\[ \vdash \Sigma \quad \text{(mcut)} \]

**Internal operations are not productive**
\( \mu \text{MALL}^\infty \) Cut elimination algorithm

- **Internal phase:** Perform internal transformations while you cannot do anything else.

- **External phase:** Build a part of the output tree whenever you can.

Theorem Internal phase always halts.

Theorem The pre-proof obtained by the cut elimination algorithm is valid.
\( \mu MALL^\infty \) Cut elimination algorithm

- **Internal phase**: Perform internal transformations while you cannot do anything else.

- **External phase**: Build a part of the output tree whenever you can.

- Repeat.
$\mu MALL \infty$ Cut elimination algorithm

- **Internal phase:** Perform internal transformations while you cannot do anything else.

- **External phase:** Build a part of the output tree whenever you can.

  Repeat.

**Theorem**

Internal phase always halts.

**Theorem**

The pre-proof obtained by the cut elimination algorithm is valid.
Cut elimination is productive

**Theorem**

Internal phase always halts.
Cut elimination is productive

**Theorem**

Internal phase always halts.

**Proof by contradiction:** Suppose that there is a proof of $F$ for which the internal phase does not halt.
Theorem

Internal phase always halts.

Proof by contradiction: Consider the trace of this divergent reduction.
Theorem
Internal phase always halts.

Proof by contradiction: No rule on $F$ is applied in the trace, otherwise the internal phase would halt.
Cut elimination is productive

Theorem

Internal phase always halts.

Proof by contradiction: We can eliminate the occurrences of $F$ from the trace. This yields a "proof" of $\vdash$. 

\[ \vdash \Sigma \]

\[ (r_F) \]

\[ \vdash \]
Cut elimination is productive

Theorem
Internal phase always halts.

Proof by contradiction: We show that the proof system is sound. Contradiction.
Cut elimination is productive (Details)

Theorem

Internal phase always halts.
Theorem
Internal phase always halts.

Proof: Suppose that the internal phase diverges for a proof $\pi \vdash \Delta$.

- Let $\theta$ be the sub-derivation of $\pi$ explored by the reduction.
- No rule is applied to a formula of $\Delta$ in $\theta$, as this would contradict the divergence of internal phase.
- Let $\overline{\theta}$ be the proof obtained from $\theta$ by dropping all the formulas from $\Delta$.
- $\overline{\theta}$ is then a proof for $\vdash$.
- We define a truth semantics for $\mu MALL^\infty$ formulas and show that the proof system is sound with respect to it. Contradiction.
Theorem

The pre-proof obtained by the cut elimination algorithm is valid.

Proof:

Let $\pi \vdash \Delta$ be the pre-proof obtained from $\pi \vdash \Delta$ by cut elimination. Suppose that a branch $b$ of $\pi \vdash \Delta$ is not valid. Let $\theta$ be the sub-derivation of $\pi \vdash \Delta$ explored by the reduction that produces $b$. Fact: Threads of $\theta$ are the threads of $b$, together with threads starting from cut formulas. The validity of $\theta$ cannot rely on the threads of $b$. $\theta_{\mu}$ is $\theta$ where we replace in $\Delta$ any $\nu$ by a $\mu$ and any $\bot, 0$ by $\top, 1$. Show that formulas containing only $\mu, \bot, 0$ and MALL connectives are false. $\theta_{\mu}$ proves a false sequent which contradicts soundness.
Theorem
The pre-proof obtained by the cut elimination algorithm is valid.

Proof: Let $\pi^\ast$ be the pre-proof obtained from $\pi \vdash \Delta$ by cut elimination. Suppose that a branch $b$ of $\pi^\ast$ is not valid.

- Let $\theta$ be the sub-derivation of $\pi$ explored by the reduction that produces $b$.
- Fact: Threads of $\theta$ are the threads of $b$, together with threads starting from cut formulas.
- The validity of $\theta$ cannot rely on the threads of $b$.
- $\theta^\mu$ is $\theta$ where we replace in $\Delta$ any $\nu$ by a $\mu$ and any $1, \top$ by $\bot, 0$.
- Show that formulas containing only $\mu, \bot, 0$ and MALL connectives are false.
- $\theta^\mu$ proves a false sequent which contradicts soundness.