Logiques à points fixes, preuves infinies & fils rebondissants

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Based on joint works with D. Baelde, A. Doumane and G. Jaber

Logics with least and greatest fixed points

- Model inductive and coinductive data structures and reasoning.
- Their proof theory is under-developed and not well-understood.
- Infinite proofs in CS:
  - **Verification device**. Complete deduction sytem giving algorithms for checking validity (Tableaux, sequent calculi)
  - **Completeness arguments**. Intermediate objects between syntax and semantics for modal μ-calculus (Kozen, Kaivola, Walukiewicz):

μ-calulus formula $\rightarrow$ Circular proof $\rightarrow$ Finite axiomatization

... but rarely as proof–program objects in themselves

Structural proof theory of least and greatest fixed points

- *Sequent calculi*: MALL *with fixed points*
  - μMALL: well-founded proofs.Rules for induction, correctness is local, cut-elim and focalization but no subformula property.
  - μMALL$^\infty$: non-well-founded proofs.Simple inference rules for fixed points, global correctness criterion, cut-elimination with subformula property and focalization. Circular fragment.

Structural proof theory of least and greatest fixed points

- *Sequent calculi*: MALL *with fixed points*
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- Aim of the talk: **Infinite proof as proof–program objects**
  - Develop such a proof-theoretical study, from a Curry-Howard perspective (focalization and cut-elimination);
  - How remove constraints on the use of cuts in infinitary proofs, *ie,* allow cuts in cycles?

Outline

Logiques à points fixes	μMALL
Preuves infinies	μMALL$^\infty$
Fils rebondissants	From straight threads to bouncing ones.

Back to Euclid's *Elements* (Book VII)

PROPOSITION 31

Any composite number is measured by some prime number.

Let A be a composite number;

I say that A is measured by some prime number.

For, since A is composite,

some number will measure it.

Let a number measure it, and let it be B.

Now, if B is prime, what was enjoined will have been done.

But if it is composite, some number will measure it.

Let a number measure it, and let it be C.

Then, since C measures B,

and B measures A,

therefore C also measures A.

And, if C is prime, what was enjoined will have been done.

But if it is composite, some number will measure it.

Thus, if the investigation be continued in this way, some prime number will be found which will measure the number before it, which will also measure A.

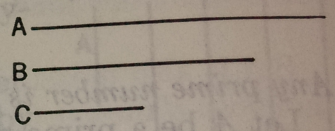
For, if it is not found, an infinite series of numbers will measure the number A, each of which is less than the other:

which is impossible in numbers.

Therefore some prime number will be found which will measure the one before it, which will also measure A.

Therefore any composite number is measured by some prime number.

Q. E. D.



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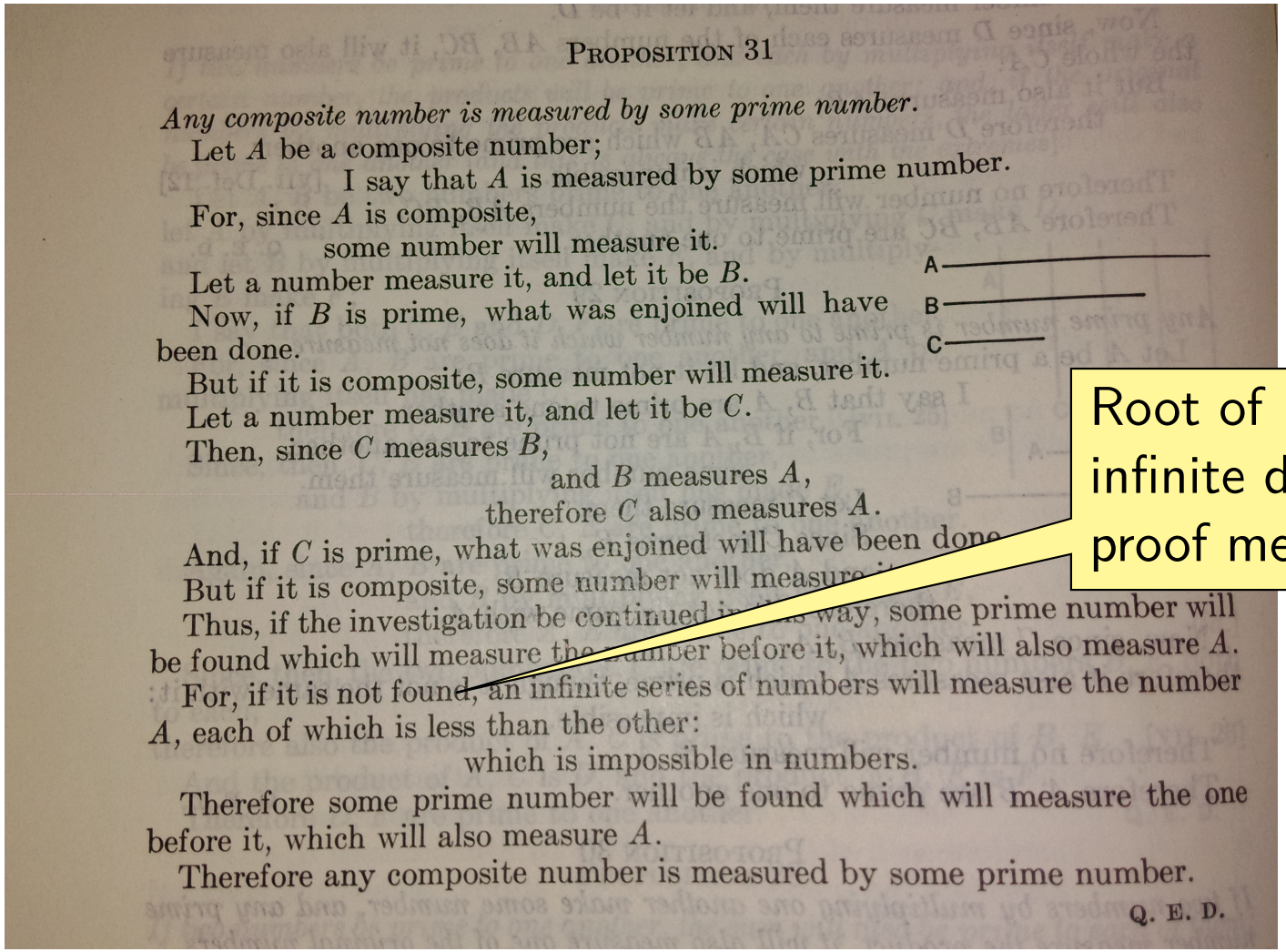

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Q. E. D.

Root of Fermat's infinite descent proof method.

Irrationality of $\sqrt{2}$ in sequent calculus

$$
\dfrac{
\dfrac{}{0 < x_0, x_0^2 = 2x_1^2 \vdash}
\qquad
\dfrac{
\dfrac{}{0 < x_1, x_1^2 = 2x_2^2 \vdash}
\qquad
\dfrac{
\dfrac{}{0 < x_2, x_2^2 = 2x_3^2 \vdash}
\qquad
\dfrac{
\dfrac{\cdots}{0 < x_3, x_4^2 = 2x_4^2 \vdash}
}{\underline{x_3 < x_2}, 0 < x_3, 4x_4^2 = 2x_3^2 \vdash}
}{0 < x_2, x_2^2 = 2x_3^2 \vdash}
}{\underline{x_2 < x_1}, 0 < x_2, 4x_3^2 = 2x_2^2 \vdash}
}{0 < x_1, x_1^2 = 2x_2^2 \vdash}
}{\underline{x_1 < x_0}, 0 < x_1, 4x_2^2 = 2x_1^2 \vdash}
}{0 < x_0, x_0^2 = 2x_1^2 \vdash}
$$

$$
\begin{array}{c}
0 < x_0, x_0^2 = 2x_1^2 \vdash \\
0 < x_1 < x_0 \wedge \\
\exists x_2 . x_0 = 2x_2
\end{array}
$$

$$
\begin{array}{c}
0 < x_1, x_1^2 = 2x_2^2 \vdash \\
0 < x_2 < x_1 \wedge \\
\exists x_3 . x_1 = 2x_3
\end{array}
$$

$$
\begin{array}{c}
0 < x_2, x_2^2 = 2x_3^2 \vdash \\
0 < x_3 < x_2 \wedge \\
\exists x_4 . x_2 = 2x_4
\end{array}
$$

$$
\begin{array}{c}
\cdots \\
\hline
0 < x_3, x_4^2 = 2x_4^2 \vdash \\
\hline
\underline{x_3 < x_2}, 0 < x_3, 4x_4^2 = 2x_3^2 \vdash
\end{array}
$$

Irrationality of $\sqrt{2}$ in sequent calculus

$$
\dfrac{
\dfrac{}{0 < x_0, x_0^2 = 2x_1^2 \vdash 0 < x_1 < x_0 \wedge \exists x_2 . x_0 = 2x_2}
\qquad
\dfrac{
\dfrac{}{0 < x_1, x_1^2 = 2x_2^2 \vdash 0 < x_2 < x_1 \wedge \exists x_3 . x_1 = 2x_3}
\qquad
\dfrac{
\dfrac{}{0 < x_2, x_2^2 = 2x_3^2 \vdash 0 < x_3 < x_2 \wedge \exists x_4 . x_2 = 2x_4}
\qquad
\dfrac{\dfrac{\cdots}{0 < x_3, x_4^2 = 2x_4^2 \vdash}}{\underline{x_3 < x_2}, 0 < x_3, 4x_4^2 = 2x_3^2 \vdash}
}{0 < x_2, x_2^2 = 2x_3^2 \vdash}
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}{0 < x_0, x_0^2 = 2x_1^2 \vdash}
$$

A natural question:

Is infinite descent equivalent to proof by induction?

μMALL: MALL **with Least and Greatest Fixed Points**

μMALL Formulas and Sequent calculus

(Baelde & Miller 2007, Baelde 2012)

μMALL formulas

$$
\begin{array}{rcll}
F & ::= & \top \mid \bot \mid F \parr F \mid F \& F & \\
 & & \mid 0 \mid 1 \mid F \otimes F \mid F \oplus F & \text{MALL } \textbf{formulas} \\
 & & \mid X \mid \mu X.F \mid \nu X.F & \textbf{least and greatest fixed point}
\end{array}
$$

- μ and ν are dual. Ex: $(\nu X.X \otimes X)^{\perp} = \mu X.X \parr X$.
- Data types encodings:

$$
\begin{array}{rcl}
\text{Nat} & := & \mu X.1 \oplus X \\
\text{List}(A) & := & \mu X.1 \oplus (A \otimes X) \\
\text{Stream}(A) & := & \nu X.A \otimes X
\end{array}
$$

μMALL Sequent Calculus $\Rightarrow$ ***See following slides***

MALL inference rules together with inference rules for μ and ν

Knaster-Tarski fixed point theorem

Knaster-Tarski theorem

Let F be a monotonic function over a complete lattice.

$$F(\mu F) \leq \mu F \qquad F(X) \leq X \Rightarrow \mu F \leq X$$

$$\nu F \leq F(\nu F) \qquad X \leq F(X) \Rightarrow X \leq \nu F$$

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$$\nu F \leq F(\nu F) \qquad X \leq F(X) \Rightarrow X \leq \nu F$$

Let F be a formula where X occurs only positively.

This gives right and left rules for μ:

$$\frac{}{F[\mu X.F] \vdash \mu X.F}\ (\mu_r) \qquad \frac{F[S] \vdash S}{\mu X.F \vdash S}\ (\mu_l)$$

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$$\frac{H \vdash F[\mu X.F/X]}{H \vdash \mu X.F}\ (\mu_r) \qquad \frac{F[S/X] \vdash S}{\mu X.F \vdash S}\ (\mu_l)$$

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This gives right and left rules for μ:

$$\frac{\Gamma \vdash \Delta, F[\mu X.F]}{\Gamma \vdash \Delta, \mu X.F}\ (\mu_r) \qquad \frac{S, \Gamma \vdash \Delta \quad F[S] \vdash S}{\mu X.F, \Gamma \vdash \Delta}\ (\mu_l)$$

Knaster-Tarski fixed point theorem

Knaster-Tarski theorem

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$$\nu F \leq F(\nu F) \qquad X \leq F(X) \Rightarrow X \leq \nu F$$

Let F be a formula where X occurs only positively.

This gives right and left rules for ν:

$$\frac{F[\nu X.F], \Gamma \vdash \Delta}{\nu X.F, \Gamma \vdash \Delta}\ (\nu_{\mathrm{l}}) \qquad \frac{\Gamma \vdash \Delta, S \quad S \vdash F[S]}{\Gamma \vdash \Delta, \nu X.F}\ (\nu_{\mathrm{r}})$$

Knaster-Tarski fixed point theorem

Knaster-Tarski theorem

Let F be a monotonic function over a complete lattice.

$$F(\mu F) \leq \mu F \qquad F(X) \leq X \Rightarrow \mu F \leq X$$

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Let F be a formula where X occurs only positively.

One-sided rules for μ and ν:

$$\frac{\Gamma \vdash \Delta, F[\mu X.F]}{\Gamma \vdash \Delta, \mu X.F}\ (\mu_r) \qquad \frac{\Gamma \vdash \Delta, S \quad S \vdash F[S]}{\Gamma \vdash \Delta, \nu X.F}\ (\nu_r)$$

μMALL$^\infty$: infinite and circular proofs

Infinitary Sequent Calculus

Consider your favourite logic $\mathscr{L}$ & add fixed points as in μMALL:

Pre-proofs are the trees **coinductively** generated by:

- $\mathscr{L}$ inference rules
- μ, ν-rules:

$$\frac{\vdash \Gamma, F[\mu X.F/X]}{\vdash \Gamma, \mu X.F}\ (\mu) \qquad \frac{\vdash \Gamma, F[\nu X.F/X]}{\vdash \Gamma, \nu X.F}\ (\nu)$$

Fischer-Ladner Subformulas: induced by fixed-point unrolling:
$F[\sigma X.F/X] \in \mathrm{FL}(\sigma X.F)$, $\sigma \in \{\mu, \nu\}$.

Circular (pre-)proofs: the regular fragment of infinite (pre-)proofs, ie finitely many sub-(pre)proofs.

μMALL$^\infty$ Infinitary Sequent Calculus

Consider your favourite logic $\mathscr{L}$ & add fixed points as in μMALL:

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MALLinference rules

$$\frac{\vdash \Gamma, F[\mu X.F/X]}{\vdash \Gamma, \mu X.F}\ (\mu) \qquad \frac{\vdash \Gamma, F[\nu X.F/X]}{\vdash \Gamma, \nu X.F}\ (\nu)$$

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Pre-proofs are unsound!! ***Need for a validity condition***

$$\dfrac{\dfrac{\dfrac{\vdots}{\vdash \mu X.X}\ (\mu)}{\vdash \mu X.X}\ (\mu) \qquad \dfrac{\dfrac{\vdots}{\vdash \nu X.X, F}\ (\nu)}{\vdash \nu X.X, F}\ (\nu)}{\vdash F}\ (\mathrm{Cut})$$

Propositional classical μ-calculus

Syntax

$$F ::= a \mid a^{\perp} \mid \top \mid \bot \mid F_1 \vee F_2 \mid F_1 \wedge F_2 \mid \mu X.F \mid \nu X.F \mid X$$

Truth semantics

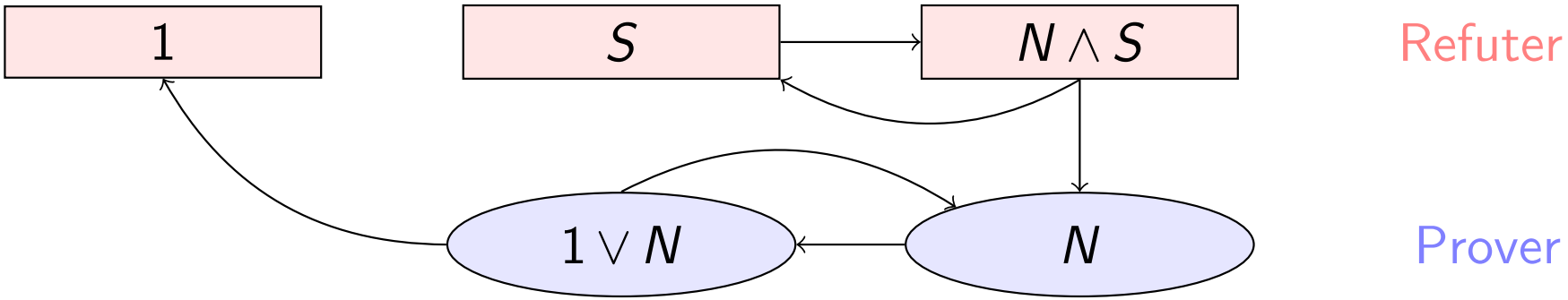
$$[\![\mu X.F]\!]^{\mathscr{E}} := \mathrm{lfp}\big(b \mapsto [\![F]\!]^{\mathscr{E}+(X\mapsto b)}\big)$$

$$[\![\nu X.F]\!]^{\mathscr{E}} := \mathrm{gfp}\big(b \mapsto [\![F]\!]^{\mathscr{E}+(X\mapsto b)}\big)$$

Examples

$$\begin{array}{rcl} [\![\mu X.X]\!]^{\mathscr{E}} & = & 0 \\ [\![\mu X.a \wedge X]\!]^{\mathscr{E}} & = & 0 \end{array} \qquad \begin{array}{rcl} [\![\nu X.X]\!]^{\mathscr{E}} & = & 1 \\ [\![\nu X.a \wedge X]\!]^{\mathscr{E}} & = & [\![a]\!]^{\mathscr{E}} \end{array}$$

Parity games



$$S := \nu X.\ (\mu Y.\ 1 \vee Y) \wedge X \qquad N := \mu Y.\ 1 \vee Y$$

Prover wins if:

- either the play is finite and ends in a Refuter node,
- or among formulas encountered infinitely often, the minimum wrt. subformula ordering is a greatest fixed point:

$$\min(\inf(p)) = \nu$$

Here, Prover wins iff 1 is reached or S is visited infinitely often.

Validity condition (1)

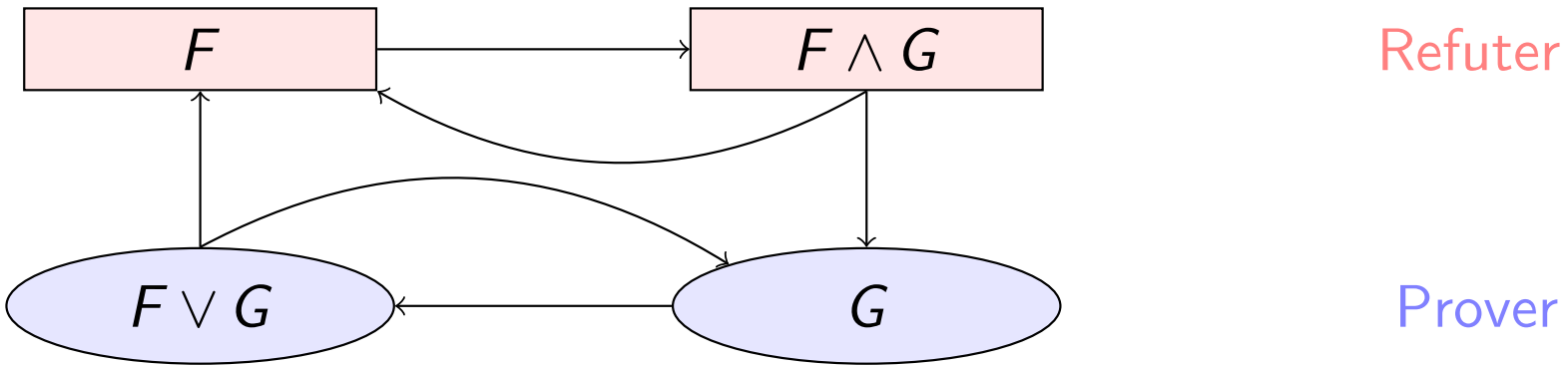
$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash 1}\,(1)}{\vdash 1 \vee N}\,(\vee_l)}{\vdash N}\,(\mu) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash 1}\,(1)}{\vdash 1 \vee N}\,(\vee_l)}{\vdash N}\,(\mu)}{\vdash 1 \vee N}\,(\vee_r)}{\vdash N}\,(\mu) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash 1}\,(1)}{\vdash 1 \vee N}\,(\vee_l)}{\vdash N}\,(\mu)}{\vdash N}\,(\mu,\vee_r)^2 \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash 1}\,(1)}{\vdash 1 \vee N}\,(\vee_l)}{\vdash N}\,(\mu)}{\vdash N}\,(\mu,\vee_r)^3 \qquad \dfrac{\vdots}{\vdash S}}{\vdash N \wedge S}\,(\wedge)}{\vdash S}\,(v)}{\vdash N \wedge S}\,(\wedge)}{\vdash S}\,(v)}{\vdash N \wedge S}\,(\wedge)}{\vdash S}\,(v)
$$

Validity condition

Every infinite branch contains a *thread* t such that $\min(\inf(t)) = v$.

Validity condition (2)

$$
\begin{aligned}
F &:= \nu X.\ X \wedge (\mu Y.\ X \vee Y) \\
G &:= \mu Y.\ F \vee Y
\end{aligned}
$$



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$$\min(\inf(p)) = \nu$$

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$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(\heartsuit)}{\vdash F, G} \quad \dfrac{(\spadesuit)}{\vdash G}}{\vdash F \wedge G, G}\,(\wedge)}{(\heartsuit) \vdash F, G}\,(\nu)}{\vdash F \vee G}\,(\vee)}{(\spadesuit) \vdash G}\,(\mu)
$$

$$
\dfrac{\dfrac{(\diamondsuit)}{\vdash F} \quad (\spadesuit) \vdash G}{\vdash F \wedge G}\,(\wedge)
$$

$$
\dfrac{\vdash F \wedge G}{(\diamondsuit) \vdash F}\,(\nu)
$$

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Every infinite branch contains a *thread* t such that $\min(\inf(t)) = \nu$.

Validity condition (2)

$$
\dfrac{\dfrac{(\Diamond)}{\vdash F} \qquad \dfrac{\dfrac{\dfrac{\dfrac{(\heartsuit)}{\vdash F, G} \quad \dfrac{(\spadesuit)}{\vdash G}}{\vdash F \wedge G, G}\,(\wedge)}{(\heartsuit) \vdash F, G}\,(\nu)}{\vdash F \vee G}\,(\vee)}{(\spadesuit) \vdash G}\,(\mu)}{\dfrac{\vdash F \wedge G}{(\Diamond) \vdash F}\,(\nu)}\,(\wedge)
$$

Validity condition

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Validity condition (2)

$$
\dfrac{\dfrac{(\heartsuit)}{\vdash F, G} \quad \dfrac{(\spadesuit)}{\vdash G}}{\vdash F \wedge G, G}\,(\wedge)
$$

$$
\dfrac{\vdash F \wedge G, G}{(\heartsuit) \vdash F, G}\,(\nu)
$$

$$
\dfrac{(\heartsuit) \vdash F, G}{\vdash F \vee G}\,(\vee)
$$

$$
\dfrac{\dfrac{(\diamondsuit)}{\vdash F} \quad \dfrac{\vdash F \vee G}{(\spadesuit) \vdash G}\,(\mu)}{\vdash F \wedge G}\,(\wedge)
$$

$$
\dfrac{\vdash F \wedge G}{(\diamondsuit) \vdash F}\,(\nu)
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$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(\heartsuit)}{\vdash F, G} \quad \dfrac{\dfrac{(\spadesuit)}{\vdash G}}{\vdash G, G}\,(\text{weak})}{\vdash F \wedge G, G}\,(\wedge)}{(\heartsuit) \vdash F, G}\,(\nu)}{\vdash F \vee G}\,(\vee)}{(\spadesuit) \vdash G}\,(\mu) \quad \dfrac{(\diamondsuit)}{\vdash F}}{\vdash F \wedge G}\,(\wedge)}{(\diamondsuit) \vdash F}\,(\nu)
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$$

Need to distinguish formula occurrences!

Validity condition

Every infinite branch contains a *thread* t such that $\min(\inf(t)) = \nu$.

Validity condition (3)

- A ***thread*** is a sequence of formulas that traces the evolution of a formula along an infinite branch.
- Cut spawns two new threads, on cut formulas: $\dfrac{\vdash \Gamma, F \quad \vdash F^{\perp}, \Delta}{\vdash \Gamma, \Delta}$
- A thread is ***valid*** if its outermost formula is a ν-formula.
- A pre-proof is ***valid*** if every branch contains a valid thread.
- A valid pre-proof is called ***proof***.
- Validity of regular pre-proofs ***is decidable***. Details

Example of a μMALL$^\omega$ proof

- **Inductive and coinductive definitions**

$$\mathsf{N} = \mu X.1 \oplus X \qquad\qquad \mathsf{S} = \nu X.N \otimes X$$

- **Proofs-programs over these data types**

$$\begin{array}{rcll} double(n) & = & 0 & \text{if } n = 0 \\ & = & succ(succ(double(m))) & \text{if } n = succ(m) \end{array}$$

Example of a μMALL$^\omega$ proof

- **Inductive and coinductive definitions**

$$\mathsf{N} = \mu X.1 \oplus X \qquad\qquad \mathsf{S} = \nu X.N \otimes X$$

- **Proofs-programs over these data types**

$$\begin{array}{rcll} double(n) & = & 0 & \text{if } n = 0 \\ & = & succ(succ(double(m))) & \text{if } n = succ(m) \end{array}$$

$$\Pi_{double} = \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{1 \vdash 1}\,(\mathrm{Ax})}{1 \vdash 1 \oplus N}\,(\oplus_1)}{1 \vdash N}\,(\mu_r) \qquad \dfrac{\dfrac{\dfrac{N \vdash N}{N \vdash 1 \oplus N}\,(\oplus_2)}{N \vdash N}\,(\mu_r)}{N \vdash 1 \oplus N}\,(\oplus_2)}{N \vdash N}\,(\mu_r)}{1 \oplus N \vdash N}\,(\oplus_1)}{N \vdash N}\,(\mu_l)$$

Circular & finitary proofs

From finitary to circular proofs

Theorem

Finitary proofs can be transformed to (valid) infinitary proofs.

The key translation step is the following:

$$\dfrac{\dfrac{\pi_1}{\vdash \Gamma, S} \qquad \dfrac{\pi_2}{\vdash S^\perp, F[S]}}{\vdash \Gamma, \nu X.F}\,(\nu) \quad \longmapsto \quad \dfrac{\dfrac{[\pi_1]}{\vdash \Gamma, S} \qquad \dfrac{\dfrac{\dfrac{[\pi_2]}{\vdash S^\perp, F[S]} \qquad \dfrac{\vdash S^\perp, \nu X.F}{\vdash F[S]^\perp, F[\nu X.F]}\,(\mathrm{r_F})}{\vdash S^\perp, F[\nu X.F]}\,(\mathrm{Cut})}{\vdash S^\perp, \nu X.F}\,(\nu)}{\vdash \Gamma, \nu X.F}\,(\mathrm{Cut})$$

From circular to finitary proofs

Brotherston-Simpson's conjecture recently (dis)proved (FOSSACS 2017, LICS 2017). No general result for μMALL$^\omega$.

μMALL$^\infty$ Cut elimination and Focalization

Before:

Additive Cut elimination only (Santocanale & Fortier):

Lattice logic [μ-calculus over ALL ($\oplus$, &, $\mathbf{0}$, $\top$).]

- Sequents are of the form $F \vdash G$.
- Infinite branches have exactly one left and one right thread.

Now:

Theorem [BDS 2016]

μMALL$^\infty$ enjoys cut-elimination and focalization.

Proofs in two steps:

- cut-elimination/focalization procedures are productive;
- the resulting pre-proof is valid.

Relaxing validity:
From straight threads to bouncing threads

Invalid but productive proofs

Motto of the previous part:

Cut-elimination is productive on valid pre-proofs and always produces valid pre-proofs.

Invalid but productive proofs

Motto of the previous part:

Cut-elimination is productive on valid pre-proofs and always produces valid pre-proofs.

However, some productive computations are invalid, e.g. convoluted identity over $S = \nu X.\ X$:

$$
\begin{array}{rl}
 & \dfrac{}{S \vdash S}\ (\mathrm{ax}) \\[2ex]
 & \dfrac{S \vdash S}{S \vdash S}\ (\nu_R) \\[2ex]
S \vdash S & \\[2ex]
\hline
 & S \vdash S\ (\mathrm{cut}) \\[2ex]
\hline
 & S \vdash S\ (\star)\ (\nu_L)
\end{array}
$$

$$
\begin{aligned}
f(\_ :: s) &= g(f(s)) \\
g(x) &= \_ :: x
\end{aligned}
$$

Allow for a more relaxed validity criterion?

Towards bouncing threads

Notations

- $s \sqsubseteq s'$ when the sequent occurrence s' is a premise of s;
- $F \sqsubseteq F'$ when F' is a sub-occurrence of F.

Definition

A ***b-thread*** is an infinite sequence $(F_i, d_i, s_i)_{i \in \omega}$ such that, for all i, $F_i \in s_i$ and one of the following holds:

- $d_i = d_{i+1} = \uparrow$, $s_i \sqsubseteq s_{i+1}$ and $F_i \sqsubseteq F_{i+1}$.
- $d_i = d_{i+1} = \downarrow$, $s_{i+1} \sqsubseteq s_i$ and $F_{i+1} \sqsubseteq F_i$.
- $d_i = \uparrow$, $d_{i+1} = \downarrow$, $s_i = s_{i+1} = \{F_i, F_{i+1}\}$ is an axiom.
- $d_i = \downarrow$, $d_{i+1} = \uparrow$, s_i and s_{i+1} premises of cut on $F_i = F_{i+1}^{\perp}$.

Example: b-thread

$$\dfrac{\dfrac{\dfrac{}{1 \vdash N}\,(0) \qquad \dfrac{\dfrac{}{N \vdash N}\,(ax)}{N \vdash N}\,(succ)}{N \vdash N} \qquad \dfrac{\dfrac{}{1 \vdash B}\,(tt) \qquad \dfrac{\dfrac{}{1 \vdash B}\,(ff) \qquad \dfrac{\vdots}{N \vdash B}}{N \vdash B}}{N \vdash B}}{N \vdash B}\,(cut)$$

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This thread gets erased/disconnected during cut elimination!

Persistent b-threads

Let $\Sigma = \{l, r, i, l^*, r^*, i^*\}$, and let M be the set of finite or infinite words over it, plus 0, quotiented by the following equations:

$$0 \cdot x = x \cdot 0 = 0$$

$$u^* \cdot u = 1 \quad \text{and} \quad v^* \cdot u = 0 \quad \text{for all} \quad u \neq v \in \{l, r, i\}$$

Definition

The ***weight*** of a thread is $w(t) = \Pi_{i \in \omega} a_i$:

- $a_i = l/r/i$ when $F_i = F_\alpha$ and $F_{i+1} = F_{\alpha a_i}$;
- $a_i = l^*/r^*/i^*$ symmetrically;
- $a_i = 1$ otherwise.

The thread is ***persistent*** if $w(t) \neq_M 0$.

Reminiscent from Geometry of Interaction, well-bracketed threads...

Visible part

Definition

A thread t admits a ***visible part*** v when:

$w(t) =_M v$ and v is a (possibly infinite) word over $\{l, r, i\}$.

Definition

When the weight admits a visible part $v = (v_j)_{j \in \lambda}$ $(\lambda \leq \omega)$ we say that the thread *visibly explores* $(G_j)_{j \in \lambda}$ defined by

$G_0 = F_0$ (the first formula in the thread) and,

G_{i+1} unique descendent of $G_i = F_\alpha$ with address αv_j.

Example: visible part

$$\dfrac{\dfrac{\dfrac{}{1 \vdash N}\,(0) \qquad \dfrac{\dfrac{}{N \vdash N}\,(ax)}{N \vdash N}\,(succ)}{N \vdash N} \qquad \dfrac{\dfrac{}{1 \vdash B}\,(tt) \qquad \dfrac{\dfrac{}{1 \vdash B}\,(ff) \qquad \dfrac{\vdots}{N \vdash B}}{N \vdash B}}{N \vdash B}}{N \vdash B}\,(cut)$$

$$\alpha\,\alpha_i\,\alpha_{ir}\,\alpha_{ir}\,\alpha_i\,\alpha\,\alpha_i\,\alpha_{ir}\,\alpha_{iri}\,\alpha_{iril}$$

Example: visible part

$$
\begin{array}{rl}
 & \dfrac{\dfrac{\dfrac{\dfrac{}{S \vdash S}\,(\mathrm{ax})}{S \vdash S}\,(\nu_R)}{S \vdash S}\,(\nu_L) \qquad \dfrac{(\star)}{S \vdash S}}{S \vdash S}\,(\mathrm{cut}) \\[2ex]
 & \dfrac{S \vdash S}{S \vdash S\ (\star)}\,(\nu_L)
\end{array}
$$

$$\alpha\alpha_i(\alpha_i\alpha\alpha_i)^\omega$$

Bouncing Validity

A b-thread is ***valid*** if:

- its weight admits an infinite visible part,
- the minimum of the visibly explored formulas is a ν formula,
- and it never bounces on its starting formula.

Bouncing Validity

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- and it never bounces on its starting formula.

An infinite branch is ***valid*** if:

- there is a valid b-thread starting from one of its sequents
- which visits the branch infinitely often.

Definition (Proof)

A ***proof*** is a pre-proof in which every infinite branch is valid.

Cut elimination with bouncing threads

Theorem

If every infinite branch of a $\mu\mathrm{MLL}^\infty$ pre-proof π is supported by a valid bouncing thread, then the cut-elimination of π is productive..

Idea: perform *just enough cut elimination*
to reduce **bouncing-valid** derivation into **straight-valid** one.

Theorem

Bouncing-valid derivation converge to straight-valid ones.

Extension to Additives (WIP): criterion for every slices of the proof.

Cut elimination with bouncing threads

Theorem

Bouncing-valid derivation converge to straight-valid ones.

Proof.

Consider head reductions of selected cuts:

bouncer = cut on which a validating thread bounces.

By contradiction, it is *productive*:

- reducing bouncers corresponds to weight simplifications;
- eventually, the visible part of some thread must be produced.

The result is *straight-valid*:

- a resulting infinite branch induces an original infinite branch;
- valid bouncing threads result in valid straight threads.

□

Conclusion

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- To sum up:
  - Fixed point logics extending MALL with finite or infinite proofs;
  - Syntactic cut elimination;
  - Focalization;
  - Relaxing threads in the multiplicative case.
- Future work:
  - Go beyond Linear Logic and handle structural rules;
  - Equivalence of circular fragment of $\mu MALL^{\infty}$ and $\mu MALL$: Translate infinitrary proofs to finitary ones;
  - Same question as above by preserving the computational content.
  - Other styles: Natural deduction, circular λ-calculus

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Thank you for your attention!

Appendix

Decidability of the validity condition

Parity automata

Definition

A *parity automaton* is a finite state word automaton,
whose states are ordered and given a parity bit ν/μ,
which accepts runs $(q_i)_{i\in\omega}$ such that $\min(\inf((q_i)_i))$ has parity ν.

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Remarks

- States are usually given a color in $\mathbb{N}$, equivalently.
- Only co-accessible states need to be ordered.

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Remarks

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- Only co-accessible states need to be ordered.

Closure properties

Parity automata can be determinized, complemented, and intersected, thus their inclusion is decidable.

Decidability of the validity condition

To slide on threads

Theorem

The validity of circular pre-proofs is decidable.

Proof.

Consider a pre-proof Π i.e. a graph with nodes $s_i = (F_i^j)_{j \in [1;n_i]}$.

- Let $\mathscr{A}_B$ be the **branch automaton** with states s_i, transitions $s_i \to^k s_j$ when s_j is the k-th premise of s_i, and which accepts all runs.
- Let $\mathscr{A}_T$ be the **thread automaton** with states F_i^j or $\bot_i$, transitions $F_i^j \to^k F_p^q$ when
  - $s_i \to^k s_p$ and
  - $F_i^j = \bot$ or the ancestor of F_p^q

  and acceptance based on the subformula ordering.

We have that Π is valid iff $\mathscr{A}_B \subseteq \mathscr{A}_T$. $\square$

MALL **proof system**

LL Sequent Calculus

$$F ::= \quad a \mid F \otimes F \mid F \oplus F \mid 1 \mid 0 \mid !F$$
$$a^{\perp} \mid F \parr F \mid F \& F \mid \perp \mid \top \mid ?F$$

$$\frac{}{\vdash A, A^{\perp}}\ (\mathrm{Ax}) \qquad \frac{\vdash \Gamma, A \quad \vdash \Delta, A^{\perp}}{\vdash \Gamma, \Delta}\ (\mathrm{Cut})$$

$$\frac{}{\vdash 1}\ [1] \quad \frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\ [\otimes] \quad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\ [\parr] \quad \frac{\vdash \Gamma}{\vdash \Gamma, \perp}\ [\perp]$$

$$\frac{\vdash \Gamma, A_i}{\vdash \Gamma, A_1 \oplus A_2}\ [\oplus_i] \quad i \in \{1, 2\} \quad \frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \& B}\ [\&] \quad \frac{}{\vdash \Gamma, \top}\ [\top]$$

$$\frac{\vdash ?\Gamma, B}{\vdash ?\Gamma, !B}\ [!] \quad \frac{\vdash \Gamma, B}{\vdash \Gamma, ?B}\ [?] \quad \frac{\vdash \Gamma}{\vdash \Gamma, ?B}\ [?w] \quad \frac{\vdash \Gamma, ?B, ?B}{\vdash \Gamma, ?B}\ [?c]$$

MALL Sequent Calculus

To circular proofs

$$F ::= \quad a \mid F \otimes F \mid F \oplus F \mid 1 \mid 0 \qquad \text{positive}$$
$$a^{\perp} \mid F \parr F \mid F \& F \mid \perp \mid \top \qquad \text{negative}$$

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$$\frac{}{\vdash 1}\ [1] \quad \frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\ [\otimes] \quad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\ [\parr] \quad \frac{\vdash \Gamma}{\vdash \Gamma, \perp}\ [\perp]$$

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Focalization in μMALL$^\infty$

$\mu MALL^{\infty}$ Focalization

1/ Reversibility of negative sequents:

- Similar to MALL reversibility except that it cannot be treated with local rule permutations as shown by the following example.

$$
\begin{array}{cl}
 & \dfrac{\dfrac{\dfrac{\vdash F, P\parr Q \qquad \dfrac{\dfrac{\pi}{\vdash F, P, Q}}{\vdash F, P\parr Q}\,(\parr)}{\vdash F\&F, P\parr Q}\,(\&)}{\vdash (F\&F)\oplus 0, P\parr Q}\,(\oplus_1)}{\vdash F, P\parr Q}\,(\nu)
\end{array}
$$

$\mu MALL^{\infty}$ Focalization

1/ Reversibility of negative sequents:

- Similar to MALL reversibility except that it cannot be treated with local rule permutations as shown by the following example.

2/ Focalization of positive sequents:

- Positive trunks are finite trees (due to the polarization of fixed points formulas);
- The rest of the proof goes as for MALL.

3 & 4/ Productivity and validity of focalization process:

- Productivity of the focusing process is essentially direct from *MALL* case (guarded calls to the focalization process);
- Preservation of validity relies on an analysis of the kind of permutations involved in focalization: valid thread cannot be infinitely postponed.

Cut elimination in $\mu\mathsf{MALL}^\infty$

The purely additive case

Lattice logic

- Consider μ-calculus over additive linear logic ($\oplus$, &, $\mathbf{0}$, $\top$).
- Sequents are of the form $F \vdash G$.
- Infinite branches have exactly one left and one right thread.

Results [Santocanale 2002, Santocanale & Fortier 2013]

- Primitive recursive functions encoded as proofs.
- Restricted cut-elimination through semantics.
- Syntactic cut elimination.

Cut elimination strategy

The following strategy is used in the additive and MALL cases:

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1. Reduce bottom-most cuts first.
2. Perform *internal* chatter while necessary, e.g.

$$\dfrac{\dfrac{\vdash \Gamma, F[\mu X.F/X]}{\vdash \Gamma, \mu X.F}\,(\mu) \quad \dfrac{\vdash F^{\perp}[\nu X.F^{\perp}/X], \Delta}{\vdash \nu X.F^{\perp}, \Delta}\,(\nu)}{\vdash \Gamma, \Delta} \longrightarrow_{\mathsf{int}} \dfrac{\vdash \Gamma, F[\mu X.F/X] \quad \vdash F^{\perp}[\nu X.F^{\perp}/X], \Delta}{\vdash \Gamma, \Delta}$$

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3. Perform *external* rules, which produce cut-free result, e.g.

$$\dfrac{\dfrac{\vdash \Gamma, F, H}{\vdash \Gamma, F \oplus G, H}\,(\oplus) \quad \vdash H^{\perp}, \Delta}{\vdash \Gamma, F \oplus G, \Delta} \longrightarrow_{\text{ext}} \dfrac{\dfrac{\vdash \Gamma, F, H \quad \vdash H^{\perp}, \Delta}{\vdash \Gamma, F, \Delta}}{\vdash \Gamma, F \oplus G, \Delta}\,(\oplus)$$

4. Repeat.

Reviewing additive cut elimination $(F \vdash G)$

Lemma (Productivity)

Cut reductions are productive, i.e. $\rightarrow_{int}$ *terminates.*

Proof sketch.

Assuming the contrary, we have an infinite internal reduction. Consider the sub-tree whose sequents take part to that reduction.

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- It has a dual branch to the left, with a right μ-thread, thus a left ν-thread by validity.
- And so on...

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- It has a dual branch to the left, with a right μ-thread, thus a left ν-thread by validity.
- And so on...
- The limit, leftmost branch, still has a right μ-thread, thus a left ν-thread, contradicting its leftmostness.

$\square$

$\mu MALL\infty$ Cut elimination procedure

- **Strategy:** "push" the cuts away from the root.
- **Cut-Cut:**

$$\dfrac{\dfrac{\vdash \Gamma, F \qquad \vdash F^{\perp}, \Delta, G}{\vdash \Gamma, \Delta, G}\,(\mathrm{Cut}) \qquad \vdash G^{\perp}, \Sigma}{\vdash \Gamma, \Delta, \Sigma}\,(\mathrm{Cut})$$

$$\updownarrow$$

$$\dfrac{\vdash \Gamma, F \qquad \dfrac{\vdash F^{\perp}, \Delta, G \qquad \vdash G^{\perp}, \Sigma}{\vdash F^{\perp}, \Delta, \Sigma}\,(\mathrm{Cut})}{\vdash \Gamma, \Delta, \Sigma}\,(\mathrm{Cut})$$

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$$\downarrow$$

$$\dfrac{\vdash \Gamma, F \qquad \vdash F^{\perp}, \Delta, G \qquad \vdash G^{\perp}, \Sigma}{\vdash \Gamma, \Delta, \Sigma}\,(\text{mcut})$$

$\mu MALL\infty$ Cut elimination procedure - External operations

$$\dfrac{\dfrac{\vdash \Delta, F, G}{\vdash \Delta, F \parr G}\,(\parr) \qquad \dots}{\vdash \Sigma, F \parr G}\,(\text{mcut}) \quad \Rightarrow \quad \dfrac{\dfrac{\vdash \Delta, F, G \qquad \dots}{\vdash \Sigma, F, G}\,(\text{mcut})}{\vdash \Sigma, F \parr G}\,(\parr)$$

$$\dfrac{\dfrac{\vdash \Delta, F \qquad \vdash \Delta, G}{\vdash \Delta, F \& G}\,(\&) \qquad \dots}{\vdash \Sigma, F \& G}\,(\text{mcut}) \quad \Rightarrow \quad \dfrac{\dfrac{\vdash \Delta, F \qquad \dots}{\vdash \Sigma, F}\,(\text{mcut}) \qquad \dfrac{\vdash \Delta, G \qquad \dots}{\vdash \Sigma, G}\,(\text{mcut})}{\vdash \Sigma, F \& G}\,(\&)$$

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External operations are productive

$\mu MALL\infty$ Cut elimination procedure - Internal operations

$$\dfrac{\dots \quad \dfrac{\vdash \Delta, F_2 \qquad \vdash \Delta, F_1}{\vdash \Delta, F_2 \& F_1}\,(\&) \qquad \dfrac{\vdash \Gamma, F_i^{\perp}}{\vdash \Gamma, F_1^{\perp} \oplus F_2^{\perp}}\,(\oplus_i)}{\vdash \Sigma}\,(\text{mcut})$$

$$\Rightarrow \quad \dfrac{\dots \quad \vdash \Delta, F_i \qquad \vdash \Gamma, F_i^{\perp}}{\vdash \Sigma}\,(\text{mcut})$$

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$$\Rightarrow \quad \dfrac{\dots \quad \vdash \Delta, F[\mu X.F/X] \qquad \vdash \Gamma, F^{\perp}[\nu X.F^{\perp}/X]}{\vdash \Sigma}\,(\text{mcut})$$

Internal operations are not productive

$\mu MALL\infty$ Cut elimination algorithm

- Internal phase: Perform internal transformations while you cannot do anything else.
- External phase: Build a part of the output tree whenever you can.

$\mu MALL\infty$ Cut elimination algorithm

- Internal phase: Perform internal transformations while you cannot do anything else.
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- Repeat.

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Theorem

Internal phase always halts.

Theorem

The pre-proof obtained by the cut elimination algorithm is valid.

Cut elimination is productive

more details

Theorem

Internal phase always halts.

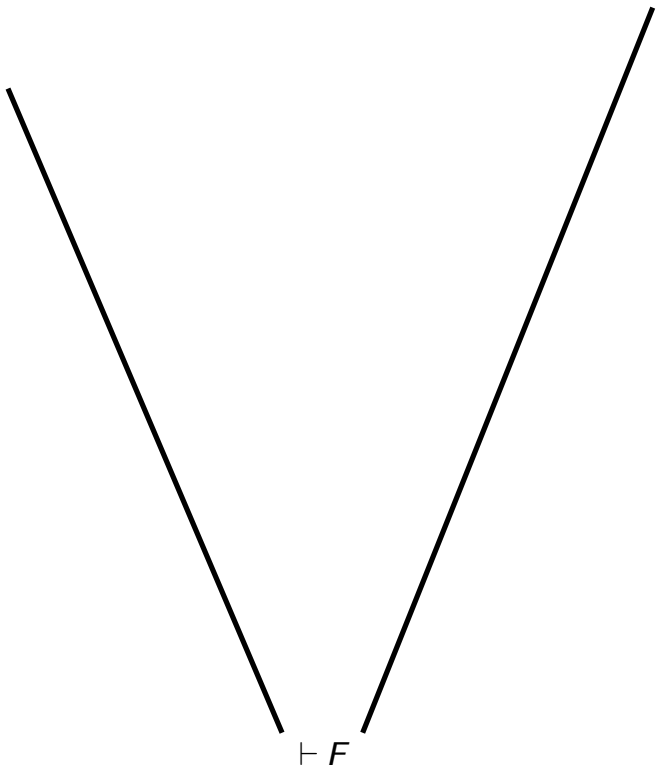
Cut elimination is productive

more details

Theorem

Internal phase always halts.

Proof by contradiction: Suppose that there is a proof of F for which the internal phase does not halt.



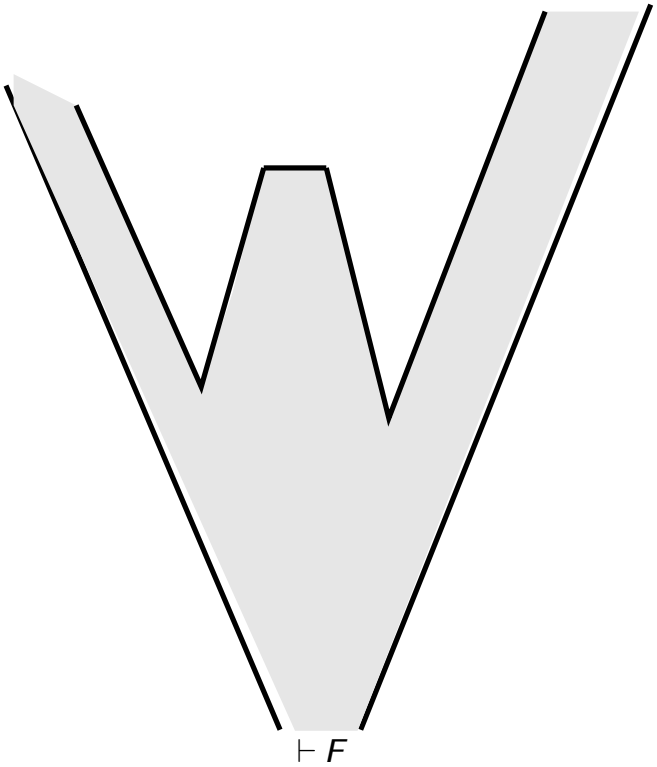
Cut elimination is productive

more details

Theorem

Internal phase always halts.

Proof by contradiction: Consider the trace of this divergent reduction.



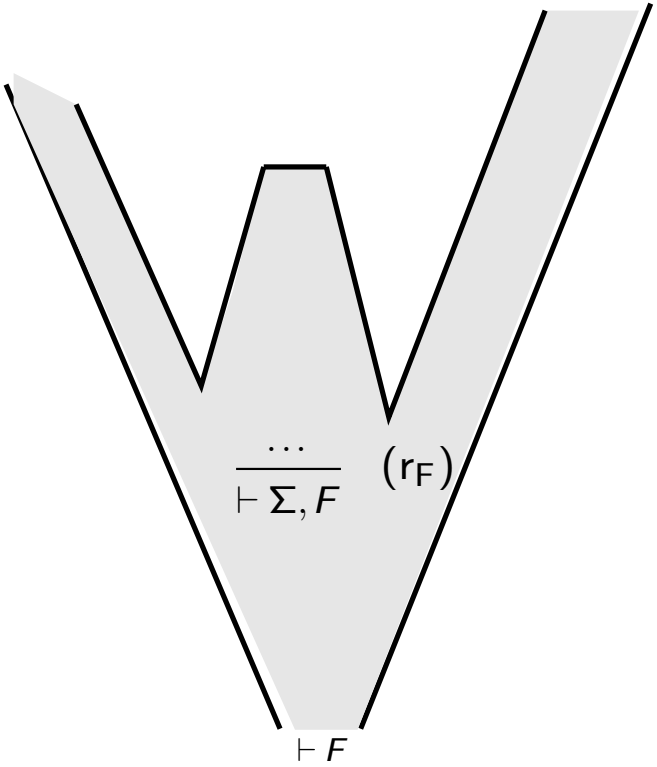
Cut elimination is productive

more details

Theorem

Internal phase always halts.

Proof by contradiction: No rule on F is applied in the trace, otherwise the internal phase would halt.



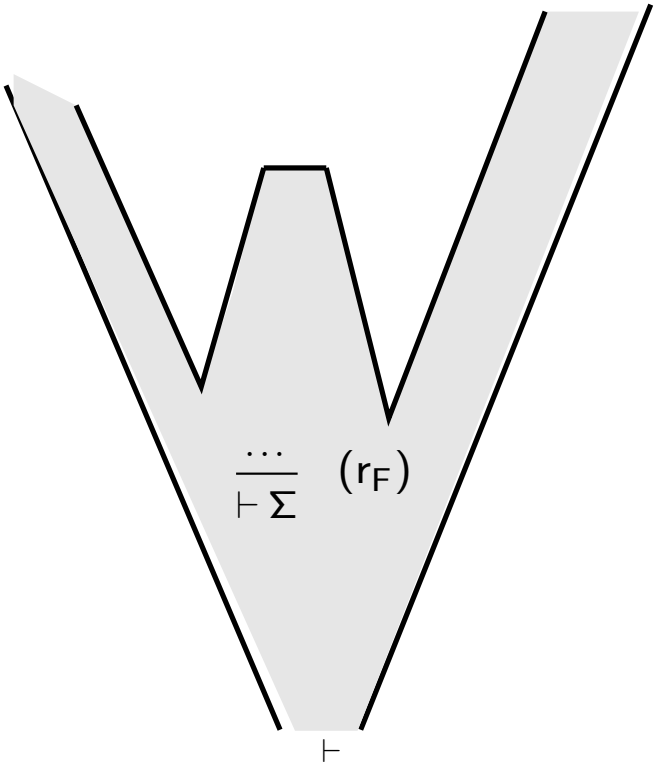
Cut elimination is productive

more details

Theorem

Internal phase always halts.

Proof by contradiction: We can eliminate the occurrences of F from the trace. This yields a "proof" of $\vdash$.



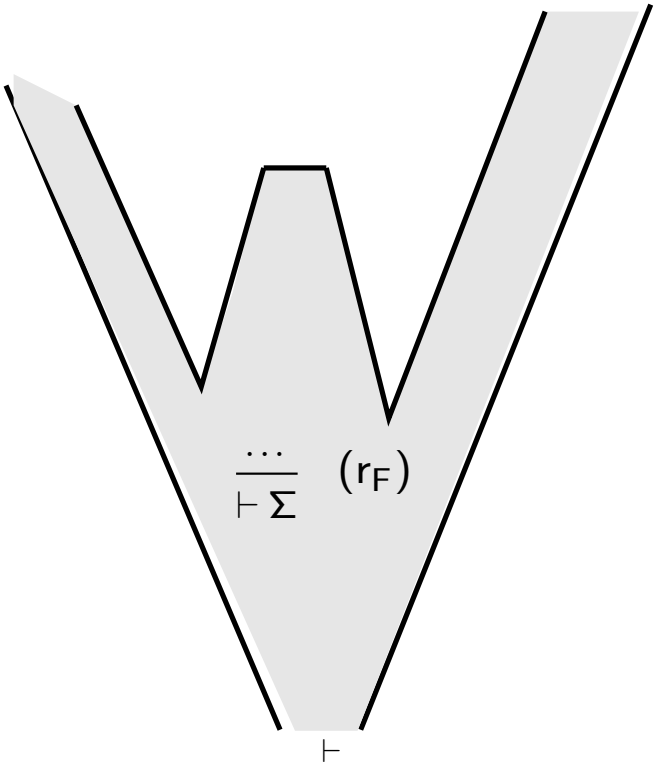
Cut elimination is productive

more details

Theorem

Internal phase always halts.

Proof by contradiction: We show that the proof system is sound. Contradiction.



Cut elimination is productive (Details)

back to the sketch

Theorem

Internal phase always halts.

Cut elimination is productive (Details)

back to the sketch

Theorem

Internal phase always halts.

Proof: Suppose that the internal phase diverges for a proof $\pi \vdash \Delta$.

- Let θ be the sub-derivation of π explored by the reduction.
- No rule is applied to a formula of Δ in θ,
  as this would contradict the divergence of internal phase.
- Let $\overline{\theta}$ be the proof obtained from θ by dropping all the formulas from Δ.
- $\overline{\theta}$ is then a proof for $\vdash$.
- We define a truth semantics for $\mu MALL^{\infty}$ formulas and show that the proof system is sound with respect to it.
  Contradiction.

Cut elimination produces a proof

back to the sketch

Theorem

The pre-proof obtained by the cut elimination algorithm is valid.

Cut elimination produces a proof

back to the sketch

Theorem

The pre-proof obtained by the cut elimination algorithm is valid.

Proof: Let $\pi^\star$ be the pre-proof obtained from $\pi \vdash \Delta$ by cut elimination. Suppose that a branch b of $\pi^\star$ is not valid.

- Let θ be the sub-derivation of π explored by the reduction that produces b.
- **Fact:** Threads of θ are the threads of b, together with threads starting from cut formulas.
- The validity of θ cannot rely on the threads of b.
- θ^μ is θ where we replace in Δ any ν by a μ and any $1, \top$ by $\bot, 0$.
- Show that formulas containing only $\mu, \bot, 0$ and *MALL* connectives are false.
- θ^μ proves a false sequent which contradicts soundness.