Syntaxe quantitative : développement de Taylor des réseaux de preuve

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Motivations/framework

- Quantitative semantics of λ-calculus and linear logic
- Commutation between cut-elmination/normalization and Taylor expansion of proof nets
- Combinatorial study of parallel cut elimination

Main result

Parallel reduction over infinite linear combinations of differential nets is well defined

Semantics of λ-calculus

Quantitative approach : think about resources

$[[M]]_{\mathcal{M}} \rightsquigarrow$ Something in a structure $\mathcal{M}$ invariant under $\to_\beta$

- Quantitative meaning of $[[M]]$ in $(\lambda x(xx))M$? *wrt* $(\lambda xx)M$?
- Interpret probabilistic reduction ?
- ...

Girard (Normal functors, 1988)

Uses of arguments $\rightsquigarrow$ degree of a monomial in a power series.

Types: $[[A]] \subseteq \mathbb{S}^{|A|}$ where $\mathbb{S}$ is a semiring
Programs : power series

Quantitative Semantics

Example 2: multirelations

$\mathbb{S}$	$\rightsquigarrow$	Boolean semiring
Types	$\rightsquigarrow$	$[[A \to B]] = \mathcal{M}_{\text{fin}}(\mid A \mid) \times \mid B \mid$
Programs	$\rightsquigarrow$	$P : A \to B \Rightarrow [[P]] \subseteq \mathcal{M}_{\text{fin}}(\mid A \mid) \times \mid B \mid$
Invariance	$\rightsquigarrow$	Composition of multirelations.

Key idea

let $M : A \to B,\ N : A$.
$([a_1, \ldots, a_k], b) \in [[M]]$ will match with k uses of the argument N in the application (MN).

Resource calculus

A quantitative syntax

$$m, n ::= x \mid \lambda x m \mid \langle m \rangle [n_1, \dots, n_k] \text{ (k-linear application)}$$

$$\langle \lambda x m \rangle [n_1, \dots, n_k] \rightarrow_\partial \sum_{\sigma \in \mathfrak{S}_n} m \left[n_{\sigma(1)}/x_1, \dots, n_{\sigma(k)}/x_k \right]$$

$$\langle \lambda x \langle x \rangle [x] \rangle [z] \rightarrow_\partial 0 \ \ {}_\partial\!\leftarrow \langle \lambda x x \rangle [z, z]$$

λ-calculus		resource calculus
$(\lambda x (x) x) z$	$\rightsquigarrow$	$\langle \lambda x \langle x \rangle [x] \rangle [z, z]$
$\downarrow_\beta$		$\downarrow_\partial$
zz	$\rightsquigarrow$	$\langle z \rangle [z] + \langle z \rangle [z]$

Multilinear Approximations

We say m is an approximation of M, and define $\lhd \subset \Delta \times \Lambda$:

- $x \lhd x$, $\quad \lambda xn \lhd \lambda xN$ if $n \lhd N$.
- $\langle m \rangle [n_1, \ldots, n_k] \lhd MN$ if $k \in \mathbb{N}$,$m \lhd M$ and $n_i \lhd N$.

Definition

We extend $\rightarrow_\partial$ to a parallel reduction $\rightrightarrows_\partial$.

Property : simulation of $\rightarrow_\beta$ with approximants

If $m \lhd M$ and $M \rightarrow M'$, $m \rightarrow 0$ or $m \rightrightarrows_\partial m' \lhd M'$.

Example

- $N \rightarrow N' \Rightarrow MN \rightarrow MN'$
- $m \lhd M, n_i \lhd N \Rightarrow \langle m \rangle [n_1, \ldots, n_k] \lhd MN$.
- $n_i \rightrightarrows_\partial n'_i \Rightarrow \langle m \rangle [n_1, \ldots, n_k] \rightrightarrows_\partial \langle m \rangle [n'_1, \ldots, n'_k] \lhd MN'$ *(ind. hyp.)*

Taylor expansion

A bridge between syntax and semantics

Semantic approach : Interpret a term/function as an infinite series of approximants.

Syntactic Taylor expansion :

$$\mathcal{T}(MN) = \sum_{k \in \mathbb{N}} \frac{1}{k!} \langle \mathcal{T}(M) \rangle [\mathcal{T}(N), \ldots, \mathcal{T}(N)]_k$$

$\mathcal{T}(\lambda x M) = \lambda x \mathcal{T}(M), \mathcal{T}(x) = x.$

Remark

$\mathcal{T}(M)$ is a weighted sum of all resource nets m s.t. $m \lhd M$

Simulation

Wanted result

Extend $\rightrightarrows_\partial$ to infinite sums of terms ($\Rightarrow_\partial$), in order to have $M \rightarrow_\beta N \Rightarrow \mathcal{T}(M) \Rightarrow_\partial \mathcal{T}(N)$, and define $NF(\mathcal{T}(M))$

Problem

Can $\Rightarrow_\partial$ be always well-defined ?

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Counterexample

$$\sum_{k \in \mathbb{N}} \langle \lambda x x \rangle [\langle \lambda x x \rangle \ldots [y]]] \ldots] \Rightarrow_\partial \infty \cdot y$$

If $\mathbb{S}$ is not a complete semiring, the reduction is not defined on all series of terms.

Some convergence results

$\Rightarrow_\partial$ and normalization are well defined and commute with Taylor expansion:

- Classical Λ: Ehrhard Regnier 2007.
- Non deterministic Λ with finite sums : Pagani, Tasson, Vaux 2016.
- Algebraic calculus: Vaux 2017.

Idea of the proof in (Vaux, CSL 2017)

$$M \quad \to_\beta \quad M'$$

$$\mathcal{T}(M) \quad \Rrightarrow_\partial \quad \mathcal{T}(M')$$

$$m_1 \searrow_\partial \; \vdots \; m' \; \nearrow^\partial \; m_k$$

Sketch

$\{appdepth(m); m \in \mathcal{T}(M)\}$ bounded by M

$\Downarrow$

$\{\#m; m \in \mathcal{T}(M), m \Rrightarrow_\partial m'\}$ bounded by $\#m'$

$\Downarrow$

$\{m_i \in \mathcal{T}(M); m_i \Rrightarrow_\partial m'\}$ is finite.

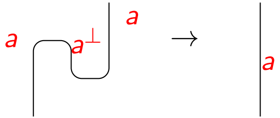
$\Downarrow$

Hence, m' has a finite coefficient in $\mathcal{T}(M')$.

MELL

Multiplicative nodes and reductions

$$A, B ::= X \mid A^{\perp} \mid A \otimes B \mid A \parr B \mid !A \mid ?A$$



MELL

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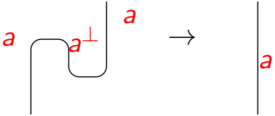
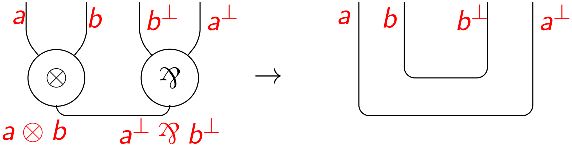
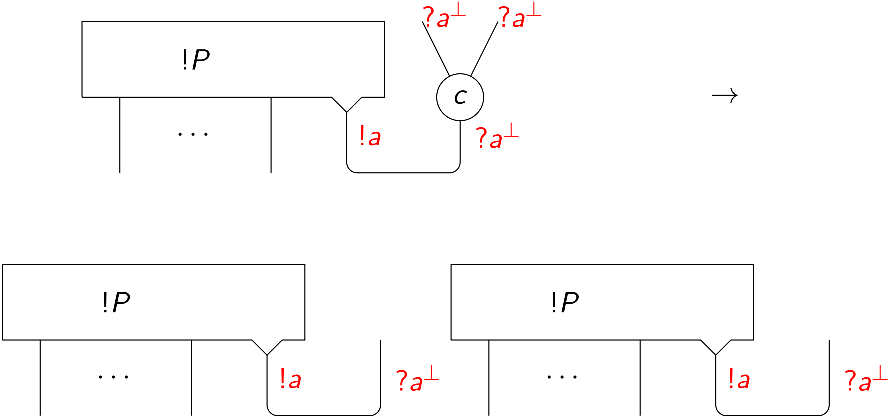




Duplication of an exponential box

An example of non linear reduction



Λ	MELL
$x, \lambda x M$	Multiplicatives
MN	Interaction of an exponential box
Several uses of the argument	Duplication of a box
Resource λ-calculus	Resource nets

Taylor expansion

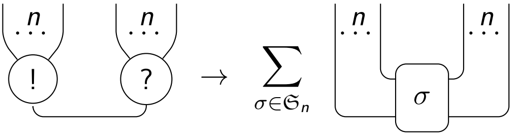
As in λ-calculus, we consider sums of approximants. An approximant of a box consists in the duplication of its content

Resource reduction

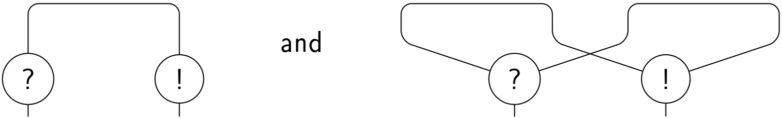
Linear fragment of differential interaction nets

Resource nets

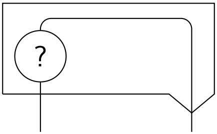
Exponential box is removed, and will be simulated by the linear constructs ? and !



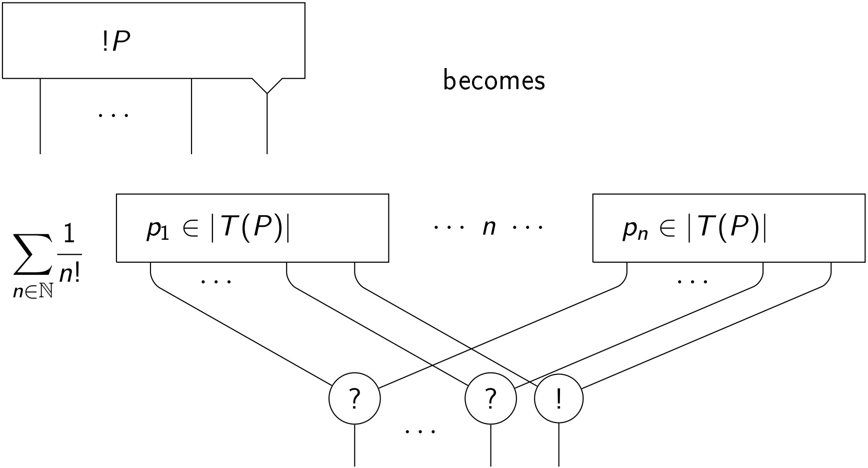
Exemples



Are approximants of :



Taylor expansion of a box



Convergence

Wanted result

Define $\Rightarrow$ over infinite sums of nets, in order to simulate exponential cut elimination and normalization of an MELL net, into its Taylor expansion

Can we define a parallel reduction over infinite sums of resource nets ?

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Counterexample

$\sum_{n\in\mathbb{N}}$  $\Rightarrow \infty \cdot$ $|$

In Λ, the convergence result is proved thanks to the following property :

If $m \in T(M)$, applicativedepth(m) is bounded by M

Key idea of our result

Applicative depth in Λ corresponds in proof nets to the number of cuts crossed by a switching path

Idea of the proof

Theorem

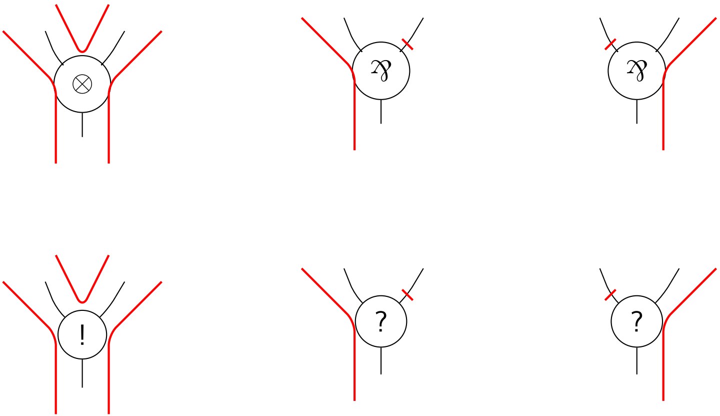
Let $P \in$MELL, q a resource net. $\{p \in |\mathcal{T}(P)|; p \rightrightarrows q\}$ is finite.

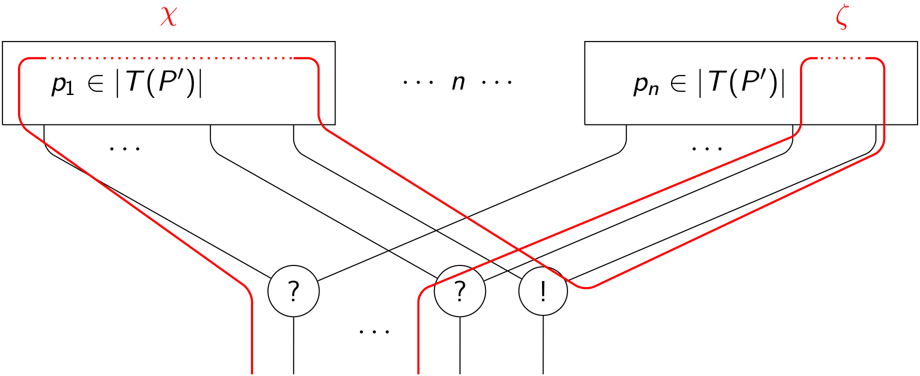
Proof.

1. If $p \in |\mathcal{T}(P)|$, the paths of p do not cross more than $2^{\#P}$ cuts
2. If $p \rightrightarrows q$, $\#p \leq f(\#q$, number of cuts on a path of $p)$
3. Then $\{\#p; p \in |\mathcal{T}(P)|, p \rightrightarrows q\}$ is bounded by p and q.
4. then $\{p \in |\mathcal{T}(P)|; p \rightrightarrows q\}$ is finite

□

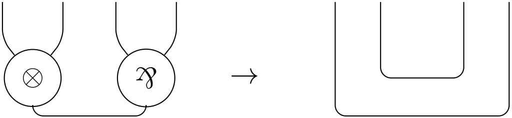
Switching paths

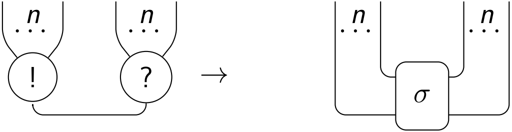




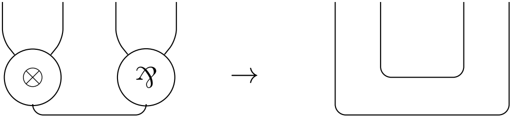
Lemme : coupures $(\chi :: \zeta) \leq 2^{\#P}$ (où $P = !(P')$)

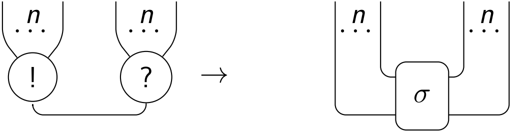
Hypothèse d'induction : coupures$(\chi) \leq 2^{\#P'}$ et coupures$(\zeta) \leq 2^{\#P}$.
$\#P \geq \#P' + 1$
$\rightarrow$ coupures$(\chi :: \zeta) \leq 2 \cdot 2^{\#P'} \leq 2^{\#P}$.





(Division de la taille par 2)





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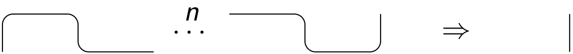

(Division de la taille par n.)

1. Quantitative semantics
2. Quantitative syntax
3. Taylor expansion
4. Linear logic proof nets
5. Contribution
   - If $p \in |\mathcal{T}(P)|$, $\text{cuts}(\text{paths}(p)) \leq 2^{\#P}$
   - If $p \rightrightarrows q$, $\#p \leq f(\#q, \text{cuts}(\text{paths}(p)))$
   - CQFD
   - Être acyclique, c'est chic

Main result

Theorem

For all MELL proof net P, if $\mathcal{T}(P) \Rightarrow \psi = \sum_{i \in I} a_i \cdot p_i$, then all resource net p has a finite coefficient in ψ

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Main technical difficulty

Bound the growth of paths(p) under parallel reduction

Main result

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For all MELL proof net P, if $\mathcal{T}(P) \Rightarrow \psi = \sum_{i \in I} a_i \cdot p_i$, then all resource net p has a finite coefficient in ψ

Main technical difficulty

Bound the growth of paths(p) under parallel reduction

Reduction

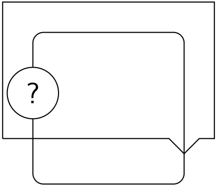
If $P \rightarrow Q$, then the reduction $\mathcal{T}(P) \Rightarrow \mathcal{T}(Q)$ is well-defined

Normalization

$$NF(\mathcal{T}(M)) = \sum_{m \in |\mathcal{T}(M)|} \mathcal{T}(M)_m \cdot nf(m) = \mathcal{T}(NF(M))$$

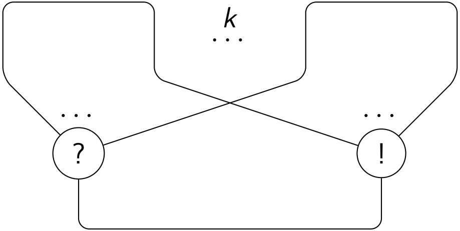
A cyclic counter-example

Taylor expansion of



gives

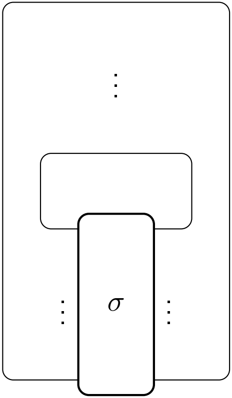
$$\sum_{k \in \mathbb{N}} \frac{1}{k!}$$



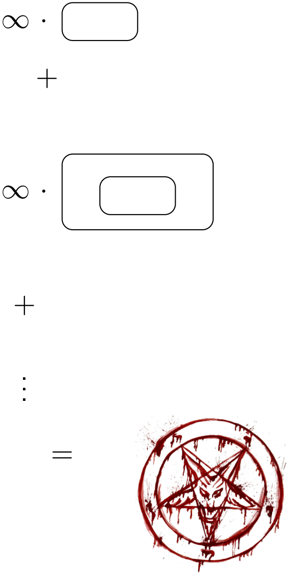
A cyclic counter-example

Or,

$$\sum_{k\in\mathbb{N},\sigma\in\mathfrak{S}_k} \frac{1}{k!}$$



$$\Rightarrow$$



Conclusion

- Quantitative semantics is a powerful approach to study important properties of various calculi
- Taylor expansion is a strong bridge between the calculus and its quantitative interpretation
- Linear logic is an efficient framework for quantitative semantics

- We study the calculus of linear logic proof nets
- We can mimick quantitative identities in the Taylor expansion setting (resource calculi)
- We show that this is sound *wrt* the underlying algebraic structure (convergence result)

Thank you