Higher Universal Algebra in Type Theory

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Joint with M. Sozeau, Y. Guiraud, A. Allioux

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- Remark : There are types which are not of *any* finite *h*-level, i.e., this filtration is not exhaustive

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Definition (Cont'd)

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6. Associative Law:

$$\text{assoc} : (xyzw : Ob)(f : Hom\,z\,w)(g : Hom\,y\,z)(h : Hom\,x\,y) \to ((f \circ g) \circ h) = (f \circ (g \circ h))$$

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- To define the composition in C/Z, we must use that composition in C is associative.
- We cannot show that the composition in C/Z is *itself* associative without more axioms!

Coherence Conditions

A sufficient condition for composition in C/Z to be associative is the well known pentagon identity:

$$\begin{tikzcd}
 & & (g_1 g_2)(g_3 g_4) \arrow[drr, "a_{g_1, g_2, g_3 g_4}"] & & \\
((g_1 g_2) g_3) g_4 \arrow[urr, "a_{g_1 g_2, g_3, g_4}"] \arrow[dr, "a_{g_1, g_2, g_3} 1_{g_4}"'] & & & & g_1(g_2(g_3 g_4)) \\
 & (g_1(g_2 g_3)) g_4 \arrow[rr, "a_{g_1, g_2 g_3, g_4}"] & & g_1((g_2 g_3) g_4) \arrow[ur, "1_{g_1} a_{g_2, g_3, g_4}"'] &
\end{tikzcd}$$

We can amend the definition of category to included this new law (which lives in a doubly iterated identity type) ...

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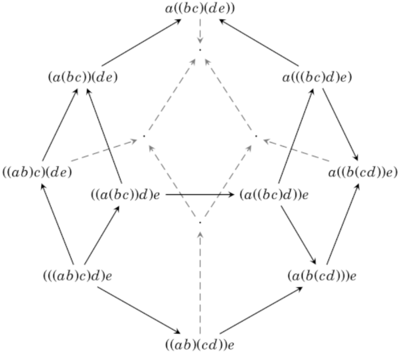
And we quickly find that in order to do so, we need *another* axiom:



The classic “coherence problem”

Now it becomes clear that this completely elementary construction is not well defined unless we add *infinitely many* laws to our definition of category

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- We can avoid this problem by supposing that our types are h-sets. (or, for example, that they have decidable equality)
- But what is the correct notion of category for a general type?

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- A number of these "higher structures" already exist (and are *useful*) in modern algebraic topology and algebraic geometry
- We need a *general* theory of algebra on types

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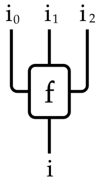
- For $i : I$, an element $f : \mathrm{Op}\, i$ represents an operation whose *output* sort is i.
- For $i : I$, $f : \mathrm{Op}\, i$ and $j : I$, an element $p : \mathrm{Param}\, i\, f\, j$ represents and *input* parameter of sort j.

Graphical Representation

- It is helpful to think of our signature as having "elements" consisting of the operations
- We can think of them as typed symbols:

$$f(i_0, i_1, i_2) : i$$

- Or we can depict them graphically:



Trees

Associated to any polynomial $P : \mathrm{Poly}\, I$ is its W-Type, that is, the type of "well-typed terms" generated by the signature.

Definition

Let $P : \mathrm{Poly}\, I$ be a polynomial. Define

$$
\begin{aligned}
W\ P &: I \to \mathit{Type} \\
\mathrm{lf} &: (i : I) \to W\ P\ i \\
\mathrm{nd} &: (i : I) \to (f : \mathrm{Op}\, P i) \\
&\quad \to (\delta : (j : J)(p : \mathrm{Param}\, f\, j) \to \mathrm{W}\ P\ j) \\
&\quad \to \mathrm{W}\ P\ i
\end{aligned}
$$

Representations of Trees

- We can represent elements of $\mathrm{W}\,P$ as terms

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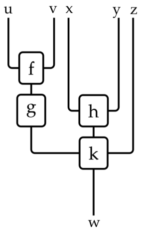
- Or graphically as actual trees:

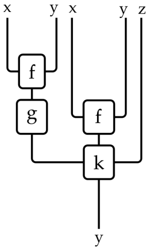


Leaves and Nodes

It is not hard to define the "type of leaves" and "type of nodes" of a give tree. These have types:

$$\text{Leaf} : \{i : I\}(w : \mathrm{W}\, P\, i)(j : I) \to \mathit{Type}$$

$$\text{Node} : \{i : I\}(w : \mathrm{W}\, P\, i)\{j : I\}(g : \mathrm{Op}\, P\, j) \to \mathit{Type}$$



$$\text{Leaf}\ w\, x = \{\bullet, \bullet\}$$

$$\text{Node}\ w\, f = \{\bullet, \bullet\}$$

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$$\begin{aligned} \text{Good:} \quad & f(g(x,y),z) = k(x,y,z) \\ \text{Bad:} \quad & f(g(x,y),z) = h(x,k(y)) \end{aligned}$$

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- The reason for these restrictions is we can then encode such relations as a multiplication operator on the signature itself.

Frames and Magmas

Definition

Let $P : \mathrm{Poly}\, I$ be a polynomial $w : \mathrm{W}\, P\, i$ a tree and $f : \mathrm{Op}\, P\, i$ and operation. A *frame* from w to f is

$$(j : I) \to \mathrm{Leaf}\, w\, j \simeq \mathrm{Param}\, P\, f\, j$$

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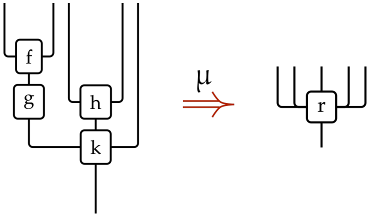
Definition

Let $P : \mathrm{Poly}\, I$ be a polynomial. A *polynomial magma* M over P is

1. A function $\mu : (i : I) \to \mathrm{W}\, P\, i \to \mathrm{Op}\, P\, i$
2. A function $\mu_f : (i : I)(w : \mathrm{W}\, P\, i) \to \mathrm{Frame}\, P\, w\, (\mu\, w)$

Visualization of Relations



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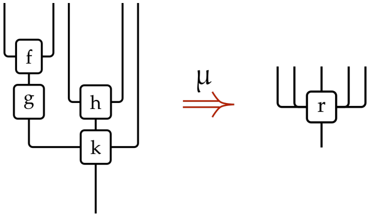

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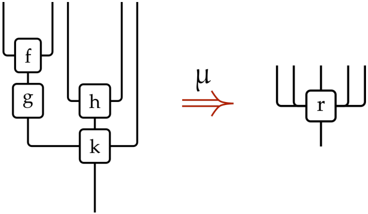

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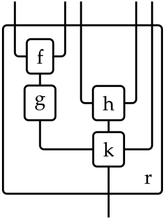

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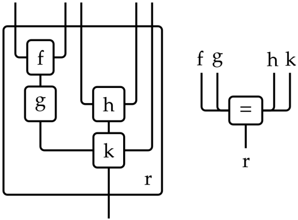

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The Slice of a Polynomial

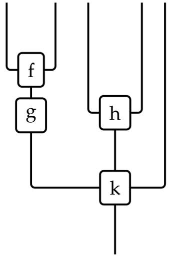
Given a polynomial $P : \mathsf{Poly}\, I$ and a magma $M : \mathsf{PolyMagma}\, P$, we can define a new polynomial $P//M : \mathsf{Poly}(\Sigma\, I\ \mathsf{Op})$ as follows:

$$\mathsf{Op}(P//M)\,(i, f) := \sum_{w : \mathsf{W}\, P\, i} \mu w = f$$

$$\mathsf{Param}(P//M)\,(i, f)(w, e)(j, g) := \mathsf{Node}\, w\, g$$

Visualizing Iterated Compositions

A tree in the slice polynomial $P//M$ can be visualized as representing a *sequence* of applications of the multiplication μ.



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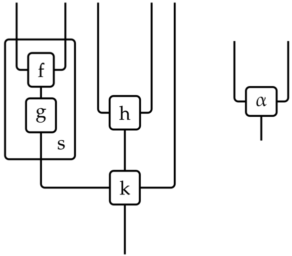

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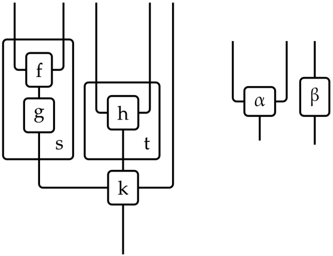

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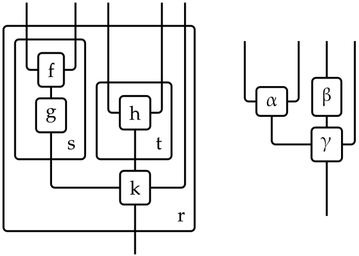
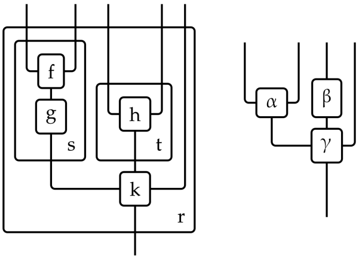


Flattening

There is a function:

$$\text{flatten} : (i : I)(f : \text{Op}\, P\, i) \to \mathsf{W}(P//M)\,(i, f) \to \mathsf{W}\, P\, i$$

which, given a *pasting diagram* extracts its "boundary":



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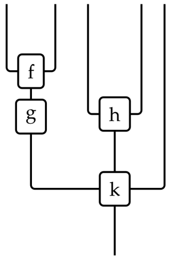

Invariance by Subdivision

A *coherence witness* Ψ for M is proof that the multiplication μ is invariant under all subdivisions.

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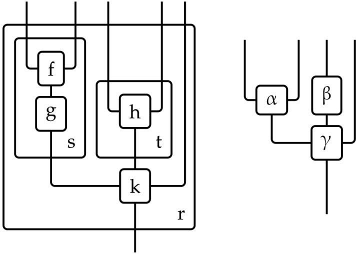

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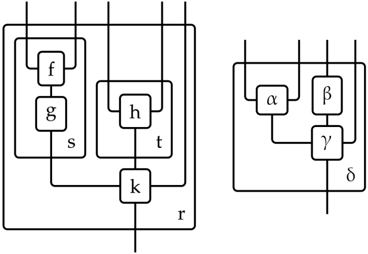

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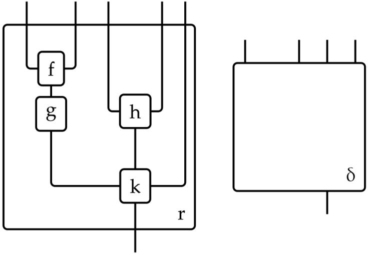
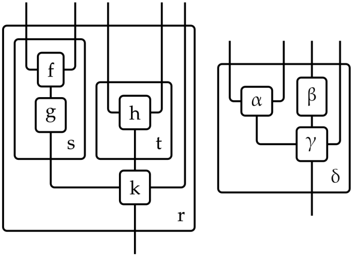


Invariance and Associativity



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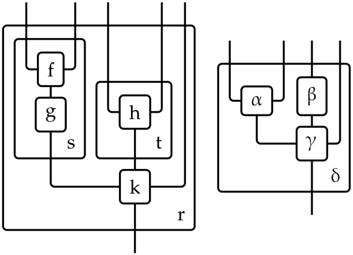

$$\mu(\mu(f, g), \mu(h), k) = r$$
$$\mu(f, g, h, k) = r$$

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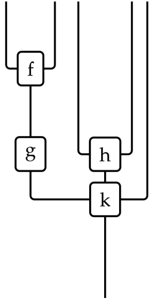

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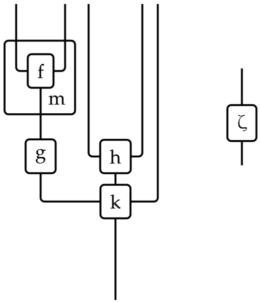

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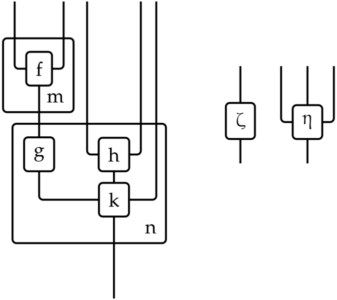

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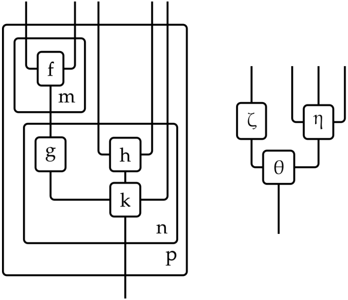

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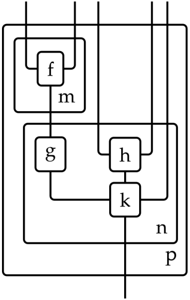

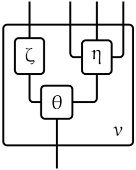

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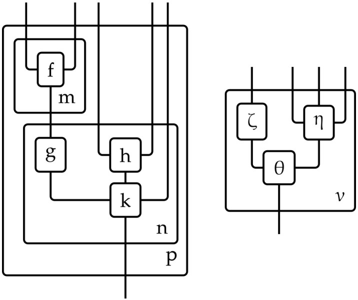

$$\mu(\mu(f), \mu(g, h, k)) = p$$
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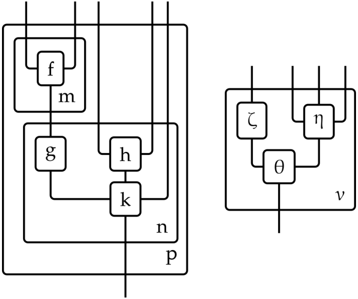

$$p = \mu(f, g, h, k) = r$$
$$\mu(\mu(f, g), \mu(h), k) = \mu(\mu(f), \mu(g, h, k))$$

Polynomial Monads

The advantage of this formulation is that we can now define a coherent structure as one for which one can find an infinite sequence of such extensions.

Definition

A coherence structure for M consists of

1. A coherence witness Ψ for M
2. Coinductively, a coherence structure for M_{Ψ}

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Definition

A *polynomial monad* is consists of

1. A polynomial $P : \mathrm{Poly}\, I$
2. A magma $M : \mathrm{PolyMagma}\, P$
3. A coherence structure C for M

Applications and future work

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- More generally, this definition gives us an internalization of *higher operads*
- Remains to explore examples and use these techniques to prove coherence theorems

Thank you!