Homology of strict ∞-categories

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General Context

The study of (the homotopy theory of) strict ∞-categories.

- Natural framework for higher rewriting theory.
- "Toy model" for the study of weak ∞-categories.

Quick intro to Homotopy Theory I

- From the point of view of category theory, two isomorphic objects in a category are *indistinguishable*.
- For example, two isomorphic groups can be considered the same as regards group theory.
- In certain contexts, we have a certain class of morphisms in a category that we want to consider as if they were isomorphisms.
- Example from algebraic topology : if there is a homotopy equivalence (or better : a weak homotopy equivalence)

$$f : X \to Y$$

  between two topological spaces, we want to consider that X and Y are "the same".

What is homotopy theory?

Definition

A *category with weak equivalences* is a pair $(\mathcal{C}, \mathcal{W})$ where $\mathcal{C}$ is a category and $\mathcal{W}$ is a class of morphisms of $\mathcal{C}$ called the *weak equivalences*.

Roughly, Homotopy Theory is the study of category with weak equivalences.

A basic construction : Localization

- Given $(\mathcal{C}, \mathcal{W})$ we have its *localization* :

$$\mathcal{C}[\mathcal{W}^{-1}].$$

  It is a category obtained from $\mathcal{C}$ where all the morphisms of $\mathcal{W}$ were forced to become isomorphisms.
- It comes with a localization functor

$$Q : \mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$$

  which is to be thought of as a sort of "quotient map".
- The construction is similar to localizations of rings.

Weak equivalences for strict ∞-categories

Consider the category $\mathbf{Cat}_\omega$ of (small) strict ∞-categories. There are two "natural" notions of weak equivalences for this category.

- $\mathcal{W}_f$: the "folk" (or "canonical") weak equivalences.
- $\mathcal{W}_g$: the "geometrical" (or "Thomason") weak equivalences.

- $(\mathbf{Cat}_\omega, \mathcal{W}_f)$ and $(\mathbf{Cat}_\omega, \mathcal{W}_g)$ are very different from the point of view of homotopy theory !
- For example, $\mathbf{Cat}_\omega[\mathcal{W}_f^{-1}]$ and $\mathbf{Cat}_\omega[\mathcal{W}_g^{-1}]$ are *non-equivalent* categories.

Vague question : Can we understand the differences between these two classes of weak equivalences?

Quick intro to Homotopy theory II

Definition

A *homotopical functor* $F : (\mathcal{C}, \mathcal{W}) \to (\mathcal{C}', \mathcal{W}')$ is a functor

$$F : \mathcal{C} \to \mathcal{C}'$$

that preserves weak equivalences, i.e.

$$F(\mathcal{W}) \subseteq \mathcal{W}'.$$

It is the "naive" notion of morphism for categories with weak equivalences.

The easy case of homotopical functors

Since a homotopical functor $F : (\mathcal{C}, \mathcal{W}) \to (\mathcal{C}', \mathcal{W}')$ preserves weak equivalences, it gives rise to a functor

$$\overline{F} : \mathcal{C}[\mathcal{W}^{-1}] \to \mathcal{C}'[\mathcal{W}'^{-1}]$$

such that the square

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\quad F \quad} & \mathcal{D} \\
\Big\downarrow{\scriptstyle Q} & \Updownarrow{\scriptstyle \cong} & \Big\downarrow{\scriptstyle Q'} \\
\mathcal{C}[\mathcal{W}^{-1}] & \xrightarrow[\overline{F}]{} & \mathcal{C}'[\mathcal{W}'^{-1}]
\end{array}$$

is commutative (up to natural isomorphism).

Derived functors

However, the usual situation is the following :

- We are given a functor

$$F : \mathcal{C} \to \mathcal{C}'$$

  that does *not* preserves weak equivalences. Yet we would like to have a "homotopy version" of that functor.

- Sometimes we still can define a functor

$$\mathcal{L}F : \mathcal{C}[\mathcal{W}^{-1}] \to \mathcal{C}'[\mathcal{W}'^{-1}].$$

  that is in a precise way "the best approximation on the left" of F by a homotopical functor.

When $\mathcal{L}F$ exists, we say that F is *left derivable* and $\mathcal{L}F$ is the *left derived functor of* F.

Remarks

- The property for $F : \mathcal{C} \to \mathcal{C}'$ to be left derivable depends on the weak equivalences $\mathcal{W}$ and $\mathcal{W}'$.
- In contrast to the case of a homotopical functor, we only have a natural transformation

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\quad F \quad} & \mathcal{C}' \\
\Big\downarrow{\scriptstyle Q} & \Uparrow & \Big\downarrow{\scriptstyle Q'} \\
\mathcal{C}[\mathcal{W}^{-1}] & \xrightarrow[\mathcal{L}F]{} & \mathcal{C}'[\mathcal{W}'^{-1}]
\end{array}$$

Example of derived functor : homotopy colimits

- Given $(\mathcal{C}, \mathcal{W})$ such that $\mathcal{C}$ is cocomplete and I a (small) category, we have

$$\operatorname*{colim}_I(-) : Fun(I, \mathcal{C}) \to \mathcal{C}.$$

In general, this functor is not homotopical.

- Usually we can derived the colimit functor :

$$\operatorname*{hocolim}_I(-) : Fun(I, \mathcal{C})[\mathcal{W}^{-1}] \to \mathcal{C}[\mathcal{W}^{-1}].$$

This is the "homotopy version" of colimits.

Abelianization of strict ∞-categories

- There is a well-known "abelianization" functor

$$(-)_{ab} : \mathbf{Cat}_\omega \to \mathrm{Ch}(\mathbb{Z})$$

  where $\mathrm{Ch}(\mathbb{Z})$ is the category of chain complexes.
- This functor is left derivable for the folk weak equivalences *and* the geometrical ones. Hence :

$$\mathcal{L}_f(-)_{ab} : \mathbf{Cat}_\omega[\mathcal{W}_f^{-1}] \to \mathcal{D}(\mathbb{Z})$$

  and

$$\mathcal{L}_g(-)_{ab} : \mathbf{Cat}_\omega[\mathcal{W}_g^{-1}] \to \mathcal{D}(\mathbb{Z})$$

(Where $\mathcal{D}(\mathbb{Z})$ is the localization of $\mathrm{Ch}(\mathbb{Z})$ with respect to quasi-isomorphisms.)

Homologies of strict ∞-categories

- We define the folk homology functor as

$$H^{folk}(-) = \mathbf{Cat}_\omega \xrightarrow{Q_f} \mathbf{Cat}_\omega[\mathcal{W}_f^{-1}] \xrightarrow{\mathcal{L}_f(-)_{ab}} \mathcal{D}(\mathbb{Z})$$

- We define the geometric homology functor as

$$H^{geom}(-) = \mathbf{Cat}_\omega \xrightarrow{Q_g} \mathbf{Cat}_\omega[\mathcal{W}_g^{-1}] \xrightarrow{\mathcal{L}_g(-)_{ab}} \mathcal{D}(\mathbb{Z})$$

Homologies of strict ∞-categories

Question :

How different are the functors $H^{folk}(-)$ and $H^{geom}(-)$?

Answering this question would help us understand the differences between the folk weak equivalences and the geometrical weak equivalences.

The case of monoids

Recall that a monoid is a particular case of (small) 1-category, hence of a (small) strict ∞-category.

Theorem (Lafont & Métayer, 2009)

Let M be a monoid. Then

$$H^{folk}(M) \cong H^{geom}(M).$$

The case of 1-categories

The previous result was then extended :

Theorem (G., 2018)

Let C be a (small) 1-category. Then

$$H^{folk}(C) \cong H^{geom}(C).$$

Although this result was expected in light of the the previous theorem, the proof uses new techniques.

2-categories : the trouble begins

Counterexample of Maltsiniotis and Ara :

There exists a (small) 2-category C for which

$$H^{folk}(C) \not\cong H^{geom}(C)$$

This leads us to the following question :

Question

Can we "simply" characterize the objects C of $\mathbf{Cat}_2$ (or even $\mathbf{Cat}_\omega$) for which

$$H^{folk}(C) \cong H^{geom}(C)?$$

A crucial point of the proof for 1-categories

In the proof of the equivalence of both homologies for 1-categories, I used the following

Magic Formula

For any (small) 1-category

$$H^{geom}(C) \cong \operatorname*{hocolim}_{C} \mathbb{Z}$$

where $\operatorname{hocolim}$ is the "homotopy colimit" (=derived version of the colimit) and $\mathbb{Z}$ is the constant functor with value $\mathbb{Z}$.

Prospect for 2-categories

question :

Does $H^{geom}(C) \cong \operatorname{hocolim}_C \mathbb{Z}$ when C is a 2-category?

Problem : What does the right-hand side mean? We would have to know what is a homotopy colimit indexed by a 2-category.

Thank you!