Dependently typed theories as “cellular” Lawvere theories

Work in progress

Chaitanya Leena Subramaniam¹
Peter LeFanu Lumsdaine²

IRIF, Université (de) Paris (7) (Diderot)
Dept. of Mathematics, Stockholms Universitet

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Dependent type theory vs. dependently typed theories

A **theory** is a set of axioms expressed in a **language**.

\(^1\)(For experts: *structural rules only.*)
Dependent type theory vs. dependently typed theories

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**Dependent type theory (DTT)** is a **language**: it is Martin-Löf’s framework of dependent types.\(^1\)

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Dependent type theory vs. dependently typed theories

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Dependent type theory (DTT) is a language:
it is Martin-Löf’s framework of dependent types.¹

A dependently typed theory is a theory:
it is a set of types, terms and equalities expressed in DTT.

¹(For experts: structural rules only.)
Rules of DTT

\[ \vdash \diamond \text{ctxt} \quad \text{EMP} \quad \frac{}{\vdash \Gamma, x : A \text{ type}} \quad \text{EXT} \]

\[ \vdash \Gamma, x : A, \Delta \text{ ctxt} \]

\[ \frac{}{\vdash \Gamma, x : A, \Delta \vdash x : A} \quad \text{VAR} \]

\[ \frac{\Gamma, \Delta \vdash J \quad \Gamma \vdash A \text{ type}}{\Gamma, x : A, \Delta \vdash J} \quad \text{WEAK} \]

\[ \frac{\Gamma, x : A, \Delta \vdash J \quad \Gamma \vdash a : A}{\Gamma, \Delta[a/x] \vdash J[a/x]} \quad \text{SUBST} \]

...not important for the rest of the talk.
“Baby” example of a dependently typed theory

“Baby” theories don’t have type dependency — e.g. the theory of **abelian groups**: 

\[
\begin{align*}
\vdash & \; G \text{ type} \\
\vdash & \; 1 : G \\
x : G \vdash & \; x^{-1} : G \\
x, y : G \vdash & \; x \cdot y : G \\
x, y, z : G \vdash & \; (x \cdot y) \cdot z = x \cdot (y \cdot z) : G \\
x, y : G \vdash & \; x \cdot y = y \cdot x : G \\
x : G \vdash & \; x \cdot 1 = x : G \\
x : G \vdash & \; x \cdot x^{-1} = 1 : G \\
\end{align*}
\]

also of monoids, groups, rings, modules, algebras, Lie algebras, bialgebras, Hopf algebras . . .
“Adult” example of a dependently typed theory

The theory of **categories**:

\[ \vdash \text{Ob type} \]

\[ x, y : \text{Ob} \vdash \text{Hom}(x, y) \text{ type} \]

\[ x : \text{Ob} \vdash 1_x : \text{Hom}(x, x) \]

\[ \ldots, f : \text{Hom}(x, y), g : \text{Hom}(y, z) \vdash g \circ f : \text{Hom}(x, z) \]

\[ x, y : \text{Ob}, f : \text{Hom}(x, y) \vdash f \circ 1_x = f = 1_y \circ f : \text{Hom}(x, y) \]

\[ \ldots \vdash (h \circ g) \circ f = h \circ (g \circ f) : \text{Hom}(x_1, x_4) \]

also of 2-categories, \( \omega \)-categories, reflexive graphs, semisimplicial sets, opetopic sets . . .
Sneak peek at the big picture

Operations in a “baby” theory

An operation in a multisorted Lawvere theory takes a finite coproduct of points as input, and outputs a point.

\[ \{ \bullet, \bullet, \circ, \circ, \circ \} \]
Sneak peek at the big picture

Operations in an “adult” theory

Every operation in a dependently typed theory takes a finite cell complex as input, and outputs a cell.

(This is related to Burroni-Leinster $T$-operads.)
Baby theories = Lawvere theories

Let $S$ be a set. Then an \textbf{$S$-sorted set} is a disjoint union of sets indexed by $S$:

$$X = \bigsqcup_{s \in S} X_s.$$ 

In other words, it is a function $X \to S$. 
Lawvere’s observation

Let $\mathcal{T}$ be a “baby” theory (i.e. with no type-dependency) with a set $S$ of types.
Then every $\mathcal{T}$-model has an underlying $S$-sorted set.

E.g. the theory of ring-module pairs: every ring-module pair $(A, M)$ has an underlying $\{a, m\}$-sorted set (i.e. pair of sets).
Lawvere’s “recognition theorem”

Lawvere (1963) gave an **algebraic** description of what a “baby” theory (i.e. no type-dependency) is.

**Lawvere’s theorem**

A baby theory (a.k.a. **multisorted Lawvere theory**) is the data of

1. a set $S$ of “types” or “sorts”
2. and a finitary monad on $[S, \text{Set}] = \text{Set}/S$.

**Recall**

A **finitary** monad is one whose underlying endofunctor preserves filtered colimits.
Our goal: A “nice” definition of dependently typed theory

We want to give an algebraic description à la Lawvere of “adult” dependently typed theories.

This description should (obviously) strictly generalise Lawvere’s.
Problem

All current definitions are syntactic, and it is not obvious to translate them into an algebraic description.

Some well-known syntactic definitions of what such a theory should be are GATs [Car78], FOLDS signatures [Pal16] and quotient inductive-inductive types (QIITs) [ACD+18].
“Recognition theorem” for dependently typed theories

Our result (L.S., LeFanu Lumsdaine)

A dependently typed theory is the data of

1. a finitely branching inverse category $I$

2. and a finitary monad on $[I, \text{Set}]$. 
“Recognition theorem” for dependently typed theories

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strictly generalising

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2. and a finitary monad on \([S, \mathbf{Set}] = \mathbf{Set}/S\).
Examples of inverse categories

- Every set $S$. 

- Every Reedy category has a (wide non-full) inverse subcategory (e.g. $\Delta^{op} +$).

- $G_1 \rightarrow G_0$ is a category $G_{2^{op}} = \ldots$
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\[
\begin{array}{c}
G_0 \\
| \\
| \\
G_1
\end{array}
\quad \Rightarrow \quad
\begin{array}{c}
\cdots \\
| \\
| \\
G_0
\end{array}
\]

\[
\begin{array}{c}
G_1 \\
| \\
| \\
G_0
\end{array}
\quad \Rightarrow \quad
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\cdots \\
| \\
| \\
G_0
\end{array}
\]

$G^{\text{op}} = \bigcirc^{\text{op}}$ (opetopes).
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- For $T : \text{Set}^I ! \text{Set}^I$ a finitary cartesian monad, every $T$-operad (à la Burroni-Leinster).
- And many more...
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- The free-strict-$\omega$-category monad on $\text{Set}^{\text{Gop}}$. 
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▶ The free-strict-ω-category monad on $\text{Set}^{\text{Op}}$.
▶ The free-weak-ω-category monad on $\text{Set}^{\text{Op}}$. 
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- The free-category monad on $\text{Graph}$.
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- The free-weak-$\omega$-category monad on $\text{Set}^{\text{G}^{\text{op}}}$.
- For $T : \text{Set}^I \to \text{Set}^I$ a finitary cartesian monad, every $T$-operad (à la Burroni-Leinster).
- And many more...
Let \( I \) be the category

\[
\begin{array}{ccc}
G_2 & \xrightarrow{s} & G_1 \\
\downarrow{t} & \uparrow{s} & \downarrow{t} \\
G_0 & \xrightarrow{s} & G_0
\end{array}
\]

with \( s \circ s = s \circ t \) and \( t \circ s = t \circ t \).
Syntactic example

Let $I$ be the category

$$
G_2 \xrightarrow{s} G_1 \xrightarrow{s} G_0
$$

with $s \circ s = s \circ t$ and $t \circ s = t \circ t$.

Then $I$ corresponds to the following type signature.

$$
\vdash G_0 \quad x, y : G_0 \vdash G_1(x, y) \quad x, y : G_0, f, g : G_1(x, y) \vdash G_2(f, g)
$$
Let $I$ be the category

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\end{array}
\]

The theory of 2-categories is a theory with this type signature.
Preliminaries
Let $I$ be a small category.
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Recall that $\text{Fin}(I)$ is the finite-colimit completion of $I^{\text{op}}$. When $I$ is a set, $\text{Fin}(I)$ is the also the finite-coproduct completion of $I$. 
Cartesian collections

The presheaf category

\[ \text{Coll}_I := \text{Set}^{I \times \text{Fin}(I)} \]

is called the category of \textit{I-collections}.

(Intuition: \( F \in \text{Coll}_I \) should be thought of as a \textit{term signature} — for each context \( \Gamma \in \text{Fin}(I) \) and each sort \( i \in I \), \( F(i, \Gamma) \) is the set of operations with input \( \Gamma \) and output sort \( i \).)
Composition of cartesian collections

$I$-collections can be composed via substitution:

\[ G \circ F(i, \Gamma) := \int_{\Theta \in \text{Fin}(I)} G(i, \Theta) \times \text{Set}^I(\Theta, F(\cdot, \Gamma)). \]

\((\text{Coll}_I, \circ, E)\) is a (non-symmetric) monoidal category, where \(E : \text{Fin}(I) \hookrightarrow \text{Set}^I\).
Cartesian collections and endofunctors on $\text{Set}^I$

The functor $\text{Lan}_E(-) : \text{Coll}_I \to [\text{Set}^I, \text{Set}^I]$ of left Kan extension along $E : \text{Fin}(I) \hookrightarrow \text{Set}^I$ is (1) fully faithful and (2) monoidal.

\[
\begin{align*}
\text{Fin}(I) & \xrightarrow{F} \text{Set}^I \\
E & \downarrow \quad \cong \\
\text{Set}^I & \xleftarrow{\text{Lan}_E F}
\end{align*}
\]

(1)

\[
\text{Lan}_E(F \circ G) \cong \text{Lan}_E F \circ \text{Lan}_E G ; \quad \text{Lan}_E E \cong \text{id}
\]

(2)
Consequence

\[ \text{Lan}_E - : \text{Mon}(\text{Coll}_I, \circ, E) \hookrightarrow \text{Mnd}(\text{Set}^I) \]

The category of monoids in \( \text{Coll}_I \) is a full subcategory of the category of monads on \( \text{Set}^I \). It is none other than the category of \textbf{finitary monads} on \( \text{Set}^I \).
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Remarks

- We have only used that \( I \) is a small category.
- \( \text{Mon}(\text{Coll}_I, \circ, E) \) is also known as the category of monads with arities (Weber) or Lawvere theories with arities (Melliès) for the arities \( E : \text{Fin}(I) \hookrightarrow \text{Set}^I \).
Contextual categories as monoids in collections
Inverse categories

Definition

An **inverse category** is:

- a small category $I$,
- whose objects are graded by “dimension” $\dim : \text{Ob}(I) \to \text{Ord}$,
- such that non-identity morphisms strictly decrease dimension,
- and that has no infinite strictly descending chains.
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- and that has no infinite strictly descending chains.

$I$ is finitely branching if the tree $i/\, I$ generated by every $i \in I$ is finite.
Main observation

Proposition (L.S., LeFanu Lumsdaine)

Let $I$ be a finitely branching inverse category. Then $\text{Fin}(I)^{\text{op}}$ is equivalent to a contextual category $C(I)$ (the free contextual category on $I$).

(Note: The structure of a contextual category does not transfer across an equivalence of categories.)
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Let $I$ be a finitely branching inverse category. Then $\text{Fin}(I)^{\text{op}}$ is equivalent to a contextual category $C(I)$ (the free contextual category on $I$).

(Note: The structure of a contextual category does not transfer across an equivalence of categories.)

- Particular case: $I$ is a set, then $\text{Fin}(I)^{\text{op}}$ is the free finite-product category on $I$. 
Proof:

1. Note that:
   
   - The Yoneda embedding factors as \( y : I^{\text{op}} \hookrightarrow \text{Fin}(I) \hookrightarrow \text{Set}^I \).
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- The Yoneda embedding factors as $\mathbf{y} : I^{\text{op}} \hookrightarrow \text{Fin}(I) \hookrightarrow \text{Set}^I$. 
- The boundary inclusions $\partial i \hookrightarrow \mathbf{y}i$ are finitely presentable (since $i/I$ is finite). 
- Every finite cell complex in $\text{Set}^I$ is finitely presentable.
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1. Note that:
   - The Yoneda embedding factors as $y : I^{\text{op}} \hookrightarrow \text{Fin}(I) \hookrightarrow \text{Set}^I$.
   - The boundary inclusions $\partial i \hookrightarrow y_i$ are finitely presentable (since $i/I$ is finite).
   - Every finite cell complex in $\text{Set}^I$ is finitely presentable.
   - Every $X \in \text{Fin}(I)$ can be written as a finite cell complex.
2. Define a **cell context** to be a finite sequence

\[ \emptyset \rightarrow X_1 \rightarrow X_2 \rightarrow \ldots \rightarrow X \]

of chosen pushouts of boundary inclusions:

\[
\begin{array}{ccc}
\partial i & \longrightarrow & X_n \\
\downarrow & & \downarrow \\
y^i & \longrightarrow & X_{n+1}.
\end{array}
\]

**Definition**

The category \( \text{Cell}_I \) has as objects the cell contexts and as morphisms, \( \text{Cell}_I(\emptyset \ldots \rightarrow X, \emptyset \ldots \rightarrow Y) := \text{Set}^I(X, Y) \).

Clearly, \( \text{Cell}_I \simeq \text{Fin}(I) \).
3. Not hard to see that $C(I) := \text{Cell}_I^{\text{op}}$ is a contextual category. (In fact, it is the free contextual category on $I$.)

Remarks

- A collection $X \in \text{Coll}_I \simeq \text{Set}^{I \times \text{Cell}_I}$ is now literally an $I$-sorted term signature.
- $C(I)$ has all finite limits.
**Definition**

An *I-contextual category* is a morphism of contextual categories $F : C(I) \to D$ such that in the (identity-on-objects, fully faithful) factorisation

\[
C(I) \xrightarrow{F_1} D_I \xleftarrow{F_2} D
\]

$F_2 : D_I \hookrightarrow D$ exhibits $D$ as the *contextual completion* of $D_I$.

A morphism of *I-contextual categories* is a morphism in the coslice $C(I)/\text{CxlCat}$.
Theorem (L.S., LeFanu Lumsdaine)

The following categories are equivalent:

1. The category $\mathcal{C}xt\text{Cat}(I)$ of $I$-contextual categories.

2. The category $\text{Mon}(\text{Coll}_I, \circ, E)$ of monoids in $I$-sorted cartesian collections.

3. The category of finitary monads on $\text{Set}^I$.

Proof.

Make use of the theory of Lawvere theories with arities [Mel10], [BMW12].
Summary, current and future work

- We introduce \textit{I-contextual categories} as algebraic objects (monoids in collections) with an underlying dependently typed theory.
- We are working on a \textit{linear} variant of this, and hoping to get a definition of \textit{dependently coloured symmetric operad/linear dependently typed theory}.
- The “base change” properties of \textit{I}-contextual categories remain to be understood.
- We would eventually like to add \textit{Id}-types to this formalism.
Thorsten Altenkirch, Paolo Capriotti, Gabe Dijkstra, Nicolai Kraus, and Fredrik Nordvall Forsberg.

Quotient inductive-inductive types.


Clemens Berger, Paul-André Mellies, and Mark Weber.

Monads with arities and their associated theories.


JW Cartmell.

*Generalised algebraic theories and contextual categories.*


Paul-André Mellies.
Segal condition meets computational effects.


Erik Palmgren.

Categories with families, folds and logic enriched type theory.