# LINEAR LOGIC WITH FIXED POINTS

## TRUTH SEMANTICS, COMPLEXITY, & PARALLEL SYNTAX

ABHISHEK DE

1ST DECEMBER 2022

# THIS IS NOT A SLIDE

- "This statement is false."
- "Set of all sets that don't contain themselves."
  $S \in S \Rightarrow S \notin S \Rightarrow S \in S \ldots$
- "Consistency of a system cannot be proved within itself."

Proof theory is the study of proofs as formal mathematical objects. In this thesis, we study the theory of infinitary and self-referencial proof techniques like infinite descent.

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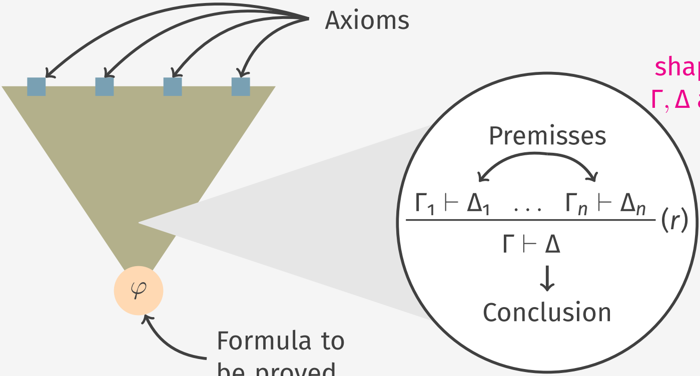
Proof theory is the study of proofs as formal mathematical objects. In this thesis, we study the theory of infinitary and self-referencial proof techniques like infinite descent.

# SEQUENT CALCULUS 101



Axioms

$\varphi$

Formula to be proved

Premisses

$$\frac{\Gamma_1 \vdash \Delta_1 \quad \dots \quad \Gamma_n \vdash \Delta_n}{\Gamma \vdash \Delta}\,(r)$$

Conclusion

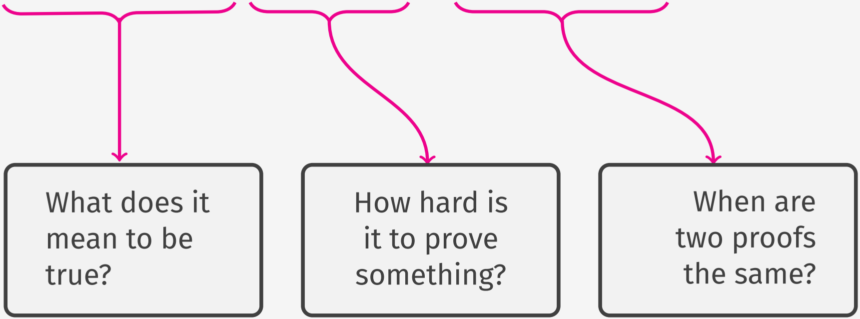
Sequents are objects of the shape $\Gamma \vdash \Delta$ where $\Gamma, \Delta$ are collections of formulas

## A special rule

$$\frac{\Gamma, \varphi \vdash \Delta \quad \Gamma' \vdash \Delta', \varphi}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,(\text{cut})$$

# LINEAR LOGIC WITH FIXED POINTS:

## Truth semantics, complexity, & parallel syntax



What does it mean to be true?

How hard is it to prove something?

When are two proofs the same?

What?

# Linear logic with fixed points:

## Truth semantics, complexity, & parallel syntax

# STRUCTURAL RULES

$$\frac{\vdash \Delta, \varphi, \varphi', \Delta'}{\vdash \Delta, \varphi', \varphi, \Delta'}(\text{ex}) \quad \frac{\vdash \Delta, \varphi, \varphi}{\vdash \Delta, \varphi}(\text{c}) \quad \frac{\vdash \Delta}{\vdash \Delta, \varphi}(\text{w})$$

- **Exchange**: sequents as lists → sequents as multisets
- **Contraction**: sequent as multisets → sequent as sets

## Substructural logics

Logics where one or more of the structural rules are absent or only allowed under controlled circumstances.

# Linear logic (MALL)

| | conjunction | disjunction | "true" | "false" |
|---|---|---|---|---|
| multiplicative | $\otimes$ | $\parr$ | $\mathbf{1}$ | $\perp$ |
| additive | $\&$ | $\oplus$ | $\top$ | $\mathbf{0}$ |

$$\frac{}{\vdash \varphi, \varphi^{\perp}}\,(\text{id}) \qquad \frac{\vdash \Gamma_1, \varphi \quad \vdash \Gamma_2, \varphi^{\perp}}{\vdash \Gamma_1, \Gamma_2}\,(\text{cut})$$

$$\frac{\vdash \Gamma, \varphi_1, \varphi_2}{\vdash \Gamma, \varphi_1 \parr \varphi_2}\,(\parr) \qquad \frac{\vdash \Gamma_1, \varphi_1 \quad \vdash \Gamma_2, \varphi_2}{\vdash \Gamma_1, \Gamma_2, \varphi_1 \otimes \varphi_2}\,(\otimes) \qquad \frac{\vdash \Gamma, \varphi_i}{\vdash \Gamma, \varphi_1 \oplus \varphi_2}\,(\oplus_i) \qquad \frac{\vdash \Gamma, \varphi_1 \quad \vdash \Gamma, \varphi_2}{\vdash \Gamma, \varphi_1 \& \varphi_2}\,(\&)$$

$$\frac{}{\vdash \mathbf{1}}\,(\mathbf{1}) \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \perp}\,(\perp) \qquad \frac{}{\vdash \Gamma, \top}\,(\top)$$

No rule for $\mathbf{0}$

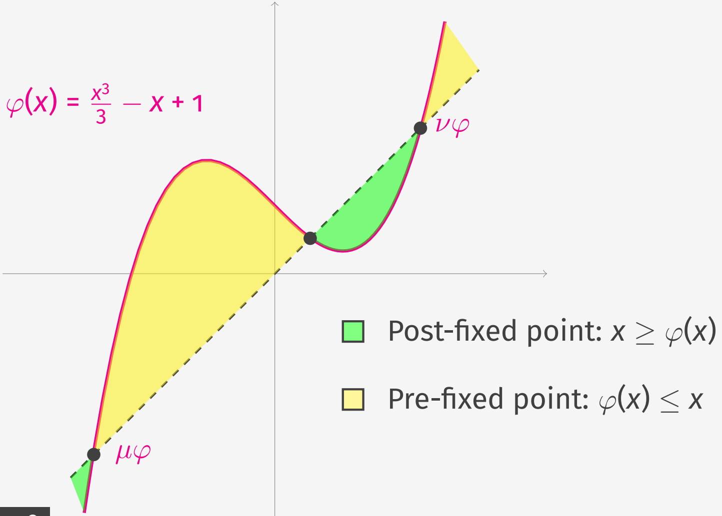
| Exchange | Contraction | Weakening |
|---|---|---|
| ✓ | × | × |

# Linear $\overbrace{\text{logic with fixed points}}^{\text{What?}}$:

## Truth semantics, complexity, & parallel syntax

# Fixed points

Fixed point of a function $\varphi$ is an $x$ that satisfies $f(x) = x$.



🟩 Post-fixed point: $x \geq \varphi(x)$

🟨 Pre-fixed point: $\varphi(x) \leq x$

$$\overbrace{\underset{\substack{\downarrow\\ \text{Fixed point operator}}}{\sigma} x.\underset{\substack{\searrow\\ \text{Function over } x}}{\varphi}}^{\text{Fixed point of that function}} = \varphi(\sigma x.\varphi)$$

**This talk**

- $\mu$ and $\nu$ operators such that $\mu x.\varphi = \neg \nu x.\neg \varphi$ for least fixed point and greatest fixed point
- Proof-theory of such logics

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## This talk

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Why?

# Linear $\overbrace{\text{logic with fixed points}}^{\text{Why?}}$:

## Truth semantics, complexity, & parallel syntax

# DIFFERENT LOGICS, DIFFERENT REASONS

- Extensions of propositional modal logics: LTL, $\mu$-calculus, $\ldots$

  to express richer specifications: "something happens infinitely often", "something happens after some time" and so on
- Extensions of first-order logic: FO[LFP], FO[IFP], $\ldots$

  to define richer classes of finite models and their descriptive complexity
- Extensions of categorical grammar: Kleene Algebra, Action algebra, $\ldots$

  to algebraically define various classes of formal languages

# WHY LINEAR LOGIC WITH FIXED POINTS?

## AUTOMATED REASONING

- On the provability level
- Prove theorems using (co)induction

## PROGRAMMING LANGUAGE THEORY

**Curry-Howard correspondence**

1. formulas $\leftrightarrow$ types.
2. proof objects $\leftrightarrow$ programs.
3. normalisation $\leftrightarrow$ computation/reduction.

Several (co)inductive types are primitive:

- $\mathbb{N} := \mu x.\mathbf{1} \oplus x$
- Lists $\mathbb{L} := \mu x.\bot \oplus (data \otimes x)$
- Streams $\mathbb{S} := \nu x.data \otimes x$

How?

# LINEAR LOGIC WITH FIXED POINTS:

## Truth semantics, complexity, & parallel syntax

# EXPLICIT (CO)INDUCTION

$$\frac{\Gamma, \psi \vdash \Delta \quad \varphi[\psi/x] \vdash \psi}{\Gamma, \mu x.\varphi \vdash \Delta}(\mu_\ell) \qquad \frac{\Gamma \vdash \varphi[\mu x.\varphi/x], \Delta}{\Gamma \vdash \mu x.\varphi, \Delta}(\mu_r)$$

- $\mu_\ell$ expresses $\mu x.\varphi$ is smaller than any pre-fixed point of $\varphi$.
- $\mu_r$ expresses that $\mu x.\varphi$ is indeed a pre-fixed point of $\varphi$. Hence it is the smallest pre-fixed point.
- Dual rules for $\nu x.\varphi$ expresses it is the largest post-fixed point.

$$\frac{\Gamma, \varphi[\nu x.\varphi/x] \vdash \Delta}{\Gamma, \nu x.\varphi \vdash \Delta}(\nu_\ell) \qquad \frac{\Gamma \vdash \psi, \Delta \quad \psi \vdash \varphi[\psi/x]}{\Gamma \vdash \nu x.\varphi, \Delta}(\nu_r)$$

**Difficult to automate!**

Cut admissibility does not guarantee subformula property.

# EXPLICIT (CO)INDUCTION

$$\frac{\Gamma, \psi \leq \psi \quad \varphi[\psi/x] \leq \psi}{\Gamma, \mu x.\varphi \leq \psi}(\mu_\ell) \qquad \frac{\varphi[\mu x.\varphi/x] \leq \varphi[\mu x.\varphi/x], \Delta}{\varphi[\mu x.\varphi/x] \leq \mu x.\varphi, \Delta}(\mu_r)$$

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# IMPLICIT (CO)INDUCTION

$$\frac{\Gamma, \varphi[\mu x.\varphi/x] \vdash \Delta}{\Gamma, \mu x.\varphi \vdash \Delta}(\mu_l) \qquad \frac{\Gamma \vdash \varphi[\mu x.\varphi/x], \Delta}{\Gamma \vdash \mu x.\varphi, \Delta}(\mu_r)$$

$$\frac{\Gamma, \varphi[\nu x.\varphi/x] \vdash \Delta}{\Gamma, \nu x.\varphi \vdash \Delta}(\nu_l) \qquad \frac{\Gamma \vdash \varphi[\nu x.\varphi/x], \Delta}{\Gamma \vdash \nu x.\varphi, \Delta}(\nu_r)$$

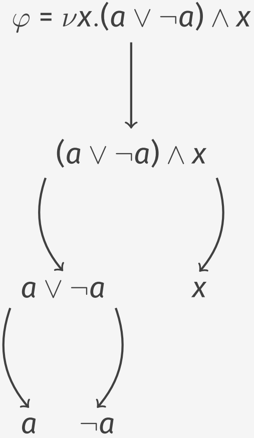
- $\mu_\ell$ and $\mu_r$ expresses that $\mu x.\varphi$ is a pre-fixpoint and post-fixpoint of $\varphi$ respectively. So, it is a fixed point.
- Similarly for $\nu_\ell$ and $\nu_r$.

**Not complete!**

$\nu x.x$ cannot be proven.

# NOTION OF SUBFORMULAS

## SUBFORMULAS



$\varphi = \nu x.(a \vee \neg a) \wedge x$

$(a \vee \neg a) \wedge x$

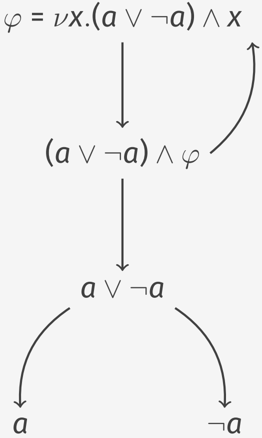
$a \vee \neg a$

$x$

$a$

$\neg a$

## FISCHER-LADNER SUBFORMULAS



$\varphi = \nu x.(a \vee \neg a) \wedge x$

$(a \vee \neg a) \wedge \varphi$

$a \vee \neg a$

$a$

$\neg a$

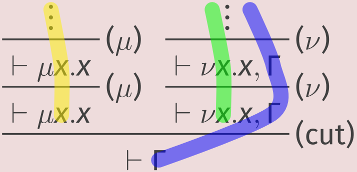
# NON-WELLFOUNDED PROOFS

- Let's allow proof trees of infinite height.
- Now $\nu x.x$ can be proved:

$$\cfrac{\cfrac{\vdots}{\vdash \nu x.x}\,(\nu)}{\vdash \nu x.x}\,(\nu)$$

## Not sound!

Any sequent can be proven now:



$$\cfrac{\cfrac{\cfrac{\vdots}{\vdash \mu x.x}\,(\mu)}{\vdash \mu x.x}\,(\mu) \qquad \cfrac{\cfrac{\vdots}{\vdash \nu x.x, \Gamma}\,(\nu)}{\vdash \nu x.x, \Gamma}\,(\nu)}{\vdash \Gamma}\,(\text{cut})$$

# CIRCULAR PROOFS

## Progress condition

Along every branch, there is a *thread* such that the smallest formula (in the subformula ordering) principal infinitely often is a $\nu$-formula.

**Circular proofs** := Non-wellfounded proofs that have finitely many distinct subtrees.

## Regularisation conjecture

Circular proofs are as powerful as non-wellfounded proofs.

# $\mu$MALL AND ITS PROOF SYSTEMS

## $\mu$MALL = MALL + fixed points

Wellfounded system := $\mu\text{MALL}^{\text{ind}}$

Circular system := $\mu\text{MALL}^{\circlearrowright}$

Non-wellfounded system := $\mu\text{MALL}^{\infty}$

# Linear logic with fixed points:

## Truth semantics, $\underbrace{\text{complexity}}_{\text{What?}}$, & parallel syntax

## Provability problem of a system $\mathcal{L}$

Given a formula $\varphi$ is it provable in $\mathcal{L}$?

## Examples

1. Classical propositional logic: decidable, co-NP complete [Cook-Levine'71]
2. First-order logic on finite models: undecidable [Trakhtenbrot'50]

- Proof-system independent
- Important to be tractable for applications like model checking, automated theorem proving...

# PROVABILITY OF LINEAR LOGIC

| Fragment of linear logic | Complexity of Provability |
|---|---|
| MLL | NP complete [Kanovich’91] |
| MALL | PSPACE complete [LMSS’90] |
| MELL | ? |
| LL | Undecidable [LMSS’90] |

- Exponentials can be encoded in $\mu$MALL. So, we expect it to be at least as hard as LL.

# LINEAR LOGIC WITH FIXED POINTS:

Truth semantics, $\underbrace{\text{complexity}}_{\text{What?}}$, & parallel syntax

Coming up: complexity of $\mu\mathrm{MALL}^{\infty}$[a]

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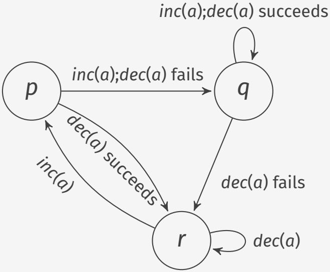
[a] Anupam Das, Abhishek De, and Alexis Saurin. *Decision Problems for Linear Logic with Least and Greatest Fixed Points (FSCD 2022)*

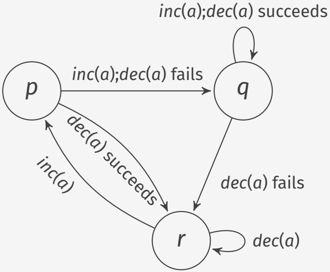
# COUNTER MACHINES

- Counter $a, b, c, \dots$ containing elements of $\mathbb{N}_0$.
- Operations *inc*(*a*) and *dec*(*a*) such that *dec*(*a*) fails if $a = 0$



- Halting problem of one-counter automata is decidable. [Folklore]
- Halting problem of two-counter automata $\Sigma^0_1$-complete [Minsky'62]

# REDUCTION TO MINSKY MACHINE



$$\boxed{[\![inc(a)]\!] := p^{\perp} \otimes (a \parr q)}$$

$$\dfrac{\vdash p, p^{\perp} \qquad \dfrac{\vdash a, a, a, q}{\vdash a, a, a \parr q}(\parr)}{\vdash a, a, p, p^{\perp} \otimes (a \parr q)}(\otimes)$$

Encode dec.

$$\boxed{\varphi := \nu x. \bot \& (\bigoplus_{I \in \mathcal{M}} [\![I]\!] \parr x)}$$

## Theorem (Thm. 6.3.2, pp. 103)

*$\vdash p, \varphi$ provable iff $\mathcal{M}$ is non-halting.*

## Proof idea

($\Leftarrow$) This relies on being able to use $[\![I]\!]$ for every $I \in \mathcal{M}$.

($\Rightarrow$) This relies on cut admissibility and focussing (the ability to apply certain rules context-freely).

# THE REGULARISATION CONJECTURE DOES NOT HOLD!

**Theorem** (Thm. 6.4.3, pp. 106)

$\mu\mathrm{MALL}^{\circlearrowright} \subsetneq \mu\mathrm{MALL}^{\infty}$

**Proof idea**

- $\mu\mathrm{MALL}^{\infty}$ is $\Pi^0_1$-hard.
- $\mu\mathrm{MALL}^{\circlearrowright}$ is in $\Sigma^0_1$.
  (circular proofs are finitely representable, hence enumerable)
- If $\mu\mathrm{MALL}^{\infty} = \mu\mathrm{MALL}^{\circlearrowright}$, then $\Pi^0_1 \subseteq \Sigma^0_1$. Contradiction!

# LINEAR LOGIC WITH FIXED POINTS:

## $\underbrace{\text{Truth semantics}}_{\text{What?}}$, complexity, & parallel syntax

# TRUTH SEMANTICS

- Establishes a semantic meaning of truth.
- Gives a mapping $[\![\bullet]\!]$ : Formulas $\rightarrow$ Mathematical Object such that a formula is provable iff its interpretation satisfies some property.
- Via CH, corresponds to *type inhabitation*.

## Example

- Truth semantics of LK : Boolean algebras
- Truth semantics of LJ : Heyting algebras
- Truth semantics of S4 : Boolean algebras with an interior operator

# TRUTH SEMANTICS OF MALL

## Phase space

A phase space is a commutative monoid $M$ along with a $\perp\!\!\!\perp \subseteq M$.
Let $X, Y \subseteq M$. Define

$$XY := \{xy \mid x \in X, y \in Y\} \qquad X^{\perp} := \{z \mid \forall x \in X. xz \in \perp\!\!\!\perp\}$$

$X$ is called a fact if $X^{\perp\perp} = X$.

We interpret formulas (and sequents) on facts.

$[\![\varphi \otimes \psi]\!] = ([\![\varphi]\!].[\![\psi]\!])^{\perp\perp} \quad [\![\varphi \& \psi]\!] = [\![\varphi]\!] \cap [\![\psi]\!]$

## Theorem (Girard'87)

*$\Gamma$ is provable in* MALL *iff for all models* $1 \in [\![\Gamma]\!]$

# SYNTACTIC MODEL

## Definition

$Pr(\varphi) := \{\Gamma \mid \vdash \Gamma, \varphi \text{ is provable}\}$

- Let $M$ = Set of all sequents.
- Let $\Gamma, \Delta \in M$. Then, $\Gamma \cdot \Delta = \Gamma, \Delta$.
- Therefore, $(M, \cdot, \varnothing)$ is a monoid.
- Let $\perp\!\!\!\perp = Pr(\perp)$ and we have a phase space.

## Lemma (Adequation lemma)

$[\![\Gamma]\!] \subseteq Pr(\Gamma)$

## Completeness proof

$\varnothing \in [\![\Gamma]\!] \Rightarrow \varnothing \in Pr(\Gamma) \Rightarrow \vdash \Gamma, \varnothing$ is provable. □

# Linear logic with fixed points:

## $\underbrace{\text{Truth semantics}}_{\text{How?}}$, complexity, & parallel syntax

Coming up: phase semantics of $\mu\text{MALL}^{\text{ind}}$ [a]

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[a] Abhishek De, Farzad Jafarrahmani, and Alexis Saurin. *Phase semantics for linear logic with least and greatest fixed points (FSTTCS'22)*

# PHASE SEMANTICS OF $\mu$MALL$^{\text{ind}}$

**Fact**

The set of facts is a complete lattice.

$\therefore$ We can interpret fixed point formulas as:

$$[\![\mu x.\varphi]\!] = lfp(\lambda X.\varphi(X)) \quad [\![\nu x.\varphi]\!] = gfp(\lambda X.\varphi(X))$$

The interpretations are facts by Knaster-Tarski theorem.

**Too liberal!**

Not every fact is an image of $[\![\bullet]\!]$. So, $[\![\varphi(X)]\!]$ doesn't necessarily correspond to the interpretation of any formula.

Sound but not complete!

# SOUNDNESS AND COMPLETENESS

Restrict to a subset of fact closed under $\mu$MALL operations.

**Theorem (Lemma 5.1.3, pp. 73 and Thm. 5.1.2, pp. 75)**

*$\Gamma$ is provable in $\mu\mathrm{MALL}^{\mathrm{ind}}$ iff* $1 \in [\![\Gamma]\!]$

**Proof idea**

($\Rightarrow$) Soundness is an easy induction on the proof.

($\Leftarrow$) For completeness, we start from the syntactic monoid but induction on formulas does not work (due to absence of subformula property)! We use Girard's *candidates of reducibility*.

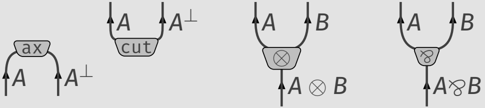
# Linear logic with fixed points:

Truth semantics, complexity, & $\underbrace{\text{parallel syntax}}_{\text{What?}}$

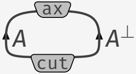
# MLL PROOF-NETS

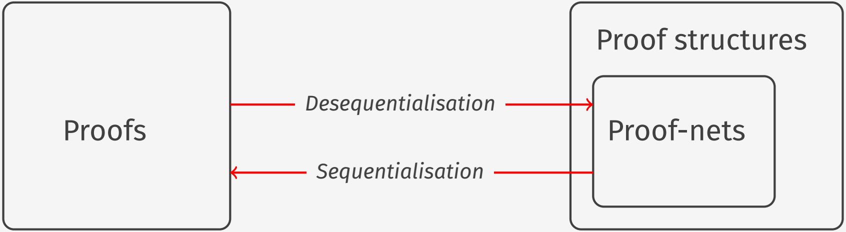
## MLL Proof Structure (Girard'87)

A directed finite multigraph composed of:



There are proof structures that represent no sequent proof





Proofs

Proof structures

Proof-nets

*Desequentialisation*

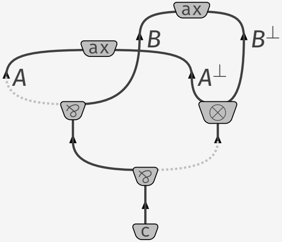
*Sequentialisation*

**Theorem**

*Two proofs are equivalent up to permutation of rules iff they have the same proof-net.*

# THEORY OF PROOF-NETS: CORRECTNESS

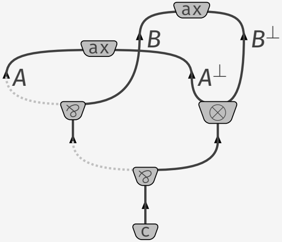
- Switch one premisse of every $\parr$.



- A proof structure is correct if it is acyclic and connected after every switching.

# Theory of proof-nets: correctness

- Switch one premisse of every ⅋.

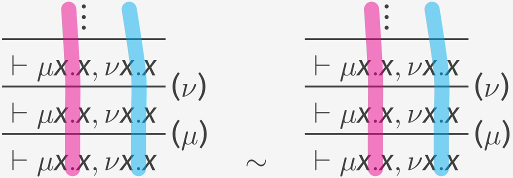


- A proof structure is correct if it is acyclic and connected after every switching.

# Linear logic with fixed points:

## Truth semantics, complexity, & $\underbrace{\text{parallel syntax}}_{\text{Why?}}$

- Proofs are sequential objects



$$\dfrac{\dfrac{\dfrac{\vdots}{\vdash \mu X.X, \nu X.X}}{\vdash \mu X.X, \nu X.X}\,(\nu)}{\vdash \mu X.X, \nu X.X}\,(\mu) \quad \sim \quad \dfrac{\dfrac{\dfrac{\vdots}{\vdash \mu X.X, \nu X.X}}{\vdash \mu X.X, \nu X.X}\,(\nu)}{\vdash \mu X.X, \nu X.X}\,(\mu)$$

- Threads are parallel objects
- Progress condition is defined using threads

To study the progress condition, it makes sense to work on more parallel proof objects.

# Linear logic with fixed points:

## Truth semantics, complexity, & parallel syntax

What?

Coming up: theory of proof-nets for $\mu$MLL$^{\infty}$[a,b]

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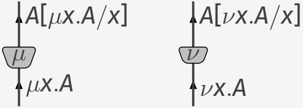
[a] Abhishek De and Alexis Saurin. *Infinets: the parallel syntax for non-wellfounded proof-theory (TABLEAUX 2019)*

[b] Abhishek De, Luc Pellissier, and Alexis Saurin. *Canonical proof-objects for coinductive programming: infinets with infinitely many cuts. (PPDP 2021)*

# $\mu$MLL$^\infty$ PROOF STRUCTURES

Allow the following types of nodes:

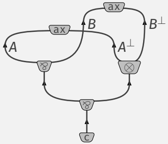


**Is that enough?**

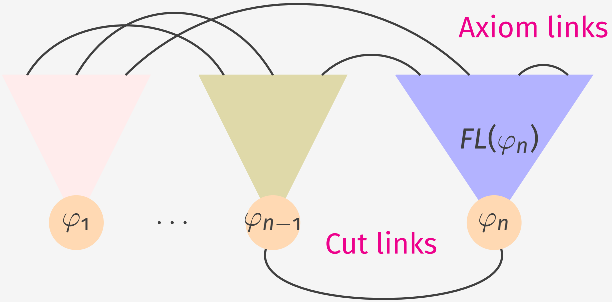
No, quotients more than equivalence by permutation.

# ALGEBRAIC PRESENTATION DUE TO CURIEN'05

Proof structure = Formula tree + axiom links + cut links

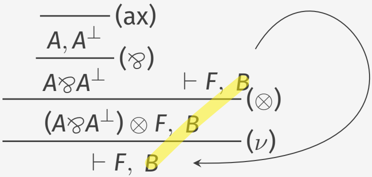


$$= (A \parr B) \parr (A^\perp \otimes B^\perp) + \{\{ll, rl\}, \{lr, rr\}\} + \varnothing$$



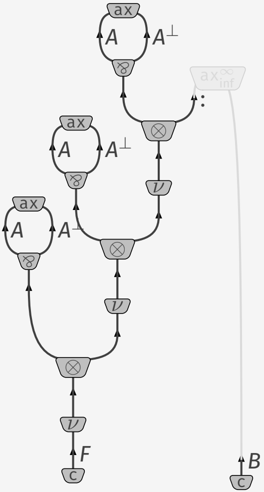
# INFINITE AXIOMS

Let $F = \nu x.(A \parr A^{\perp}) \otimes x$.



$$\dfrac{\dfrac{\dfrac{\dfrac{}{A, A^{\perp}}\,(\mathrm{ax})}{A \parr A^{\perp}}\,(\parr) \qquad \vdash F, B}{(A \parr A^{\perp}) \otimes F, B}\,(\otimes)}{\vdash F, B}\,(\nu)$$

Infinite axioms are invariants of infinite branches in proofs.



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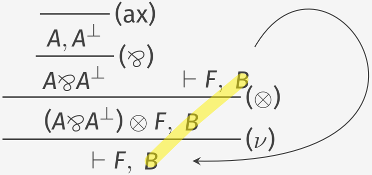
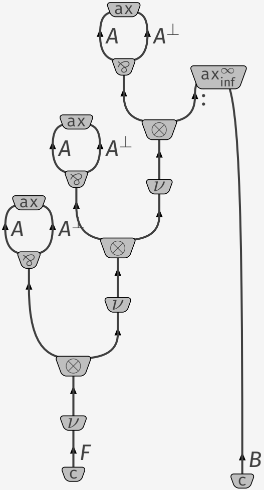
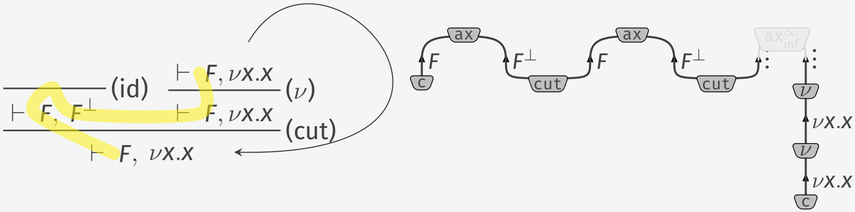


$$\cfrac{\cfrac{\cfrac{\cfrac{}{A, A^{\perp}}\,(\text{ax})}{A \parr A^{\perp}}\,(\parr) \qquad \vdash F, B}{(A \parr A^{\perp}) \otimes F, B}\,(\otimes)}{\vdash F, B}\,(\nu)$$

Infinite axioms are invariants of infinite branches in proofs.



# SIMPLE PROOF STRUCTURES



We restrict ourselves to *simple* proofs and proof structures that do not contain such paths.

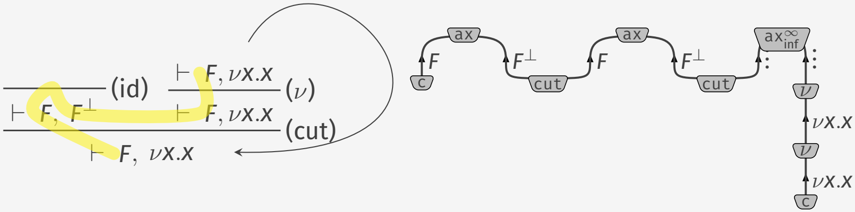
## Theorem (Prop. 8.2.1, pp. 135)

*Simple proofs desequentialise to simple proof structures. If 2 proofs are equal up to rule perm., then they desequentialise to the same structure.*

## Is that enough to ensure DR-correctness?

No, we can encode mix and weakening.

# SIMPLE PROOF STRUCTURES



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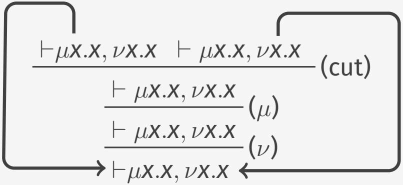
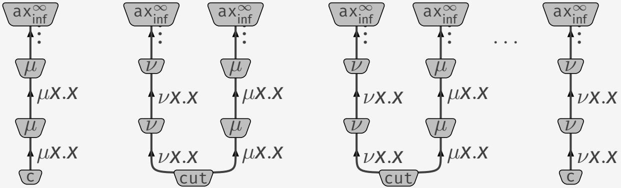
# Crucially disconnected



$$\dfrac{\dfrac{\dfrac{\vdash \mu x.x, \nu x.x \quad \vdash \mu x.x, \nu x.x}{\vdash \mu x.x, \nu x.x}\,(\text{cut})}{\vdash \mu x.x, \nu x.x}\,(\mu)}{\vdash \mu x.x, \nu x.x}\,(\nu)$$



DR-correctness is not right notion!

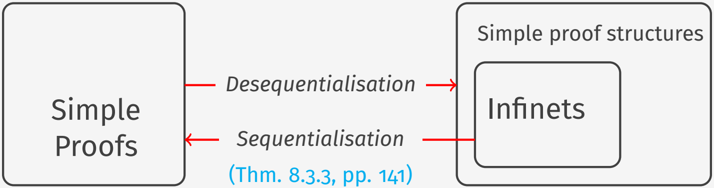
## Can DR-correct simple proof structures be sequentialised? No!

### Lock-free

Every node depends on at most finitely many other nodes.

### Infinets

DR-correct & lock-free simple proof structures.



Simple Proofs

*Desequentialisation* →

← *Sequentialisation*

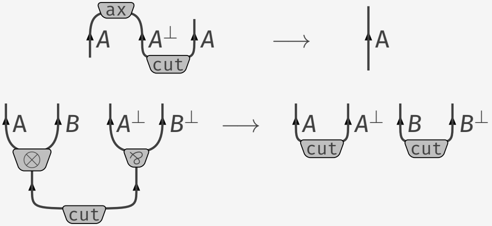
(Thm. 8.3.3, pp. 141)

Simple proof structures

Infinets

# THEORY OF PROOF-NETS: DYNAMICS

Cut reductions steps form a graph rewriting system:

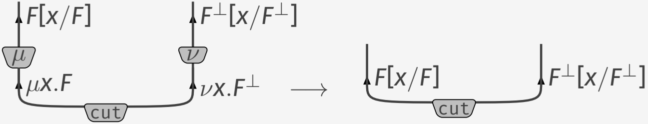


**Theorem (Girard'87)**

*Every step preserves correctness. This system is confluent and terminating.*

# TOWARDS $\mu$MLL$^\infty$ CUT ELIMINATION

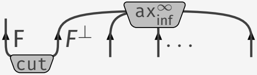
- Cut reduction is now an infinite rewriting system.
- Termination replaced by *productivity*: finite prefixes of the limit should be produced in finite time.
- New rules for new operators:



- Need a notion of *fairness* so that a rule is applied on every cut in finite time.

# THE CUT/AX RULE

No reduction rule from here:



## Two solutions

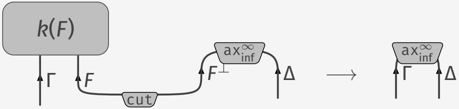
1. Work in a fragment where this case does not occur.
2. Devise a new rule for this case.

# CUT ELIMINATION WITH NEW RULE

## Kingdom

Given $F$, $k(F)$ is the smallest sub-infinet with $F$ as the conclusion.

$$\dfrac{\begin{array}{c}\pi'\\ \bigtriangledown\\ \vdash \Gamma, F^{\perp}\end{array} \qquad \dfrac{\dfrac{\vdots}{\vdash \nu x.x, F}\,(\nu)}{\vdash \nu x.x, F}\,(\nu)}{\vdash \Gamma, \nu x.x}\,(\text{cut})$$



## Theorem (Thm. 9.3.1 , pp. 167)

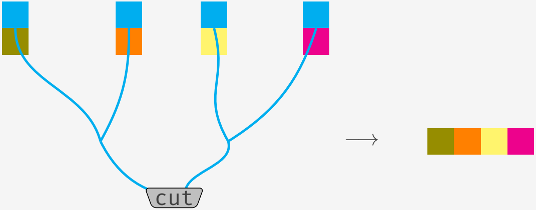
*(Fair) reduction sequences starting from progressing infinets converges to (cut-free) progressing infinets.*

# CUT ELIMINATION IN AXIOM-FREE INFINETS

## Axiom-free infinets

No finite axioms, no formulas in infinite axioms (only infinite branches).

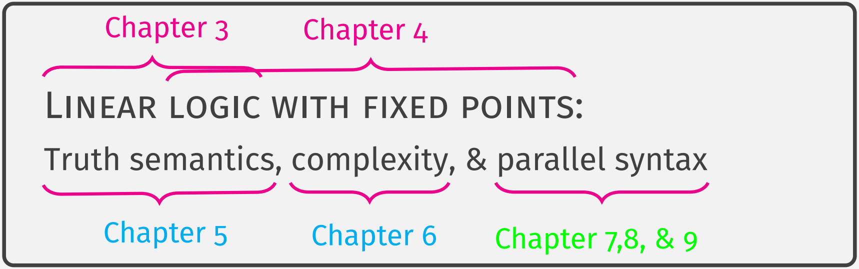
Guess the normal form:



## Theorem (Thm. 9.1.2, pp. 154)

*Fair reduction sequences starting from progressing axiom-free infinets converges to the normal form.*

# SUMMARY OF CONTRIBUTIONS



Chapter 3 Chapter 4

LINEAR LOGIC WITH FIXED POINTS:

Truth semantics, complexity, & parallel syntax

Chapter 5 Chapter 6 Chapter 7,8, & 9

### EXTENSIONAL

- $\mu\mathsf{MALL}^{\mathrm{ind}}$ and $\mu\mathsf{MALL}^{\circlearrowright}$ are $\Sigma^0_1$-complete
- $\mu\mathsf{MALL}^{\infty}$ is $(\Sigma^0_1 \cup \Pi^0_1)$-hard
- $\mu\mathsf{MALL}^{\circlearrowright} \subsetneq \mu\mathsf{MALL}^{\infty}$
- Phase semantics of $\mu\mathsf{MALL}^{\mathrm{ind}}$

### INTENTIONAL

- The theory of proof-nets for $\mu\mathsf{MLL}^{\infty}$
- Cut-elimination on $\mu\mathsf{MLL}^{\infty}$ proof-nets

# FUTURE DIRECTIONS

- Complexity of $\mu$MALL$^{\infty}$.
- Phase semantics of $\mu$MALL$^{\circlearrowright}$ and $\mu$MALL$^{\infty}$.
- Brotherston-Simpson conjecture for $\mu$MALL.
- Devise bouncing thread progress condition for $\mu$MLL$^{\infty}$ proof-nets and prove cut-elimination on these proof-nets.

# TOWARDS THE BROTHERSTON-SIMPSON HYPOTHESIS

**Brotherston-Simpson hypothesis**

Explicit (co)induction is as powerful as implicit (co)induction.

$\Rightarrow \mu\mathsf{MALL}^{\mathsf{ind}} \overset{?}{=} \mu\mathsf{MALL}^{\circlearrowright}$

**Idea**

$\vdash \Gamma$ in $\mu\mathsf{MALL}^{\mathsf{ind}}$ iff for all models $1 \in [\![\Gamma]\!]$ iff $\vdash \Gamma$ in $\mu\mathsf{MALL}^{\circlearrowright}$

# An infinitary calculus

What if we approximate lfp and gfp by their $\omega$-th approximation?

$$[\![\mu x.\varphi]\!] = \left(\bigcup_{n\geq 0} [\![\varphi^n(0)]\!]\right)^{\perp\perp} \qquad [\![\nu x.\varphi]\!] = \bigcap_{n\geq 0} [\![\varphi^n(\top)]\!]$$

This gives us an idea for new inference rules for fixed points:

$$\frac{\vdash \Gamma, \overbrace{\varphi(\varphi(\cdots(\varphi}^{n}(0))\cdots)}{\vdash \Gamma, \mu x.\varphi}\,(\mu_\omega) \qquad \frac{\vdash \Gamma, \top \quad \vdash \Gamma, \varphi(\top) \quad \vdash \Gamma, \varphi(\varphi(\top)) \quad \ldots}{\vdash \Gamma, \nu x.\varphi}\,(\nu_\omega)$$

We call this system $\mu\mathrm{MALL}_\omega$.

# AN INFINITARY CALCULUS

**Theorem (DJS'22)**

*The new intepretation is sound and complete wrt $\mu$MALL$_\omega$.*
*$\mu$MALL$_\omega$ admits cuts.*

| | |
|---|---|
| **Advantage** | Completeness is easy since there is a (sort-of) subformula property. |
| **Disadvantage** | Does not prove the same theorems as $\mu$MALL$^{\text{ind}}$ . |

# BOUNCING

## Bouncing thread progress condition (Baelde et al.'22)

Along every branch, there is a *bouncing thread* such that the smallest formula (in the subformula ordering) principal infinitely often is a $\nu$-formula.

$$\begin{tikzcd}
\pi \arrow[d] & \sim & \pi' \arrow[d] \\
\pi_0 & & \text{diverges}
\end{tikzcd}$$

- Need to make the bouncing thread progress condition stable under permutation of rules.