A Deeper Study of $\lambda!$-Calculus Simulations

A Little History of the Embeddings in the $\lambda!$-Calculus

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Université Paris Cité, Paris

International Workshop on Higher-Order Rewriting

Rome, July 04, 2023

Call-by-Name and Call-by-Value

Different Models of Computation:

Call-by-Name

Call-by-Name and Call-by-Value

Different Models of Computation:

Call-by-Name

Well studied

Different Models of Computation:

Call-by-Name

Well studied

Not used

Call-by-Name and Call-by-Value

Different Models of Computation:

Call-by-Name

Call-by-Value

Well studied

Not used

Call-by-Name and Call-by-Value

Different Models of Computation:

Call-by-Name

Well studied

Not used

Call-by-Value

Not understood

Call-by-Name and Call-by-Value

Different Models of Computation:

Call-by-Name

Well studied

Not used

Call-by-Value

Not understood

Very much used

Call-by-Name and Call-by-Value

Call-by-Name

Call-by-Value

Call-by-Name and Call-by-Value

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Call-by-Value

Call-by-Name and Call-by-Value

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(Terms) $t, u \quad ::= \quad x \mid \lambda x.t \mid t\ u$

Call-by-Name and Call-by-Value

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(Terms) $t, u \quad ::= \quad x \mid \lambda x.t \mid t\ u$

$(\lambda x.t)\ u$

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(Terms) $t, u \quad ::= \quad x \mid \lambda x.t \mid t\ u$

$$(\lambda x.t)\ u \quad \mapsto_\beta \quad t\{x := u\}$$

Call-by-Name and Call-by-Value

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$$N \quad ::= \quad \Box \mid N\ t \mid \lambda x.N$$

Call-by-Name and Call-by-Value

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NAME

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Call-by-Value

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(Terms) $t, u \quad ::= \quad v \mid t\,u$

Call-by-Name and Call-by-Value

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Call-by-Name and Call-by-Value

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Call-by-Value

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(Terms) $t, u \quad ::= \quad v \mid t\,u$

(Values) $v \quad ::= \quad x \mid \lambda x.t$

$$(\lambda x.t)\,v \quad \mapsto_{\beta_v} \quad t\{x := v\}$$

Call-by-Name and Call-by-Value

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$$V \quad ::= \quad \Box \mid V\,t \mid t\,V$$

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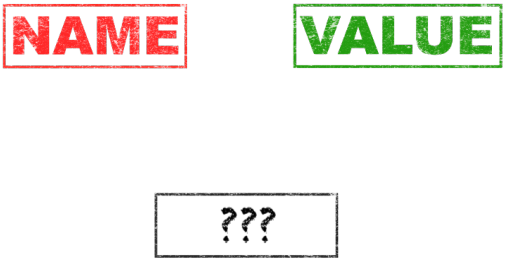
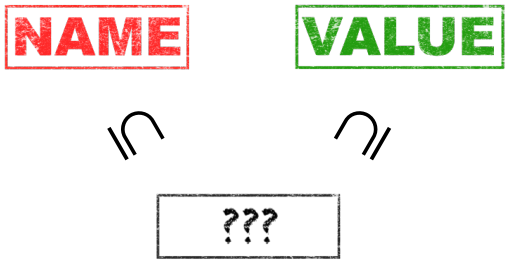
$$(\lambda x.t)\ v \quad \mapsto_{\beta_v} \quad t\{x := v\}$$

$$V \quad ::= \quad \Box \mid V\ t \mid t\ V$$


NAME
VALUE


NAME
VALUE
???


NAME
VALUE
???

Bang-Calculus

Bang-Calculus

(Terms) $t, u \quad ::= \quad x \mid \lambda x.t \mid t\ u$

Bang-Calculus

BANG

(Terms) $t, u \quad ::= \quad x \mid \lambda x.t \mid t\,u$

$\mid \; !t$ (value)

Bang-Calculus

(Terms) t, u ::= $x \mid \lambda x.t \mid t\ u$
| $!t$ (value)
| $\mathtt{der}(t)$ (computation)

Bang-Calculus

(Terms) $t, u \quad ::= \quad x \mid \lambda x.t \mid t\ u$

$\mid\ !t$ (value)

$\mid\ \mathtt{der}(t)$ (computation)

$(\lambda x.t)\ !u$

A Unifying Framework: Bang-Calculus [EhrhardGuerrieri'16][GuerrieriManzonetto'19]

Bang-Calculus

BANG

$$\textbf{(Terms)} \quad t, u \quad ::= \quad x \mid \lambda x.t \mid t\,u$$
$$\mid\ !t \qquad \text{(value)}$$
$$\mid\ \mathtt{der}(t) \qquad \text{(computation)}$$

$$(\lambda x.t)\ {\color{red}!u} \quad \mapsto_\beta \quad t\{x := u\}$$

A Unifying Framework: Bang-Calculus [EhrhardGuerrieri'16][GuerrieriManzonetto'19]

Bang-Calculus

BANG

$$
\begin{array}{llll}
\textbf{(Terms)} & t, u & ::= & x \mid \lambda x.t \mid t\,u \\
& & & \mid\ !t \qquad \text{(value)} \\
& & & \mid \mathtt{der}(t) \qquad \text{(computation)}
\end{array}
$$

$$
\begin{array}{lll}
(\lambda x.t)\,!u & \mapsto_\beta & t\{x := u\} \\
\mathtt{der}(!t) & &
\end{array}
$$

Bang-Calculus

$$
\begin{array}{lrll}
\textbf{(Terms)} & t, u & ::= & x \mid \lambda x.t \mid t\ u \\
& & \mid & !t \qquad\qquad \text{(value)} \\
& & \mid & \mathtt{der}(t) \qquad\qquad \text{(computation)}
\end{array}
$$

$$
\begin{array}{rcl}
(\lambda x.t)\,!u & \mapsto_{\beta} & t\{x := u\} \\
\mathtt{der}(!t) & \mapsto_{!} & t
\end{array}
$$

Bang-Calculus

$$\textbf{(Terms)}\quad t, u \;::=\; x \mid \lambda x.t \mid t\,u \qquad \text{(value)}$$
$$\mid\; !t \qquad \text{(value)}$$
$$\mid\; \mathtt{der}(t) \qquad \text{(computation)}$$

$$(\lambda x.t)\,!u \;\mapsto_{\beta}\; t\{x := u\}$$
$$\mathtt{der}(!t) \;\mapsto_{!}\; t$$

$$S \;::=\; \Box \mid \lambda x.S \mid S\,t \mid t\,S \mid \mathrm{der}\,(S)$$


t^N :
NAME
⟶
BANG

$$t^{N} : \text{NAME} \longrightarrow \text{BANG}$$

$$\begin{aligned} x^{N} &:= x \\ (\lambda x.t)^{N} &:= \lambda x.\, t^{N} \\ (t\,u)^{N} &:= t^{N}\, u^{N} \end{aligned}$$

$$t^{N} : \quad \text{NAME} \longrightarrow \text{BANG}$$

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$$t^{V} : \text{VALUE} \longrightarrow \text{BANG}$$

$$t^{N} : \quad \text{NAME} \longrightarrow \text{BANG}$$

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t^{N} : NAME $\longrightarrow$ BANG

$$
\begin{aligned}
x^{N} &:= x \\
(\lambda x.t)^{N} &:= \lambda x.\, t^{N} \\
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\end{aligned}
$$

t^{V} : VALUE $\longrightarrow$ BANG

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\begin{aligned}
x^{N} &:= x \\
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t\,u^{N} &:= t^{N}\,!\,u^{N}
\end{aligned}
$$

t^{V} : VALUE $\longrightarrow$ BANG

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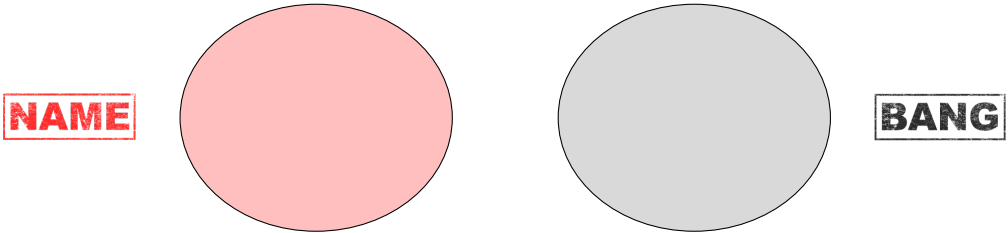
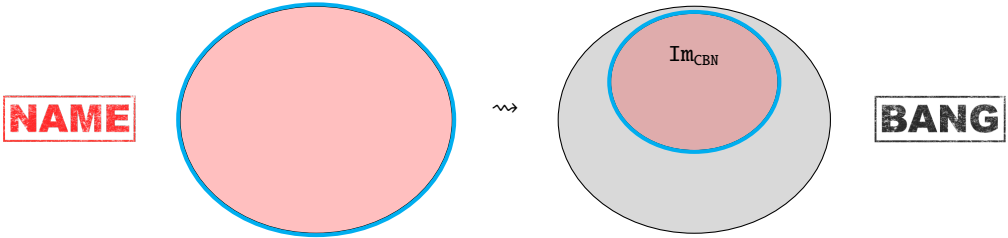
t\,u^{V} &:= \mathrm{der}(t^{V})\,u^{V}
\end{aligned}
$$

Bang Calculus: A Subsuming Paradigm



Bang Calculus: A Subsuming Paradigm



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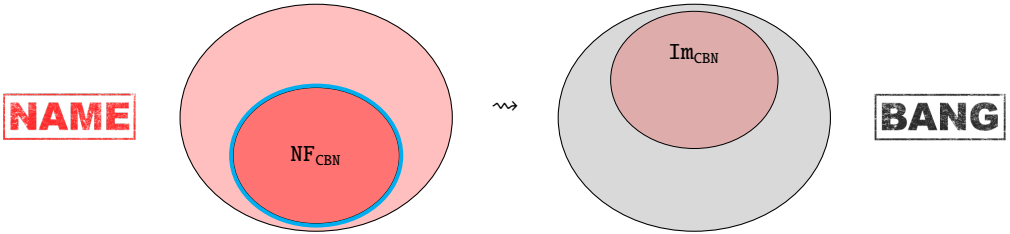
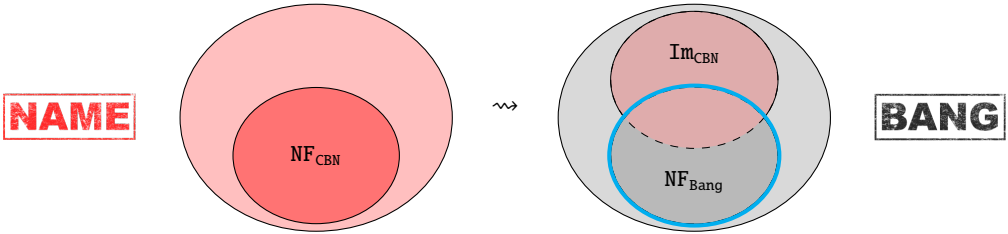


Bang Calculus: A Subsuming Paradigm



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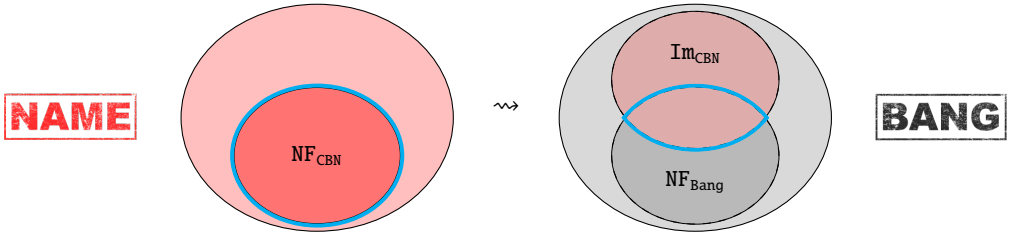
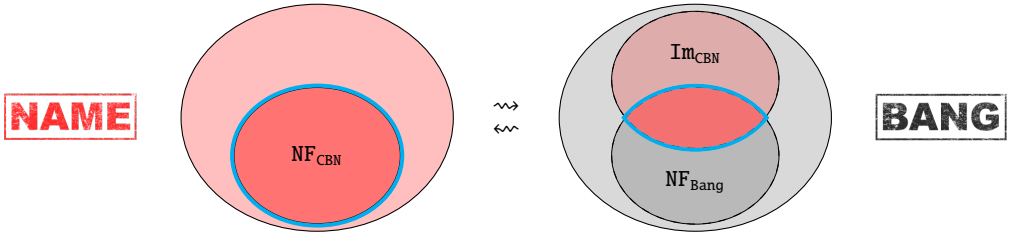


Static Properties: [EhrhardGuerrieri'16][GuerrieriManzonetto'19]

NAME t normal form $\Rightarrow$ t^{N} normal form BANG

Bang Calculus: A Subsuming Paradigm



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Bang Calculus: A Subsuming Paradigm

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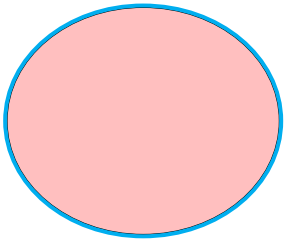

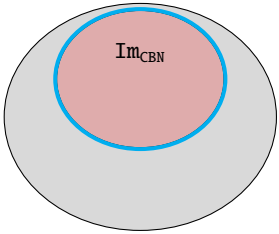

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Bang Calculus: A Subsuming Paradigm

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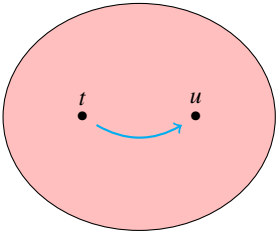

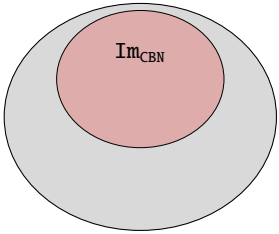

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NAME $t \twoheadrightarrow u$ BANG

Bang Calculus: A Subsuming Paradigm

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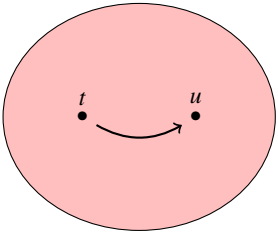

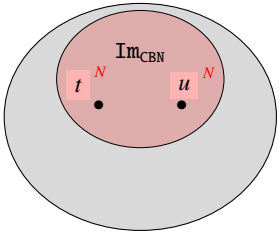

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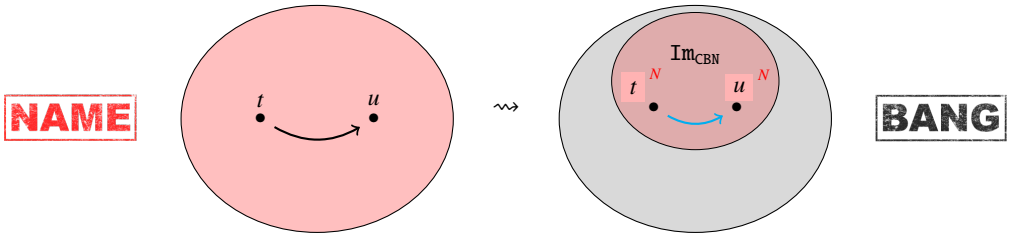

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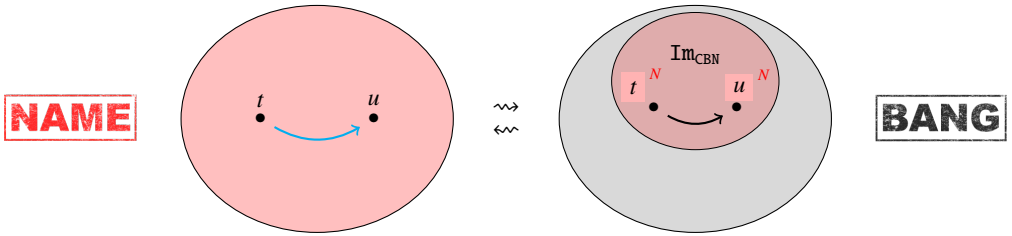
NAME $t \twoheadrightarrow u$ $\Rightarrow$ $t^{N} \twoheadrightarrow u^{N}$ BANG

Bang Calculus: A Subsuming Paradigm



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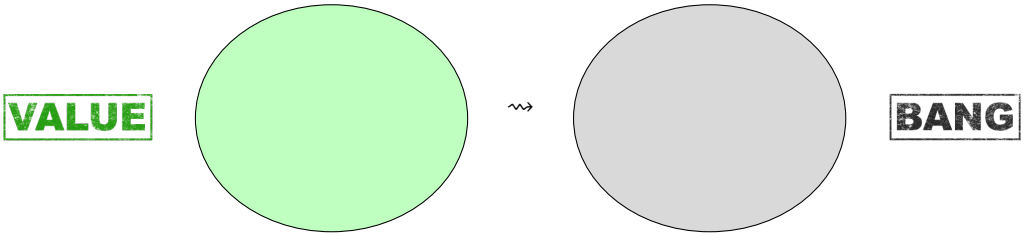
Bang Calculus: A Subsuming Paradigm



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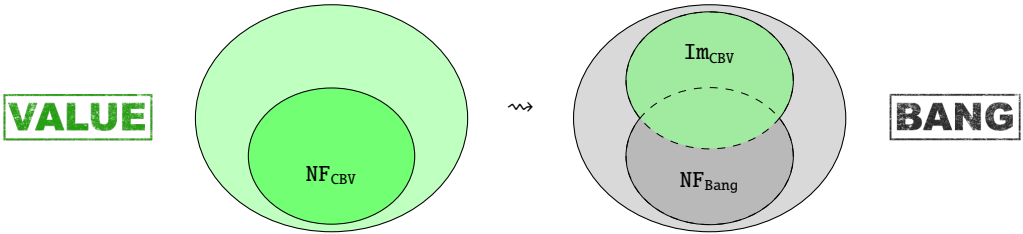
NAME $t \twoheadrightarrow u$ $\Leftrightarrow$ $t^{N} \twoheadrightarrow u^{N}$ BANG

Bang Calculus: A Subsuming Paradigm



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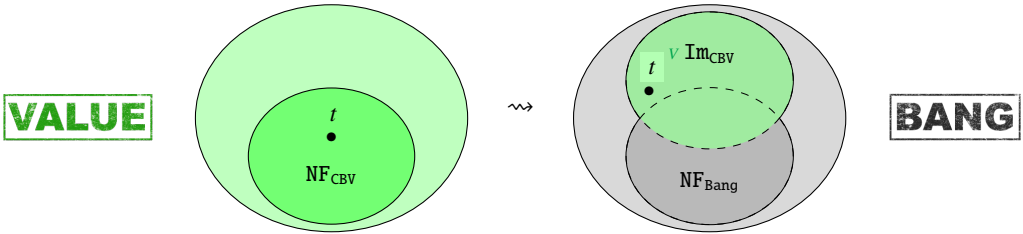

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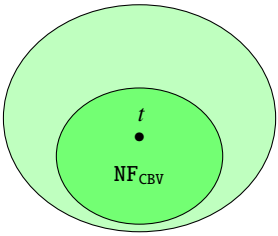
NAME $t \twoheadrightarrow u$ $\Leftrightarrow$ $t^{N} \twoheadrightarrow u^{N}$ BANG

Bang Calculus: A Subsuming Paradigm

VALUE



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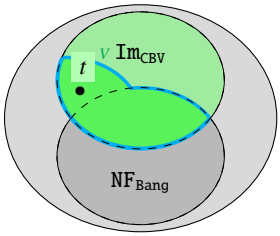


BANG

Static Properties: [EhrhardGuerrieri'16][GuerrieriManzonetto'19]

NAME	t normal form	$\Leftrightarrow$	t^N normal form	BANG
VALUE	t normal form	$\Rightarrow\!\!\!\!\Rightarrow$	t^V normal form	

Dynamic Properties: [EhrhardGuerrieri'16][GuerrieriManzonetto'19]

NAME	$t \twoheadrightarrow u$	$\Leftrightarrow$	$t^N \twoheadrightarrow u^N$	BANG

Counterexamples

$x\,y$

Counterexamples

$x\,y$

Counterexamples

$x\,y \rightsquigarrow x\,y^{V}$

Counterexamples

$$x\,y \rightsquigarrow \mathrm{der}\,(x^{\,v})\,y^{\,v}$$

Counterexamples

$x\,y \quad \rightsquigarrow \quad \mathrm{der}\,(!x)\;y^{\,V}$

Counterexamples

$$x\,y \quad \rightsquigarrow \quad \mathrm{der}\,(!x)\,(!y)$$

Counterexamples

$x\,y$ $\rightsquigarrow$ $\mathrm{der}\,(!x)\,(!y)$

$x\,(!y)$

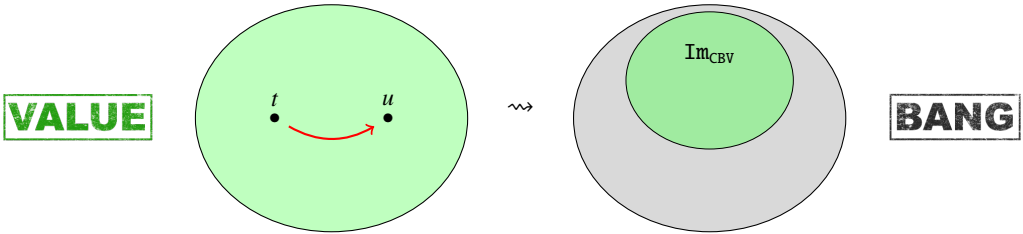
Counterexamples

$x\,y$ $\rightsquigarrow$ $\mathrm{der}\,(!x)\,(!y)$

$x\,(!y)$

Substituability

Bang Calculus: A Subsuming Paradigm



Static Properties: [EhrhardGuerrieri'16][GuerrieriManzonetto'19]

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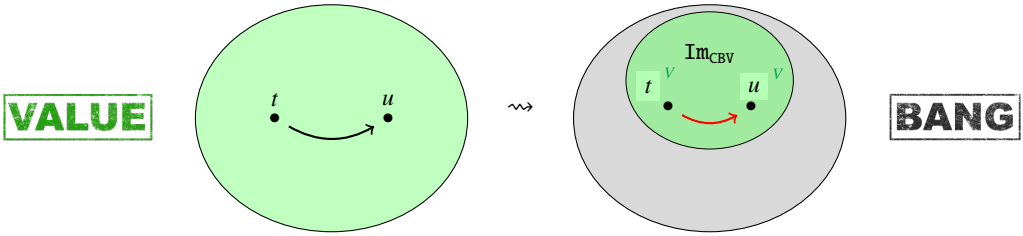
Dynamic Properties: [EhrhardGuerrieri'16][GuerrieriManzonetto'19]

NAME	$t \twoheadrightarrow u$	$\Leftrightarrow$	$t^{N} \twoheadrightarrow u^{N}$	BANG
VALUE	$t \twoheadrightarrow u$			

Bang Calculus: A Subsuming Paradigm



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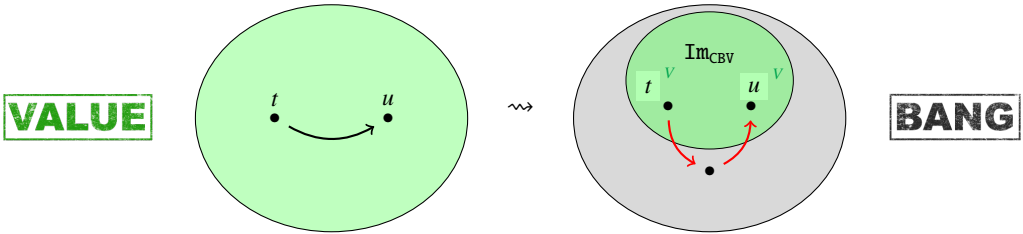
VALUE	$t \twoheadrightarrow u$	$\Rightarrow$	$t^V \twoheadrightarrow u^V$	

Bang Calculus: A Subsuming Paradigm



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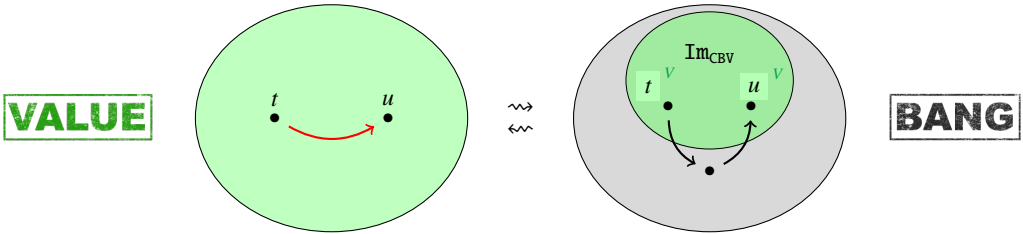
Bang Calculus: A Subsuming Paradigm



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NAME	$t \twoheadrightarrow u$	$\Leftrightarrow$	$t^N \twoheadrightarrow u^N$	BANG
VALUE	$t \twoheadrightarrow u$	$\Leftrightarrow$	$t^V \twoheadrightarrow u^V$	

Mismatch between the syntax and the semantics

Blocked redexes:

$$(\lambda x.\Delta)\,(y\,y)\,\Delta$$

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Explicit substitutions: [AccattoliKesner'10]

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$$(\lambda x.\Delta)\,(y\,y)\,\Delta \quad \to \quad \Delta[x\backslash y\,y]\,\Delta \quad \to \quad zz[z\backslash\Delta][x\backslash y\,y] \quad \to \quad ...$$

Different Models of Computation:

Call-by-Name

NAME

(Terms) $t, u \quad ::= \quad x \mid \lambda x.t \mid t\,u$

$$(\lambda x.t)\,u \quad \mapsto_\beta \quad t\{x := u\}$$

$$N \quad ::= \quad \Box \mid N\,t \mid \lambda x.N$$

Call-by-Value

VALUE

(Terms) $t, u \quad ::= \quad v \mid t\,u$
(Values) $v \quad ::= \quad x \mid \lambda x.t$

$$(\lambda x.t)\,v \quad \mapsto_{\beta_v} \quad t\{x := v\}$$

$$V \quad ::= \quad \Box \mid V\,t \mid t\,V$$

Different Models of Computation:

Call-by-Name

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(Terms) $t, u \quad ::= \quad x \mid \lambda x.t \mid t\,u \mid t[x\backslash u]$

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$$(\lambda x.t)\,v \quad \mapsto_{\beta_v} \quad t\{x := v\}$$

$$V \quad ::= \quad \Box \mid V\,t \mid t\,V$$

Different Models of Computation:

Call-by-Name

NAME

(Terms) $t, u \quad ::= \quad x \mid \lambda x.t \mid t\,u \mid t[x\backslash u]$

$$L\,\langle \lambda x.t\rangle\,u \quad \mapsto_{d\beta} \quad L\,\langle t[x\backslash u]\rangle$$

$$N \quad ::= \quad \Box \mid N\,t \mid \lambda x.N$$

Call-by-Value

VALUE

(Terms) $t, u \quad ::= \quad v \mid t\,u$

(Values) $v \quad ::= \quad x \mid \lambda x.t$

$$(\lambda x.t)\,v \quad \mapsto_{\beta_v} \quad t\{x := v\}$$

$$V \quad ::= \quad \Box \mid V\,t \mid t\,V$$

Different Models of Computation:

Call-by-Name

NAME

(Terms) $t, u \quad ::= \quad x \mid \lambda x.t \mid t\,u \mid t[x\backslash u]$

$$L\langle\lambda x.t\rangle\, u \quad \mapsto_{d\beta} \quad L\langle t[x\backslash u]\rangle$$

$$\begin{array}{lll} L & ::= & \Box \mid L[x\backslash u] \\ N & ::= & \Box \mid N\,t \mid \lambda x.N \end{array}$$

Call-by-Value

VALUE

$$\begin{array}{lll} \textbf{(Terms)} \quad t, u & ::= & v \mid t\,u \\ \textbf{(Values)} \quad v & ::= & x \mid \lambda x.t \end{array}$$

$$(\lambda x.t)\, v \quad \mapsto_{\beta_v} \quad t\{x := v\}$$

$$V \quad ::= \quad \Box \mid V\,t \mid t\,V$$

Different Models of Computation:

Call-by-Name

NAME

(Terms) $t, u \quad ::= \quad x \mid \lambda x.t \mid t\,u \mid t[x\backslash u]$

$$\begin{array}{rcl} L\langle\lambda x.t\rangle\, u & \mapsto_{d\beta} & L\langle t[x\backslash u]\rangle \\ t[x\backslash u] & \mapsto_{s} & t\{x := u\} \end{array}$$

$$\begin{array}{rcl} L & ::= & \Box \mid L[x\backslash u] \\ N & ::= & \Box \mid N\,t \mid \lambda x.N \end{array}$$

Call-by-Value

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$$\begin{array}{rcl} \textbf{(Terms)} \quad t, u & ::= & v \mid t\,u \\ \textbf{(Values)} \quad v & ::= & x \mid \lambda x.t \end{array}$$

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Call-by-Value

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$$(\lambda x.t)\,v \quad \mapsto_{\beta_v} \quad t\{x := v\}$$

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Different Models of Computation:

Call-by-Name

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(Terms) $t, u \quad ::= \quad x \mid \lambda x.t \mid t\, u \mid t[x\backslash u]$

$$\begin{array}{rcl} L\langle \lambda x.t\rangle\, u & \mapsto_{d\beta} & L\langle t[x\backslash u]\rangle \\ t[x\backslash u] & \mapsto_{s} & t\{x := u\} \end{array}$$

$$\begin{array}{rcl} L & ::= & \Box \mid L[x\backslash u] \\ N & ::= & \Box \mid N\, t \mid \lambda x.N \end{array}$$

Call-by-Value

VALUE

$$\begin{array}{rcll} \textbf{(Terms)} & t, u & ::= & v \mid t\, u \mid t[x\backslash u] \\ \textbf{(Values)} & v & ::= & x \mid \lambda x.t \end{array}$$

$$L\langle \lambda x.t\rangle\, u \quad \mapsto_{d\beta} \quad L\langle t[x\backslash u]\rangle$$

$$\begin{array}{rcl} L & ::= & \Box \mid L[x\backslash u] \\ V & ::= & \Box \mid V\, t \mid t\, V \end{array}$$

Different Models of Computation:

Call-by-Name

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(Terms) $t, u \quad ::= \quad x \mid \lambda x.t \mid t\, u \mid t[x\backslash u]$

$$\begin{array}{rcl} L\langle\lambda x.t\rangle\, u & \mapsto_{d\beta} & L\langle t[x\backslash u]\rangle \\ t[x\backslash u] & \mapsto_{s} & t\{x := u\} \end{array}$$

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Call-by-Value

VALUE

(Terms) $t, u \quad ::= \quad v \mid t\, u \mid t[x\backslash u]$
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Different Models of Computation:

Call-by-Name

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Call-by-Value

VALUE

(Terms) $t, u \quad ::= \quad v \mid t\, u \mid t[x\backslash u]$
(Values) $v \quad ::= \quad x \mid \lambda x.t$

$$\begin{array}{rcl} L\langle\lambda x.t\rangle\, u & \mapsto_{d\beta} & L\langle t[x\backslash u]\rangle \\ t[x\backslash L\langle v\rangle] & \mapsto_{sV} & L\langle t\{x := v\}\rangle \end{array}$$

$$\begin{array}{rcl} L & ::= & \Box \mid L[x\backslash u] \\ V & ::= & \Box \mid V\, t \mid t\, V \end{array}$$

Different Models of Computation:

Call-by-Name

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(Terms) $t, u \quad ::= \quad x \mid \lambda x.t \mid t\,u \mid t[x\backslash u]$

$$\begin{array}{rcl} L\langle\lambda x.t\rangle\, u & \mapsto_{d\beta} & L\langle t[x\backslash u]\rangle \\ t[x\backslash u] & \mapsto_{s} & t\{x := u\} \end{array}$$

$$\begin{array}{rcl} L & ::= & \Box \mid L[x\backslash u] \\ N & ::= & \Box \mid N\,t \mid \lambda x.N \mid N[x\backslash t] \end{array}$$

Call-by-Value

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(Terms) $t, u \quad ::= \quad v \mid t\,u \mid t[x\backslash u]$
(Values) $v \quad ::= \quad x \mid \lambda x.t$

$$\begin{array}{rcl} L\langle\lambda x.t\rangle\, u & \mapsto_{d\beta} & L\langle t[x\backslash u]\rangle \\ t[x\backslash L\langle v\rangle] & \mapsto_{sV} & L\langle t\{x := v\}\rangle \end{array}$$

$$\begin{array}{rcl} L & ::= & \Box \mid L[x\backslash u] \\ V & ::= & \Box \mid V\,t \mid t\,V \mid V[x\backslash t] \mid t[x\backslash V] \end{array}$$

Bang-Calculus

BANG

(Terms) $t, u \quad ::= \quad x \mid \lambda x.t \mid t\,u$

$\mid\ !t$ (value)

$\mid\ \mathrm{der}\,(t)$ (computation)

$$\lambda x.t\,!u \quad \mapsto_{\beta} \quad t\{x := u\}$$

$$\mathrm{der}\,(!t) \quad \mapsto_{!} \quad t$$

$$S \quad ::= \quad \Box \mid \lambda x.S \mid S\ t \mid t\,S \mid \mathrm{der}\,(S)$$

Bang-Calculus

BANG

(Terms) $t, u \quad ::= \quad x \mid \lambda x.t \mid t\,u \mid t[x\backslash u]$

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Bang-Calculus

BANG

$$
\begin{array}{lrll}
\textbf{(Terms)} & t, u & ::= & x \mid \lambda x.t \mid t\,u \mid t[x\backslash u] \\
& & \mid & !t \qquad \text{(value)} \\
& & \mid & \mathrm{der}\,(t) \qquad \text{(computation)}
\end{array}
$$

$$
\begin{array}{rcl}
L\,\langle \lambda x.t\rangle\, u & \mapsto_{\beta} & L\,\langle t[x\backslash u]\rangle \\
t[x\backslash L\,\langle !u\rangle] & \mapsto_{s!} & L\,\langle t\{x := u\}\rangle \\
\mathrm{der}\,(!t) & \mapsto_{!} & t
\end{array}
$$

$$
\begin{array}{rcl}
L & ::= & \Box \mid L[x\backslash u] \\
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Bang-Calculus

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\begin{array}{lrll}
\textbf{(Terms)} & t, u & ::= & x \mid \lambda x.t \mid t\,u \mid t[x\backslash u] \\
& & \mid & !t \qquad\qquad \text{(value)} \\
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L\,\langle \lambda x.t\rangle\, u & \mapsto_{\beta} & L\,\langle t[x\backslash u]\rangle \\
t[x\backslash L\,\langle !u\rangle] & \mapsto_{s!} & L\,\langle t\{x := u\}\rangle \\
\mathrm{der}\,(L\,\langle !t\rangle) & \mapsto_{!} & L\,\langle t\rangle
\end{array}
$$

$$
\begin{array}{rcl}
L & ::= & \Box \mid L[x\backslash u] \\
S & ::= & \Box \mid \lambda x.S \mid S\ t \mid t\,S \mid \mathrm{der}\,(S)
\end{array}
$$

Bang-Calculus

BANG

$$
\begin{array}{lrll}
\textbf{(Terms)} & t, u & ::= & x \mid \lambda x.t \mid t\,u \mid t[x\backslash u] \\
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\end{array}
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L\,\langle \lambda x.t\rangle\, u & \mapsto_{\beta} & L\,\langle t[x\backslash u]\rangle \\
t[x\backslash L\,\langle !u\rangle] & \mapsto_{s!} & L\,\langle t\{x := u\}\rangle \\
\mathrm{der}\,(L\,\langle !t\rangle) & \mapsto_{!} & L\,\langle t\rangle
\end{array}
$$

$$
\begin{array}{rcl}
L & ::= & \Box \mid L[x\backslash u] \\
S & ::= & \Box \mid \lambda x.S \mid S\ t \mid t\,S \mid \mathrm{der}\,(S) \mid S\,[x\backslash t] \mid t[x\backslash S\,]
\end{array}
$$

$t^{N} : \text{NAME} \longrightarrow \text{BANG}$

$$
\begin{aligned}
x^{N} &:= x \\
(\lambda x.t)^{N} &:= \lambda x.\, t^{N} \\
(t\,u)^{N} &:= t^{N}\, !\, u^{N}
\end{aligned}
$$

$t^{V} : \text{VALUE} \longrightarrow \text{BANG}$

$$
\begin{aligned}
x^{V} &:= !x \\
(\lambda x.t)^{V} &:= !\lambda x.\, t^{V} \\
(t\,u)^{V} &:= \mathrm{der}\,(t^{V})\, u^{V}
\end{aligned}
$$

t^{N} : NAME $\longrightarrow$ BANG

$$
\begin{array}{rcl}
x^{N} & := & x \\
(\lambda x.t)^{N} & := & \lambda x.\, t^{N} \\
(t\,u)^{N} & := & t^{N}\, !\, u^{N} \\
(t[x\backslash u])^{N} & := & t^{N}[x\backslash\ u^{N}]
\end{array}
$$

t^{V} : VALUE $\longrightarrow$ BANG

$$
\begin{array}{rcl}
x^{V} & := & !x \\
(\lambda x.t)^{V} & := & !\lambda x.\, t^{V} \\
(t\,u)^{V} & := & \mathrm{der}\,(t^{V})\, u^{V}
\end{array}
$$

$$t^{N} : \text{NAME} \longrightarrow \text{BANG}$$

$$\begin{aligned} x^{N} &:= x \\ (\lambda x.t)^{N} &:= \lambda x.\, t^{N} \\ (t\,u)^{N} &:= t^{N}\, !\, u^{N} \\ t[x\backslash u]^{N} &:= t^{N}[x\backslash !\, u^{N}] \end{aligned}$$

$$t^{V} : \text{VALUE} \longrightarrow \text{BANG}$$

$$\begin{aligned} x^{V} &:= !x \\ (\lambda x.t)^{V} &:= !\lambda x.\, t^{V} \\ (t\,u)^{V} &:= \mathrm{der}\,(t^{V})\, u^{V} \end{aligned}$$

$$t^{N} : \text{NAME} \longrightarrow \text{BANG}$$

$$\begin{aligned} x^{N} &:= x \\ \lambda x.t^{N} &:= \lambda x.\, t^{N} \\ t\,u^{N} &:= t^{N}\, !\, u^{N} \\ t[x\backslash u]^{N} &:= t^{N}[x\backslash !\, u^{N}] \end{aligned}$$

$$t^{V} : \text{VALUE} \longrightarrow \text{BANG}$$

$$\begin{aligned} x^{V} &:= !x \\ \lambda x.t^{V} &:= !\lambda x.\, t^{V} \\ t\,u^{V} &:= \mathrm{der}\,(t^{V})\, u^{V} \\ t[x\backslash u]^{V} &:= t^{V}[x\backslash u^{V}] \end{aligned}$$

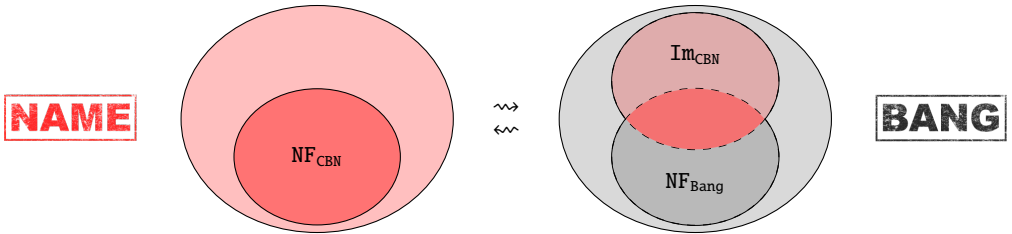
$$t^{N} : \text{NAME} \longrightarrow \text{BANG}$$

$$\begin{aligned} x^{N} &:= x \\ (\lambda x.t)^{N} &:= \lambda x.\, t^{N} \\ (t\,u)^{N} &:= t^{N}\, !\, u^{N} \\ t[x\backslash u]^{N} &:= t^{N}[x\backslash !\, u^{N}] \end{aligned}$$

$$t^{V} : \text{VALUE} \longrightarrow \text{BANG}$$

$$\begin{aligned} x^{V} &:= !x \\ (\lambda x.t)^{V} &:= !\lambda x.\, t^{V} \\ (t\,u)^{V} &:= \mathrm{der}\,(t^{V})\, u^{V} \quad + \texttt{superdevelopment} \\ t[x\backslash u]^{V} &:= t^{V}[x\backslash u^{V}] \end{aligned}$$

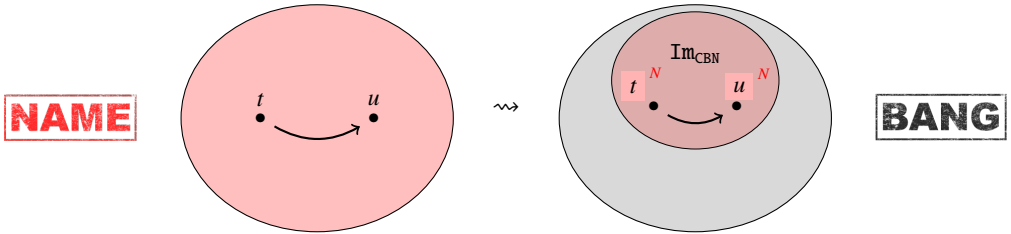
Distant Bang Calculus: A Subsuming Paradigm



Static Properties: [BucciarelliKesnerRiosViso'20,'23]

NAME t normal form $\Leftrightarrow$ t^{N} normal form BANG

Distant Bang Calculus: A Subsuming Paradigm



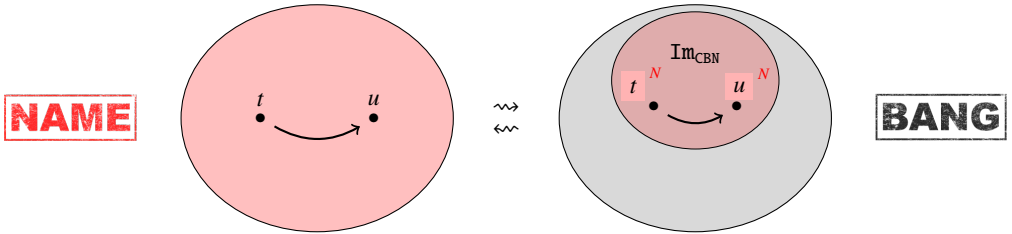
Static Properties: [BucciarelliKesnerRiosViso'20,'23]

NAME t normal form $\Leftrightarrow$ t^{N} normal form BANG

Dynamic Properties: [BucciarelliKesnerRiosViso'20,'23]

NAME $t \twoheadrightarrow u$ $\Rightarrow$ $t^{N} \twoheadrightarrow u^{N}$ BANG

Distant Bang Calculus: A Subsuming Paradigm



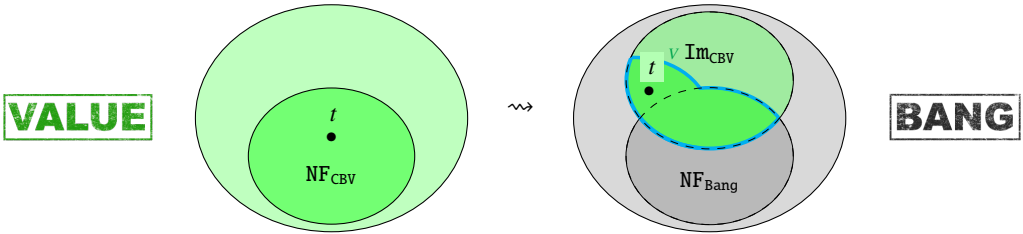
Static Properties: [BucciarelliKesnerRiosViso'20,'23]

NAME t normal form $\Leftrightarrow$ t^{N} normal form BANG

Dynamic Properties: [BucciarelliKesnerRiosViso'20,'23]

NAME $t \twoheadrightarrow u \Leftrightarrow t^{N} \twoheadrightarrow u^{N}$ BANG

Distant Bang Calculus: A Subsuming Paradigm



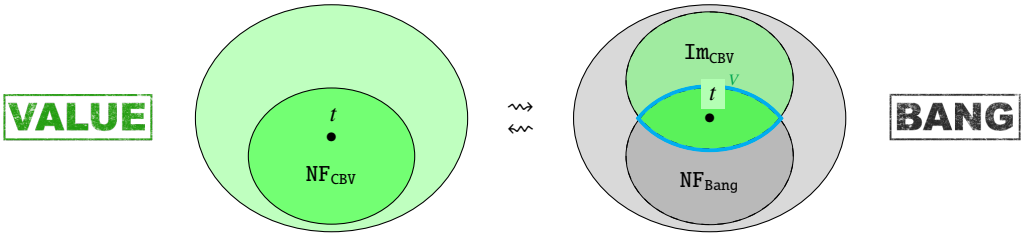
Static Properties: [BucciarelliKesnerRíosViso'20,'23]

NAME	t normal form	$\Leftrightarrow$	t^N normal form	BANG
VALUE	t normal form	$\Rightarrow\!\!\!\!\Rightarrow$	t^V normal form	

Dynamic Properties: [BucciarelliKesnerRíosViso'20,'23]

NAME	$t \twoheadrightarrow u$	$\Leftrightarrow$	$t^N \twoheadrightarrow u^N$	BANG

Distant Bang Calculus: A Subsuming Paradigm



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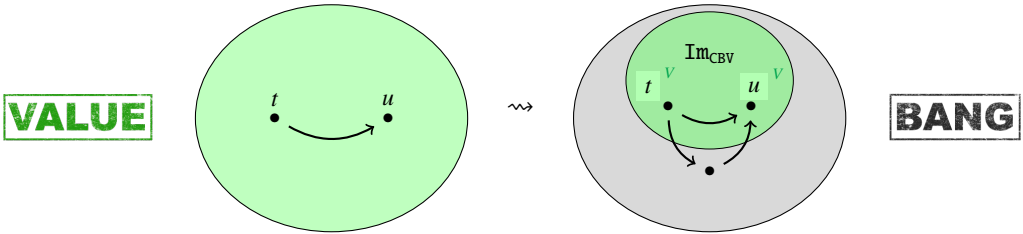

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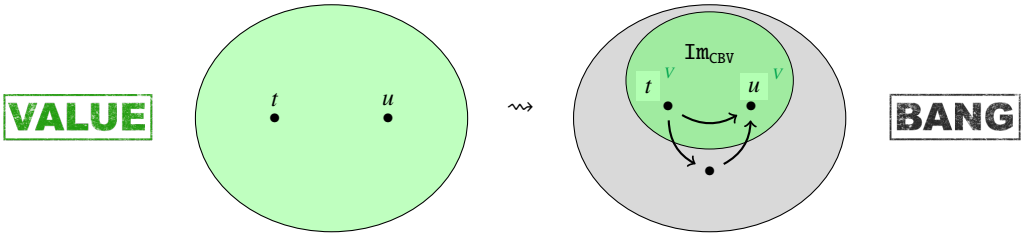

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Counterexamples

$x\,y$ $\rightsquigarrow$ $\mathrm{der}\,(!x)\,(!y)$

$x\,(!y)$

$(\lambda x.\Omega)\,y$

Substituability

Counterexamples

$x\,y \rightsquigarrow \mathrm{der}\,(!x)\,(!y)$

VALUE

$x\,(!y)$

BANG

$(\lambda x.\Omega)\,y \rightsquigarrow (\lambda x.\Omega)\,y\ ^{V}$

VALUE

BANG

Substituability

Counterexamples

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BANG

$(\lambda x.\Omega)\,y \rightsquigarrow \mathrm{der}\,(\,\lambda x.\Omega\,^{V}\,)\;y\,^{V}$

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$(\lambda x.\Omega)\,y \rightsquigarrow (\lambda x.\ \Omega\ ^{V})\,!y$

VALUE

$(\lambda x.(z!z)[z\backslash!\ \Delta\ ^{V}])\,(!y)$

BANG

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Substituability VS Accessibility

CbN and CbV Embeddings

$$t^{N} : \text{NAME} \longrightarrow \text{BANG}$$

$$\begin{aligned}
x^{N} &:= x \\
(\lambda x.t)^{N} &:= \lambda x.\, t^{N} \\
(t\,u)^{N} &:= t^{N}\, !\, u^{N} \\
t[x\backslash u]^{N} &:= t^{N}[x\backslash !\, u^{N}]
\end{aligned}$$

$$t^{V} : \text{VALUE} \longrightarrow \text{BANG}$$

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x^{V} &:= !x \\
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(t\,u)^{V} &:= \mathrm{der}\,(t^{V})\, u^{V} \quad + \texttt{superdevelopment} \\
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Distant Bang Calculus: A Subsuming Paradigm

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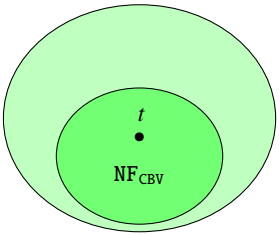

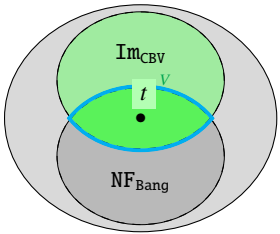

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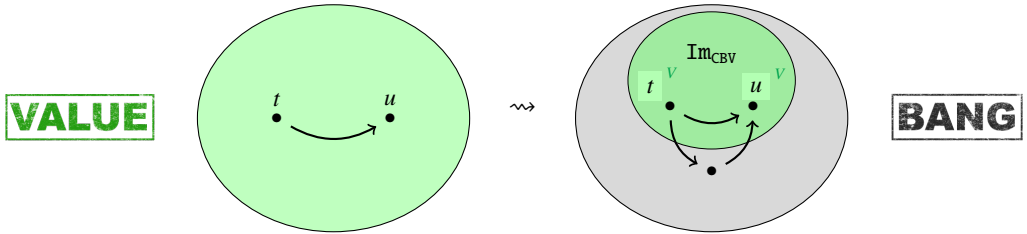

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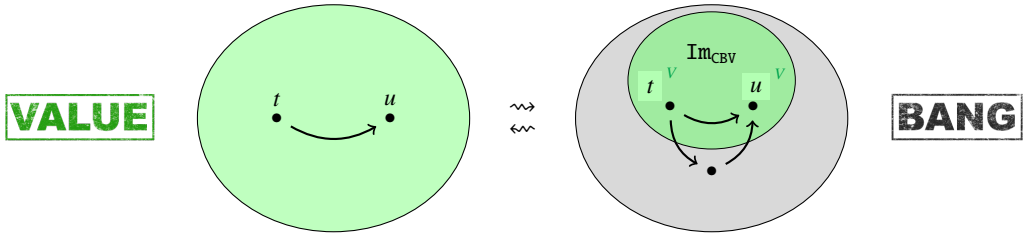

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  - Simulation
  - Reverse simulation

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Thank you for your attention !