

Genericity Through Stratification

Victor Arrial

Université Paris Cité
Paris

Giulio Guerrieri

University of Sussex
Brighton

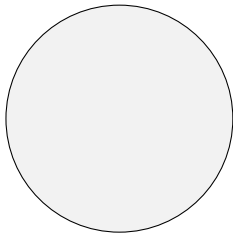
Delia Kesner

Université Paris Cité
Paris

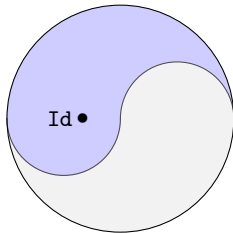
Logic in Computer Science
(LICS)

Tallinn, July 8, 2024

Meaningfulness: a Question of Taste ?

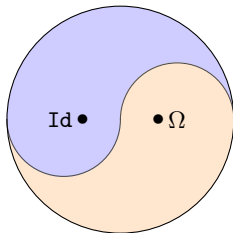


MEANINGFUL



Meaningfulness: a Question of Taste ?

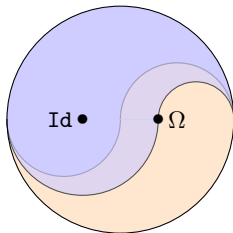
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Meaningfulness: a Question of Taste ?

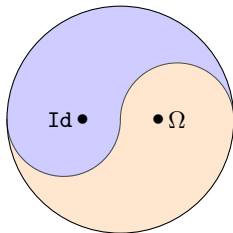
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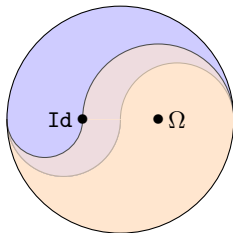
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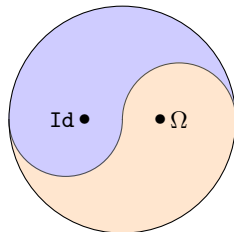
MEANINGFUL



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Meaningfulness: a Question of Taste ?

MEANINGFUL



MEANINGLESS

Key properties:

- (Operational and Logical Characterizations)
- Genericity Lemmas
- Consistency when equating all meaningless terms

Call-by-Name

NAME

Call-by-Value

VALUE

Call-by-Name

NAME

$(\lambda x.y \Omega) \Omega$

Call-by-Value

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Call-by-Name

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Call-by-Value

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Call-by-Value

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Call-by-Value

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Call-by-Name

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MEANINGFUL

\exists testing context T , $T\langle t \rangle \rightarrow^* \text{Obs}$

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$\exists T, T\langle t \rangle \rightarrow^* \text{Id}$

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MEANINGLESS

$\lambda x.\Omega, \Omega$

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$\text{Id}, \lambda x.\Omega$

MEANINGLESS

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Call-by-Name

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Call-by-Value

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Lemma (Lifting)

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$(\lambda x.y \ \Omega) \ \perp$

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$y \ \perp$

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Call-by-Value

VALUE

$(\lambda x.y \ \Omega) \ \Omega$

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Lemma (Lifting)

VALUE

t

$\forall I$

$t \longrightarrow u$

Call-by-Name

NAME

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\downarrow

$y \ \Omega$

\swarrow

$(\lambda x. y \ \Omega) \ \perp$

\downarrow

$y \ \perp$

\swarrow

Call-by-Value

VALUE

$(\lambda x. y \ \Omega) \ \Omega$

\downarrow

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\downarrow

\vdots

$(\lambda x. \perp) \ \Omega$

\downarrow

$(\lambda x. \perp) \ \Omega$

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\vdots

Lemma (Lifting)

VALUE

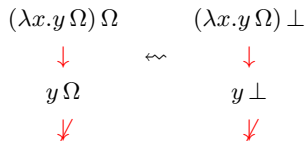
$t \longrightarrow u$

$\forall I \quad \{ \quad \forall I$

$\mathbf{t} \longrightarrow \mathbf{u}$

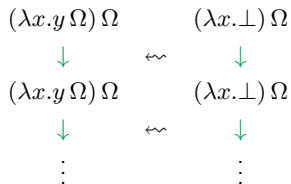
Call-by-Name

NAME



Call-by-Value

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Lemma (Lifting)

VALUE



Meaningful (Quasi) Approximation:

$$t \text{ \textcolor{brown}{MEANINGLESS} } \rightsquigarrow \mathcal{QA}(t) := \perp$$

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$$t \text{ \textcolor{brown}{MEANINGLESS}} \rightsquigarrow \mathcal{QA}(t) := \perp$$

Lemma (Approximation)

VALUE

$$\begin{array}{ccc} t & \xrightarrow{\quad\quad\quad} & u \\ \text{VI} & & \text{VI} \\ \mathcal{QA}(t) & & \mathcal{QA}(u) \end{array}$$

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Meaningful (Quasi) Approximation:

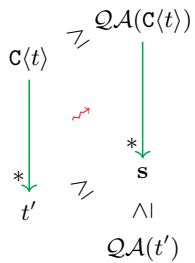
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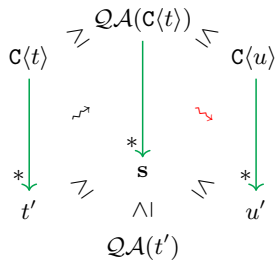
Lemma (Approximation)

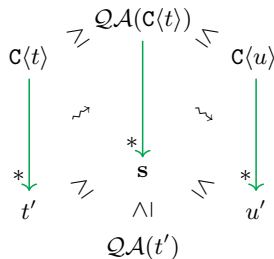
VALUE

$$\begin{array}{ccc} t & \xrightarrow{\quad} & u \\ \text{VI} & \downarrow & \text{VI} \\ \mathcal{QA}(t) & \xrightarrow{\quad*} & \textcolor{red}{u} \geq \mathcal{QA}(u) \end{array}$$

$$\begin{array}{c} \mathbf{C}\langle t \rangle \\ \downarrow * \\ t' \end{array}$$



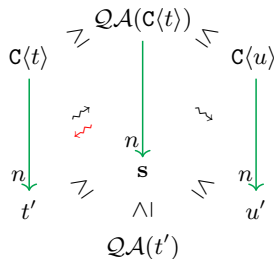




Theorem ((Surface) Genericity)

VALUE

Let t be **MEANINGLESS** such that $C\langle t \rangle$ has a \rightarrow -normal form. Then, there exists $n \in \mathbb{N}$ such that for any $u \in \Lambda$, there exists \rightarrow -normal forms t', u' such that $C\langle t \rangle \rightarrow^* t'$ and $C\langle u \rangle \rightarrow^* u'$.

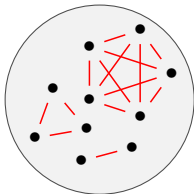


Theorem (Quantitative (Surface) Genericity)

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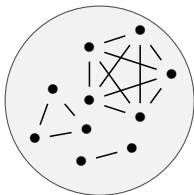
Theories: **Equivalence** relations on Λ .



Theories of the λ -calculus

Theories: Equivalence relations on Λ .

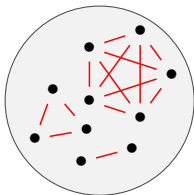
λ_v -**theories:** Contextually closed theory containing β_v .



Theories of the λ -calculus

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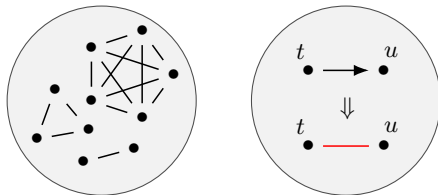
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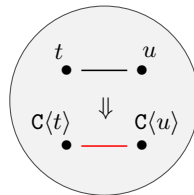
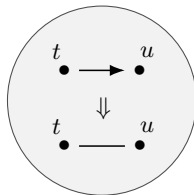
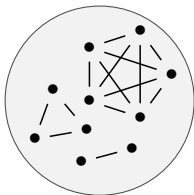
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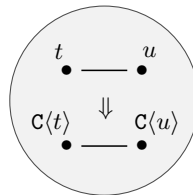
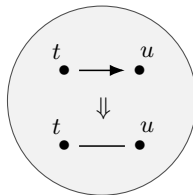
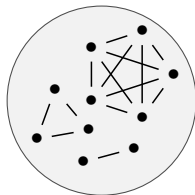
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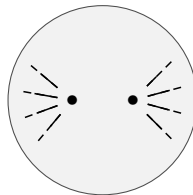
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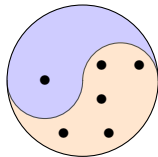
Consistent: There exists two **distinct** points.



Theory \mathcal{H}_v : Smallest Sensible λ_v -Theory

Sensible Theory: λ_v -theory equating all **MEANINGLESS** terms.

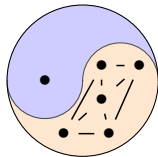
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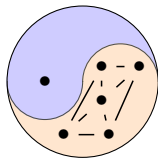


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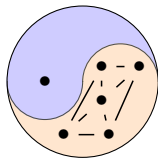
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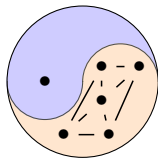
Theorem (Full Genericity)

VALUE

Let $C\langle t \rangle \rightarrow_{\omega}^* s$ with t **MEANINGLESS** and s a \rightarrow_{ω} -normal form, then for any $u \in \Lambda$, $C\langle u \rangle \rightarrow_{\omega}^* s$.

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Theorem

VALUE

The theory \mathcal{H}_v is consistent.

Theory \mathcal{H}_v^* : $t \multimap u$

Theory \mathcal{H}_v^* : $t \sim u$ when $\forall C, C\langle t \rangle$ **MEANINGFUL** iff $C\langle u \rangle$ **MEANINGFUL**.

Theory \mathcal{H}_v^* : $t \text{ --- } u$ when $\forall C, C\langle t \rangle$ **MEANINGFUL** iff $C\langle u \rangle$ **MEANINGFUL**.

Corollary

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The theory \mathcal{H}_v^ extends \mathcal{H}_v .*

Theory \mathcal{H}_v^* : Maximal Consistent Extension of \mathcal{H}_v

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Summary:

- Novel simple technique to prove Stratified Quantitative Genericity
- Generalizes Surface and Full Genericity
- Consistency of theories \mathcal{H} and \mathcal{H}^* (and coincides with observational equivalence).
- Applies to both CBN and CBV without any trick

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Further questions and future work:

- Dynamic approximations for other properties
- Meaningfulness in Call-by-Need
- Criteria or key elements of meaningfulness ?

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History of meaningfulness in call-by-value \rightsquigarrow **TLLA - Today 16h50**

Meaningfulness in a unifying paradigm \rightsquigarrow **FSCD - Friday 14h00**

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