Quantitative Inhabitation for Different Lambda Calculi in a Unifying Framework

Victor Arrial [1]   Giulio Guerrieri [2,3]   Delia Kesner [1,4]

[1] Université Paris Cité, Paris   [2] Aix Marseille Univ, Marseille

[3] Edinburgh Research Centre, Huawei, Edinburgh

[4] Institut Universitaire de France

Marseille - I2M, May 4, 2023

Different Models of Computation:

Call-by-Name — NAME

Call-by-Value — VALUE

Unifying Frameworks:

- Call-by-Push-Value **[Levy'99]**
- **Distant Bang Calculus [EG'16] [BKRV'20]**:

$$\begin{array}{rcll} t, u & ::= & x \mid \lambda x.t \mid tu & \\ & \mid & !\,t & \texttt{Values} \\ & \mid & \mathtt{der}(\,t\,) & \texttt{Computations} \\ & \mid & t\,[\,x := u\,] & \texttt{Let} \end{array}$$

BANG

Le λ!-calcul : Syntax and Operational Semantics

$$t, u \ ::= \ x \in \mathcal{V} \mid tu \mid \lambda x.u$$

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$$t, u \ ::= \ x \in \mathcal{V} \mid tu \mid \lambda x.u \mid !u \quad \text{Value}$$

Le λ!-calcul : Syntax and Operational Semantics

$$t, u ::= x \in \mathcal{V} \mid tu \mid \lambda x.u \mid !u \mid \mathtt{der}(u)$$

Le λ!-calcul : Syntax and Operational Semantics

$$t, u \ ::= \ x \in \mathcal{V} \mid tu \mid \lambda x.u \mid !u \mid \mathtt{der}(u) \mid u[x := v]$$

Le λ!-calcul : Syntax and Operational Semantics

$$t, u \ ::= \ x \in \mathcal{V} \mid tu \mid \lambda x.u \mid !u \mid \mathtt{der}(u) \mid u[x := v]$$

Reduction :

Le λ!-calcul : Syntax and Operational Semantics

$$t, u \ ::= \ x \in \mathcal{V} \mid tu \mid \lambda x.u \mid !u \mid \mathtt{der}(u) \mid u[x := v]$$

Reduction :

$$(\lambda x.t)\, u$$

Le λ!-calcul : Syntax and Operational Semantics

$$t, u ::= x \in \mathcal{V} \mid tu \mid \lambda x.u \mid {!u} \mid \mathtt{der}(u) \mid u[x := v]$$

Reduction :

$$(\lambda x.t)\, u \quad \mapsto_{dB} \qquad t[x := u]$$

Le λ!-calcul : Syntax and Operational Semantics

$$t, u \ ::= \ x \in \mathcal{V} \mid tu \mid \lambda x.u \mid !u \mid \texttt{der}(u) \mid u[x := v]$$

Reduction :

$$(\lambda x.t)\, u \quad \mapsto_{dB} \qquad t[x := u]$$

$$t[x := \quad (!u)]$$

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$$t, u ::= x \in \mathcal{V} \mid tu \mid \lambda x.u \mid !u \mid \mathtt{der}(u) \mid u[x := v]$$

Reduction :

$$\begin{array}{rcl} (\lambda x.t)\, u & \mapsto_{dB} & t[x := u] \\ t[x := (!u)] & \mapsto_{s!} & t\{x := u\} \end{array}$$

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Contexts :

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Contexts :

$$\mathtt{S} \ ::= \ \diamond \mid \lambda x.\mathtt{S} \mid \mathtt{S}\ t \mid t\ \mathtt{S} \mid \mathtt{S}[x := t] \mid t[x := \mathtt{S}] \mid \mathtt{der}(\mathtt{S})$$

Le λ!-calcul : Syntax and Operational Semantics

$$t, u \ ::= \ x \in \mathcal{V} \mid tu \mid \lambda x.u \mid !u \mid \mathtt{der}(u) \mid u[x := v]$$

Reduction :

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$$\begin{array}{rcl} \mathtt{L} & ::= & \diamond \mid \mathtt{L}[x := t] \\ \mathtt{S} & ::= & \diamond \mid \lambda x.\mathtt{S} \mid \mathtt{S}\ t \mid t\ \mathtt{S} \mid \mathtt{S}[x := t] \mid t[x := \mathtt{S}] \mid \mathtt{der}(\mathtt{S}) \end{array}$$

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Reduction :

$$\begin{array}{rcl} \mathtt{L}\langle \lambda x.t \rangle\, u & \mapsto_{dB} & \mathtt{L}\langle t[x := u] \rangle \\ t[x := (!u)] & \mapsto_{s!} & t\{x := u\} \\ \mathtt{der}(!t) & \mapsto_{d!} & t \end{array}$$

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Le $\lambda!$-calcul : Syntax and Operational Semantics

$$t, u \ ::= \ x \in \mathcal{V} \mid tu \mid \lambda x.u \mid !u \mid \mathtt{der}(u) \mid u[x := v]$$

Reduction :

$$\begin{array}{rcl} \mathtt{L}\langle \lambda x.t \rangle\, u & \mapsto_{dB} & \mathtt{L}\langle t[x := u] \rangle \\ t[x := \mathtt{L}\langle !u \rangle] & \mapsto_{s!} & \mathtt{L}\langle t\{x := u\} \rangle \\ \mathtt{der}(\mathtt{L}\langle !t \rangle) & \mapsto_{d!} & \mathtt{L}\langle t \rangle \end{array}$$

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Contexts :

$$\begin{array}{lcl} \mathtt{L} & ::= & \diamond \mid \mathtt{L}[x := t] \\ \mathtt{S} & ::= & \diamond \mid \lambda x.\mathtt{S} \mid \mathtt{S}\; t \mid t\; \mathtt{S} \mid \mathtt{S}[x := t] \mid t[x := \mathtt{S}] \mid \mathtt{der}(\mathtt{S}) \\ \mathtt{F} & ::= & \diamond \mid \lambda x.\mathtt{F} \mid \mathtt{F}\; t \mid t\; \mathtt{F} \mid \mathtt{F}[x := t] \mid t[x := \mathtt{F}] \mid \mathtt{der}(\mathtt{F}) \mid !\mathtt{F} \end{array}$$

Distant Bang: A Subsuming Paradigm

t^N : NAME $\rightarrow$ BANG

t^V : VALUE $\rightarrow$ BANG

Static Properties: **[BKRV'20]**

NAME	t normal form	$\Leftrightarrow$	t^N normal form	BANG
VALUE	t normal form	$\Leftrightarrow$	t^V normal form	

Dynamic Properties: **[BKRV'20]**

NAME	$t \twoheadrightarrow u$	$\Leftrightarrow$	$t^N \twoheadrightarrow u^N$	BANG
VALUE	$t \twoheadrightarrow u$	$\Leftrightarrow$	$t^V \twoheadrightarrow u^V$	

Can we do the same thing with inhabitation ?

$$t^N : \quad \text{NAME} \rightarrow \text{BANG}$$

$$
\begin{aligned}
x^N &:= \\
(\lambda x.t)^N &:= \\
(tu)^N &:= \\
(t[x := u])^N &:=
\end{aligned}
$$

t^N : NAME $\to$ BANG

$$x^N := x$$

$$(\lambda x.t)^N := \lambda x.\, t^N$$

$$(tu)^N :=$$

$$(t[x := u])^N :=$$

$$t^{N} : \text{NAME} \to \text{BANG}$$

$$
\begin{aligned}
x^{N} &:= x \\
(\lambda x.t)^{N} &:= \lambda x.\, t^{N} \\
(tu)^{N} &:= t^{N}\, !\, u^{N} \\
(t[x := u])^{N} &:= t^{N}[x := \, !\, u^{N}]
\end{aligned}
$$

$\boxed{t}^{N}$: **NAME** $\rightarrow$ **BANG**

$$\boxed{x}^{N} := x$$
$$\boxed{\lambda x.t}^{N} := \lambda x.\boxed{t}^{N}$$
$$\boxed{tu}^{N} := \boxed{t}^{N}\,!\,\boxed{u}^{N}$$
$$\boxed{t[x := u]}^{N} := \boxed{t}^{N}[x := !\,\boxed{u}^{N}]$$

$\boxed{t}^{V}$: **VALUE** $\rightarrow$ **BANG**

$$\boxed{x}^{V} :=$$
$$\boxed{\lambda x.t}^{V} :=$$
$$\boxed{tu}^{V} :=$$
$$\boxed{t[x := u]}^{V} :=$$

$$t^N : \text{NAME} \to \text{BANG}$$

$$\begin{aligned}
x^N &:= x \\
(\lambda x.t)^N &:= \lambda x.\, t^N \\
(tu)^N &:= t^N \,!\, u^N \\
(t[x := u])^N &:= t^N [x := !\, u^N]
\end{aligned}$$

$$t^V : \text{VALUE} \to \text{BANG}$$

$$\begin{aligned}
x^V &:= \\
(\lambda x.t)^V &:= \\
(tu)^V &:= \quad t^V \; u^V \\
(t[x := u])^V &:= t^V [x := u^V]
\end{aligned}$$

t^{N} : **NAME** $\rightarrow$ **BANG**

$$x^{N} := x$$

$$(\lambda x.t)^{N} := \lambda x.\, t^{N}$$

$$(tu)^{N} := t^{N}\, !\, u^{N}$$

$$(t[x := u])^{N} := t^{N}[x := !\, u^{N}]$$

t^{V} : **VALUE** $\rightarrow$ **BANG**

$$x^{V} := !\, x$$

$$(\lambda x.t)^{V} := !\, \lambda x.\, t^{V}$$

$$(tu)^{V} := t^{V}\, u^{V}$$

$$(t[x := u])^{V} := t^{V}[x := u^{V}]$$

$$
\begin{aligned}
\boxed{t}^{N} :\quad & \textbf{NAME} && \rightarrow && \textbf{BANG} \\
\boxed{x}^{N} & && := && x \\
\boxed{\lambda x.t}^{N} & && := && \lambda x.\boxed{t}^{N} \\
\boxed{tu}^{N} & && := && \boxed{t}^{N}\,!\,\boxed{u}^{N} \\
\boxed{t[x := u]}^{N} & && := && \boxed{t}^{N}[x := !\,\boxed{u}^{N}]
\end{aligned}
$$

$$
\begin{aligned}
\boxed{t}^{V} :\quad & \textbf{VALUE} && \rightarrow && \textbf{BANG} \\
\boxed{x}^{V} & && := && !\,x \\
\boxed{\lambda x.t}^{V} & && := && !\,\lambda x.\boxed{t}^{V} \\
\boxed{tu}^{V} & && := && \mathtt{der}(\boxed{t}^{V})\,\boxed{u}^{V} \\
\boxed{t[x := u]}^{V} & && := && \boxed{t}^{V}[x := \boxed{u}^{V}]
\end{aligned}
$$

Three Typing Systems: **[BKRV'20]**

NAME : $\mathcal{N}$ VALUE : $\mathcal{V}$ BANG : $\mathcal{B}$

Static Properties: **[BKRV'20]**

NAME $\Gamma \vdash_{\mathcal{N}} t : \sigma \iff \Gamma \vdash_{\mathcal{B}} t^{N} : \sigma$

VALUE $\Gamma \vdash_{\mathcal{V}} t : \sigma \iff \Gamma \vdash_{\mathcal{B}} t^{V} : \sigma$

BANG

Non-Idempotent Intersection Types = Multitypes

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Notation : $[\sigma_1 \, , \, \cdots \, , \sigma_n]$ symbolises $\sigma_1 \cap \, \cdots \, \cap \sigma_n$

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Sequent : $\Gamma \vdash t : \sigma$

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Sequent : $\Gamma \vdash t : \sigma$

Rules :

$$\frac{\Gamma_1 \vdash t : \mathcal{M} \Rightarrow \sigma \qquad \Gamma_2 \vdash u : \mathcal{M}}{\Gamma_1 + \Gamma_2 \vdash tu : \sigma} \; \texttt{(app)}$$

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$$\frac{(\Gamma_i \vdash t : \sigma_i)_{i \in I}}{+_i \Gamma_i \vdash !t : [\sigma_i]_{i \in I}} \; \texttt{(bang)}$$

Non-Idempotent Intersection Types = Multitypes

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Rules :

$$\frac{\Gamma_1 \vdash t : \mathcal{M} \Rightarrow \sigma \qquad \Gamma_2 \vdash u : \mathcal{M}}{\Gamma_1 + \Gamma_2 \vdash tu : \sigma} \ \texttt{(app)}$$

$$\frac{}{\emptyset \vdash !t : [\,]} \ \texttt{(bang)}$$

Non-Idempotent Intersection Types = Multitypes

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Example :

Non-Idempotent Intersection Types = Multitypes

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Sequent : $\Gamma \vdash t : \sigma$

Rules :

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$$\frac{(\Gamma_i \vdash t : \sigma_i)_{i \in I}}{+_i \Gamma_i \vdash !t : [\sigma_i]_{i \in I}} \ \texttt{(bang)}$$

Example : $\emptyset \vdash \lambda x.x!x : [\tau,\ [\tau] \Rightarrow \sigma] \Rightarrow \sigma$

Solving the Inhabitation Problem - Methodology

Instead of **just one** solution:

$$\Gamma \vdash \mathbf{t} : \sigma$$

We want to compute **all** solutions:

$$\texttt{Sol}(\Gamma,\sigma) \ := \ \{\, \mathbf{t} \mid \Gamma \vdash \mathbf{t} : \sigma \,\}$$

Problem

- The set $\texttt{Sol}(\Gamma,\sigma)$ is either empty of infinite **BANG**

We compute a **finite** generator:

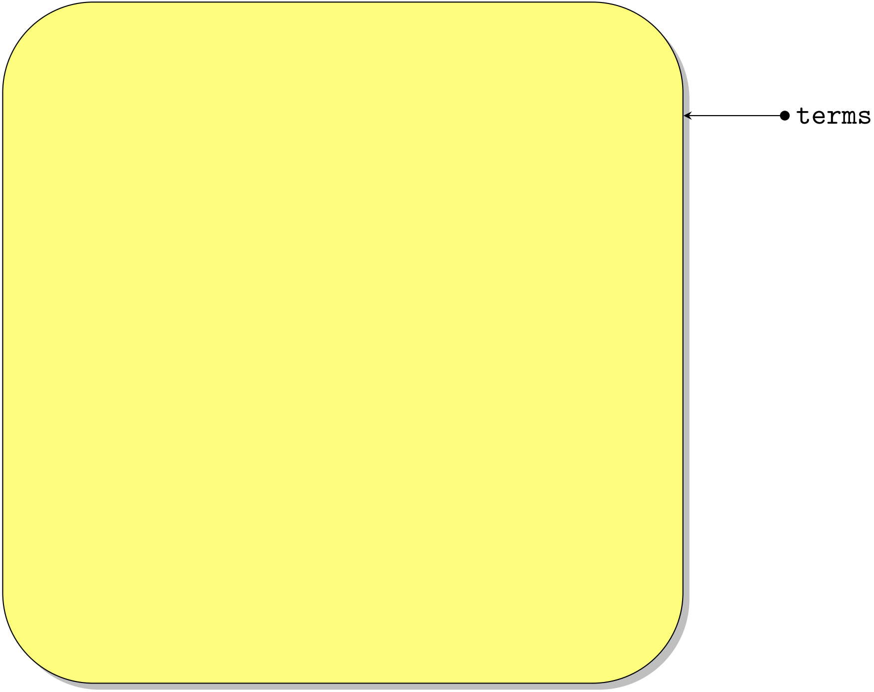
$$\texttt{Basis}(\Gamma,\sigma)$$

Which is **correct** and **complete**:

$$\texttt{span}(\,\texttt{Basis}(\Gamma,\sigma)\,) \ = \ \texttt{Sol}(\Gamma,\sigma)$$

Theorem

- *For any typing* (Γ,σ), $\texttt{Basis}_{\mathcal{B}}(\Gamma,\sigma)$ ***exists***, *is* ***finite***, ***correct*** *and* ***complete***. **BANG**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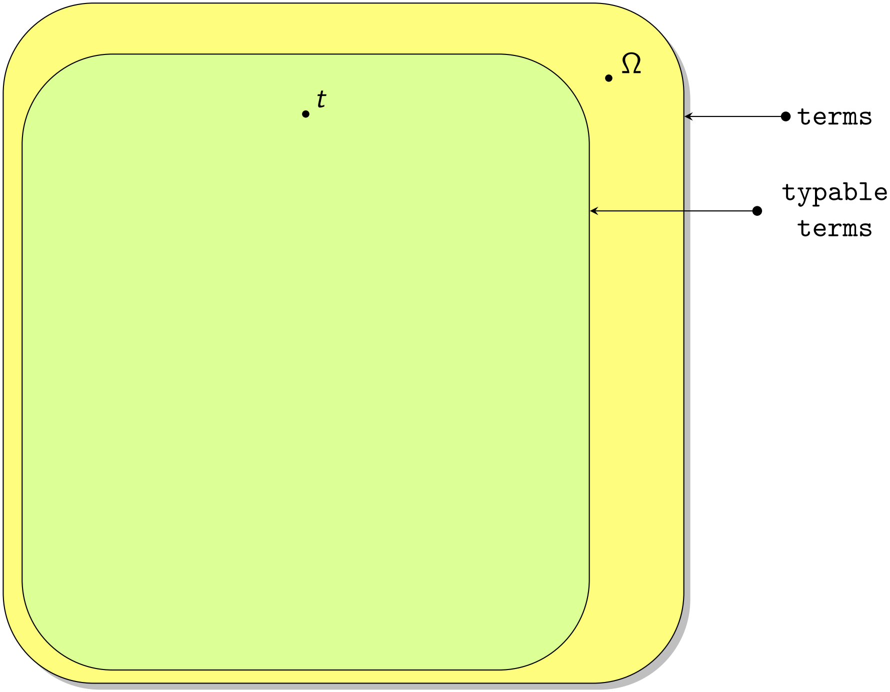
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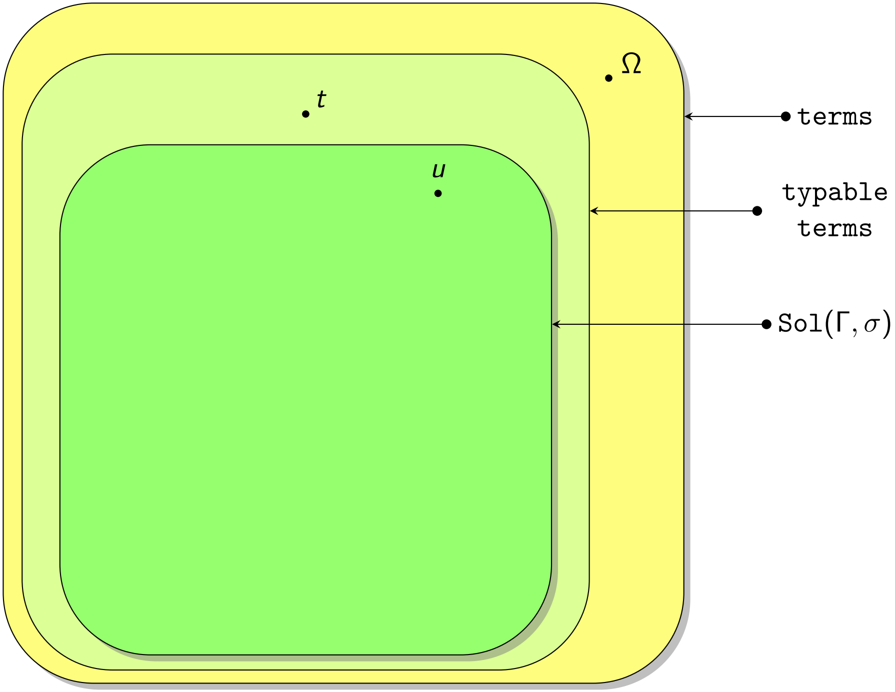
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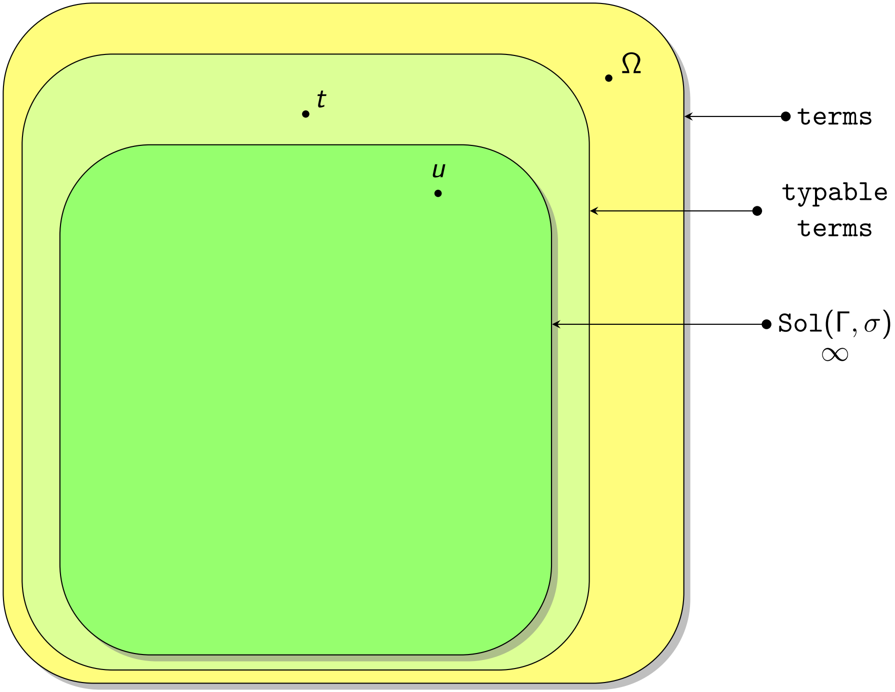

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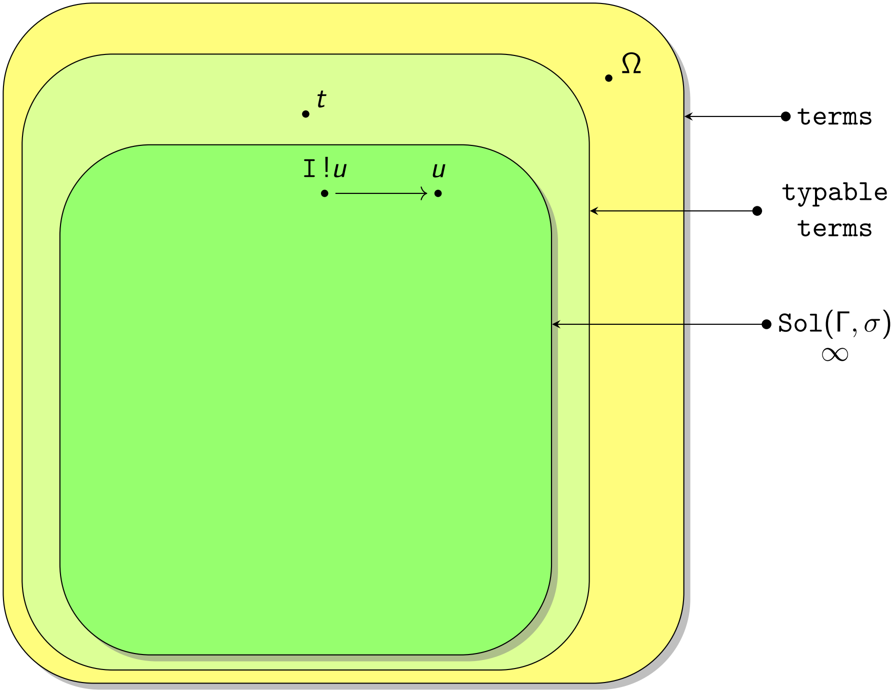

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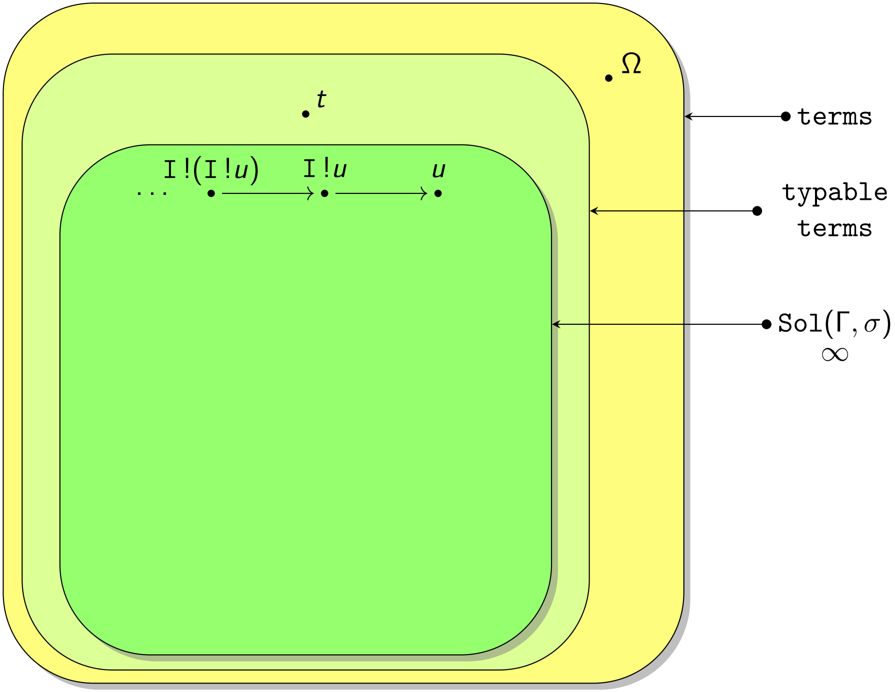

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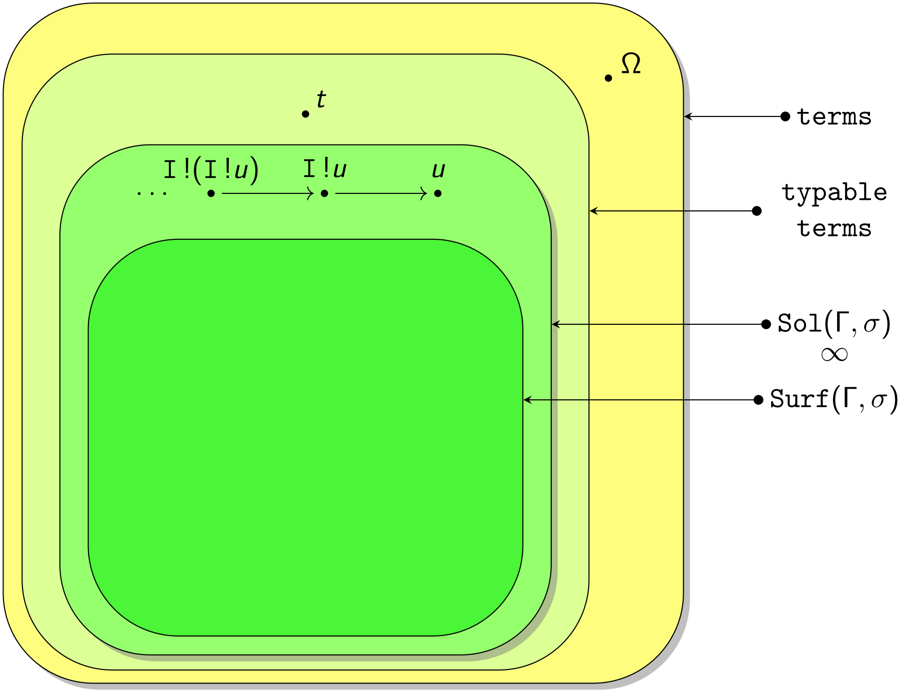
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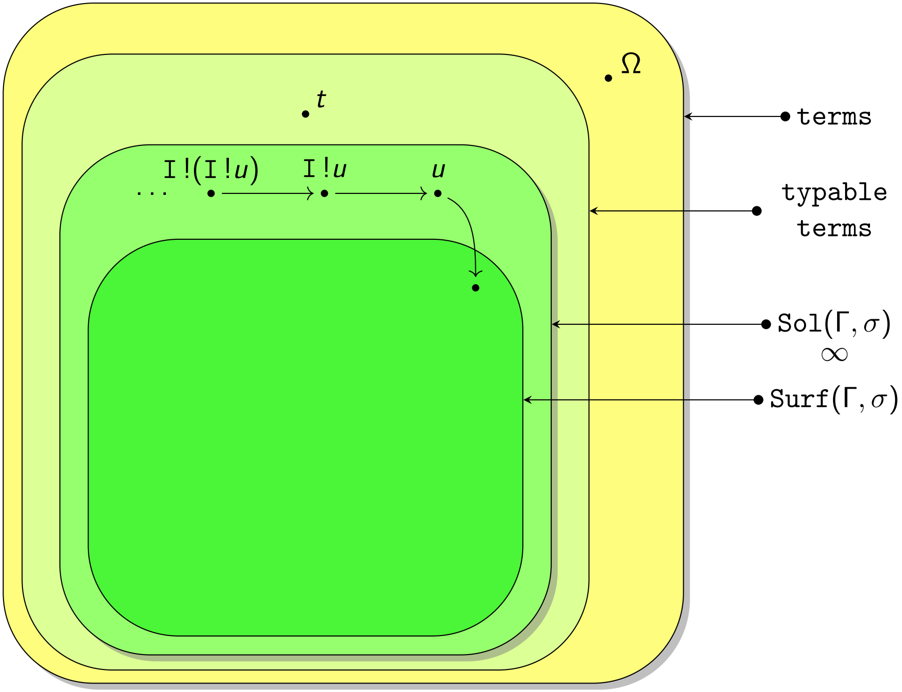

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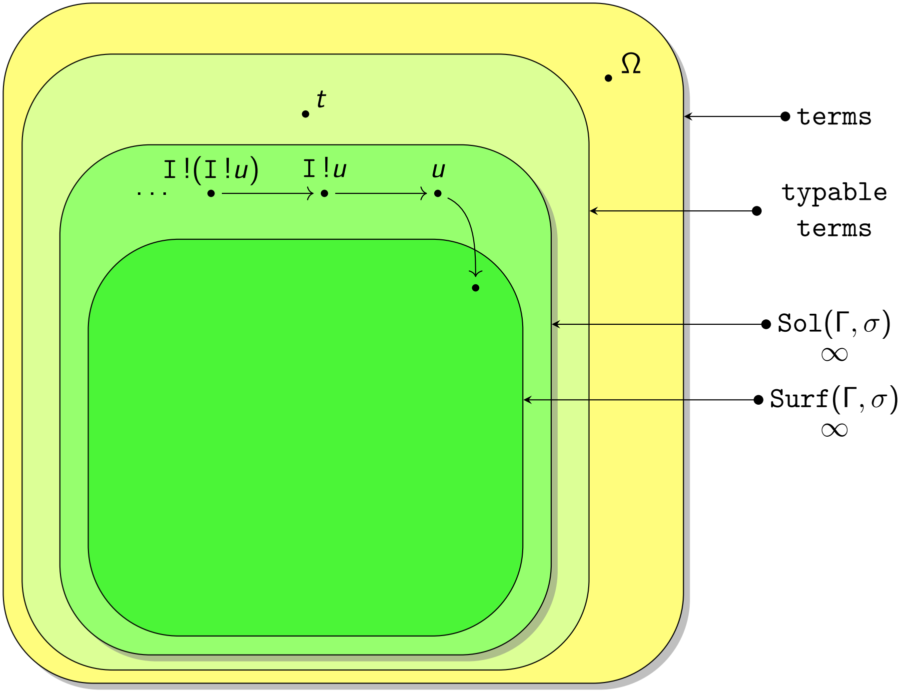

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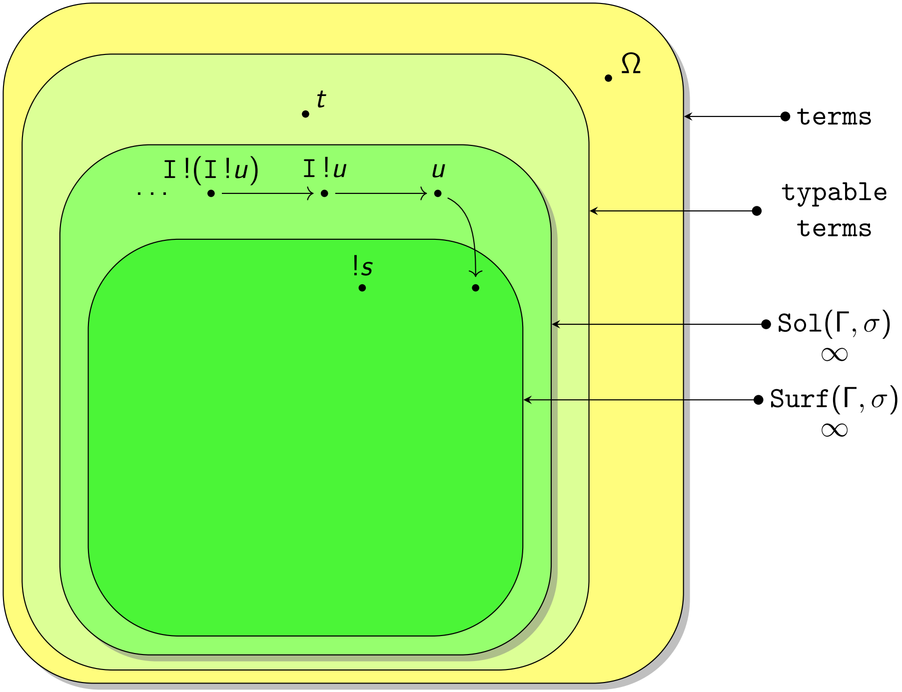

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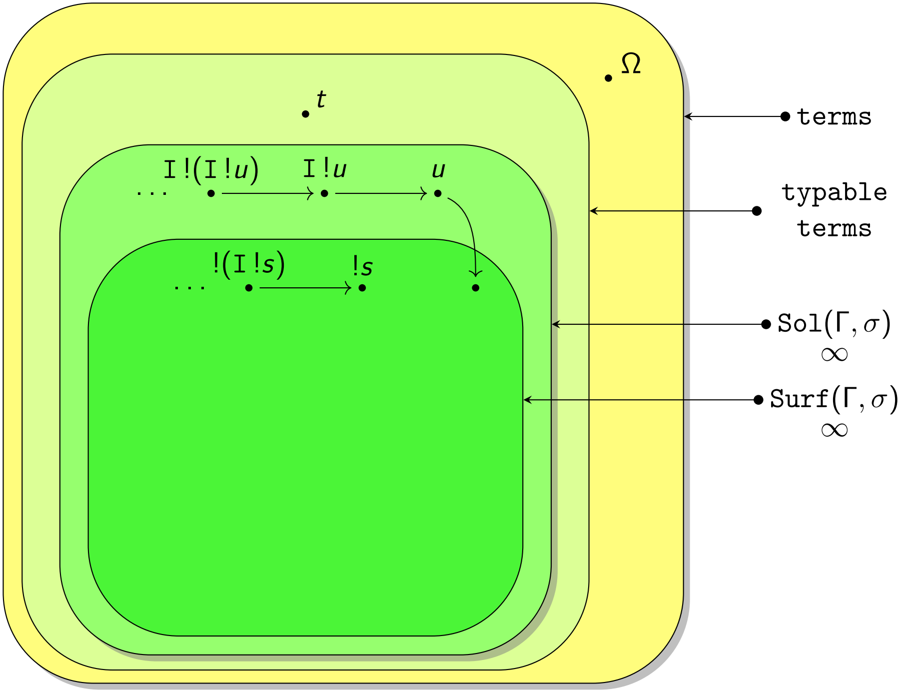
terms
typable
terms
Sol(Γ, σ)
∞
Surf(Γ, σ)
∞



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typable terms
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Sol(Γ, σ)
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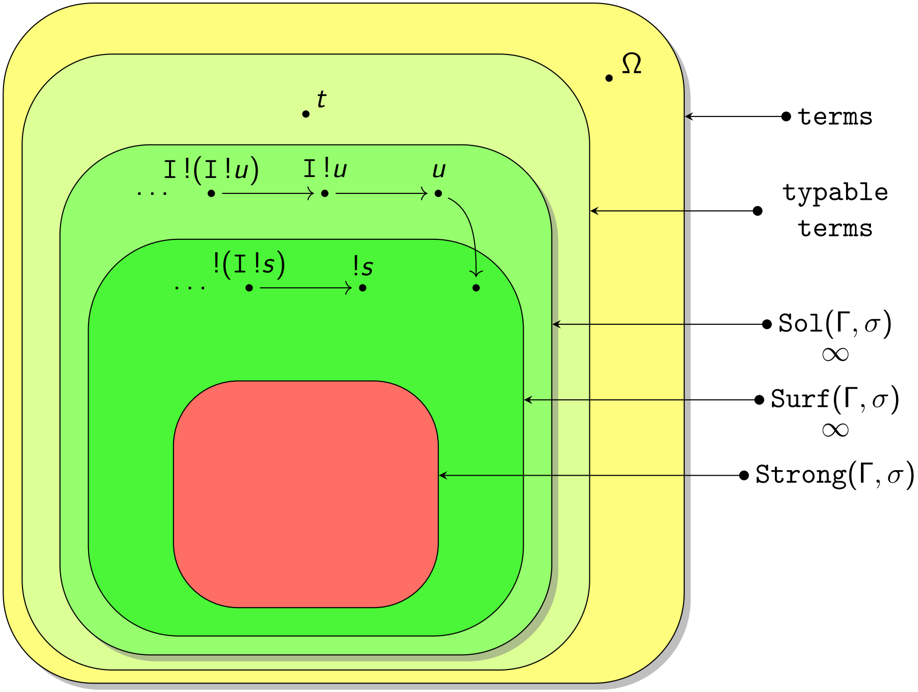
Surf(Γ, σ)
∞



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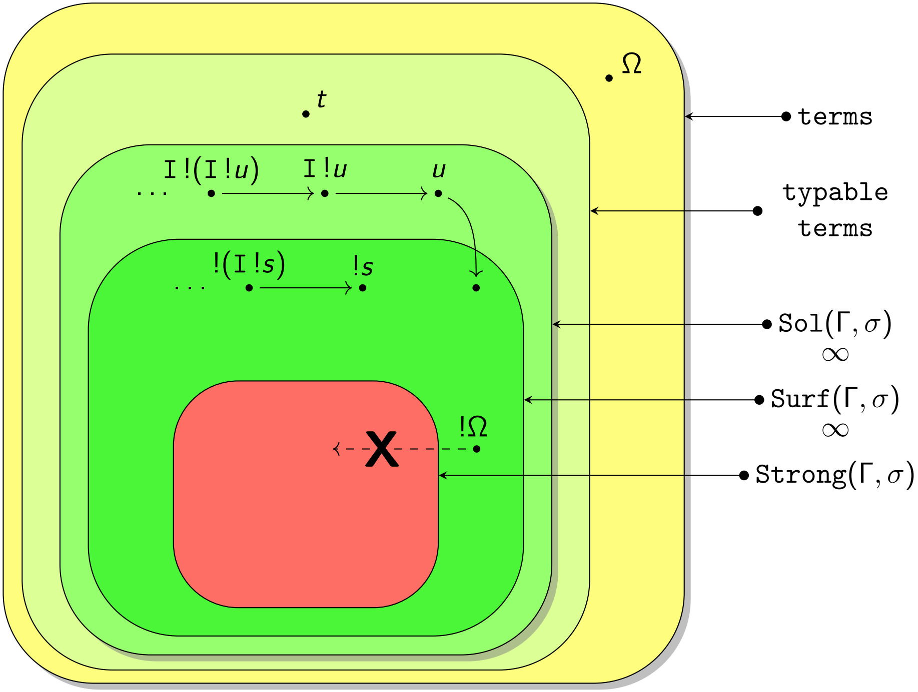
Surf(Γ, σ)
∞
Strong(Γ, σ)



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!s
Sol(Γ, σ) ∞
Surf(Γ, σ) ∞
!Ω
X
Strong(Γ, σ)

$$\frac{(\Gamma_i \vdash t : \sigma_i)_{i\in I}}{+_i \Gamma_i \vdash t : [\sigma_i]_{i\in I}}\ (\texttt{bg})$$

$$\frac{\dfrac{\dfrac{}{x{:}[[\tau]\Rightarrow[\,]\Rightarrow\sigma] \vdash x : [\tau] \Rightarrow [\,] \Rightarrow \sigma} \qquad \dfrac{\dfrac{}{y{:}[\tau_1] \vdash y : \tau_1} \quad \dfrac{}{y{:}[\tau_2] \vdash y : \tau_2}}{y{:}[\tau_1,\tau_2] \vdash\, !y : [\tau_1, \tau_2]}\,\texttt{bg}}{x{:}[[\tau]\Rightarrow[\,]\Rightarrow\sigma], y{:}[\tau] \vdash x\,!y : [\,] \Rightarrow \sigma} \qquad \dfrac{}{\emptyset \vdash\, !\Omega : [\,]}\,\texttt{bg}}{x{:}[[\tau]\Rightarrow[\,]\Rightarrow\sigma], y{:}[\tau] \vdash x(!y)(!\Omega) : \sigma}$$

$$\frac{(\Gamma_i \vdash t : \sigma_i)_{i\in I}}{+_i \Gamma_i \vdash t : [\sigma_i]_{i\in I}}\ (\texttt{bg})$$

$$\dfrac{\dfrac{\dfrac{}{x:[[\tau]\Rightarrow[\,]\Rightarrow\sigma] \vdash x : [\tau] \Rightarrow [\,] \Rightarrow \sigma} \qquad \dfrac{\dfrac{}{y:[\tau_1] \vdash y : \tau_1} \quad \dfrac{}{y:[\tau_2] \vdash y : \tau_2}}{y:[\tau_1,\tau_2] \vdash\ !y : [\tau_1, \tau_2]}\,\texttt{bg}}{x:[[\tau]\Rightarrow[\,]\Rightarrow\sigma], y:[\tau] \vdash x\,!y : [\,] \Rightarrow \sigma} \qquad \dfrac{}{\emptyset \vdash\ !\Omega : [\,]}\,\texttt{bg}}{x:[[\tau]\Rightarrow[\,]\Rightarrow\sigma], y:[\tau] \vdash x(!y)(!\Omega) : \sigma}$$

$$\frac{}{\emptyset \vdash t : {\color{red}[\,]}}\ (\mathtt{bg})$$

$$\dfrac{\dfrac{\dfrac{}{x{:}[[\tau]\Rightarrow[\,]\Rightarrow\sigma] \vdash x : [\tau] \Rightarrow [\,] \Rightarrow \sigma} \qquad \dfrac{\dfrac{}{y{:}[\tau_1] \vdash y : \tau_1} \quad \dfrac{}{y{:}[\tau_2] \vdash y : \tau_2}}{y{:}[\tau_1,\tau_2] \vdash\ !y : [\tau_1, \tau_2]}\,\mathtt{bg}}{x{:}[[\tau]\Rightarrow[\,]\Rightarrow\sigma], y{:}[\tau] \vdash x\,!y : [\,] \Rightarrow \sigma} \qquad \dfrac{}{\emptyset \vdash\ !\Omega : [\,]}\,\mathtt{bg}}{x{:}[[\tau]\Rightarrow[\,]\Rightarrow\sigma], y{:}[\tau] \vdash x(!y)(!\Omega) : \sigma}$$

$$\frac{(\Gamma_i \vdash t : \sigma_i)_{i\in I}}{+_i \Gamma_i \vdash t : [\sigma_i]_{i\in I}}\ (\texttt{bg})$$

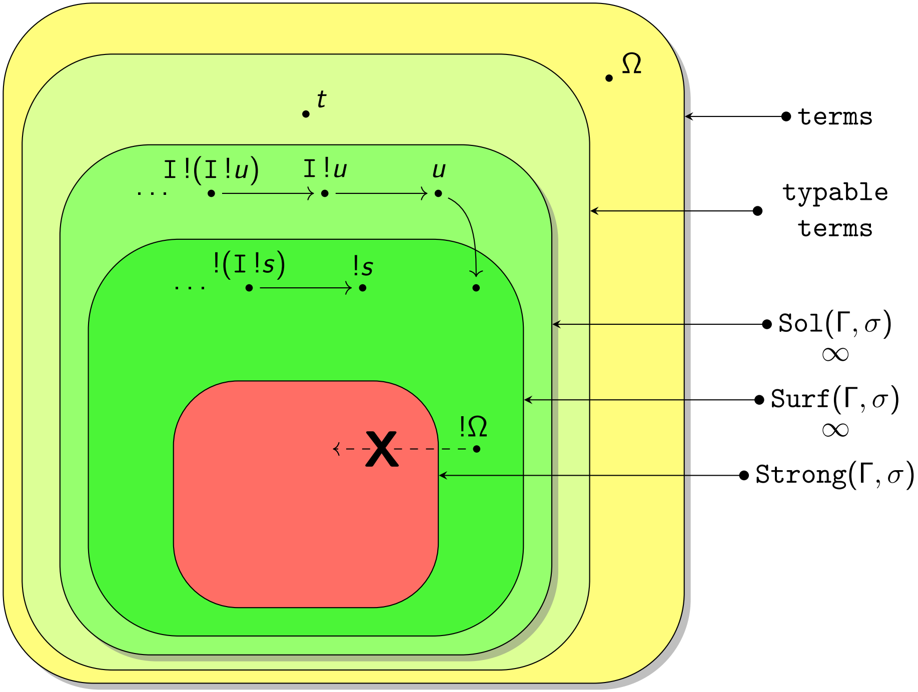
$$\dfrac{\dfrac{\dfrac{}{x{:}[[\tau]\Rightarrow[\,]\Rightarrow\sigma] \vdash x : [\tau]\Rightarrow[\,]\Rightarrow\sigma} \qquad \dfrac{\dfrac{}{y{:}[\tau_1] \vdash y : \tau_1} \quad \dfrac{}{y{:}[\tau_2] \vdash y : \tau_2}}{y{:}[\tau_1,\tau_2] \vdash !y : [\tau_1,\tau_2]}\,\texttt{bg}}{x{:}[[\tau]\Rightarrow[\,]\Rightarrow\sigma],y{:}[\tau] \vdash x\,!y : [\,]\Rightarrow\sigma} \qquad \dfrac{}{\emptyset \vdash !\Omega : [\,]}\,\texttt{bg}}{x{:}[[\tau]\Rightarrow[\,]\Rightarrow\sigma],y{:}[\tau] \vdash x(!y)(!\Omega) : \sigma}$$

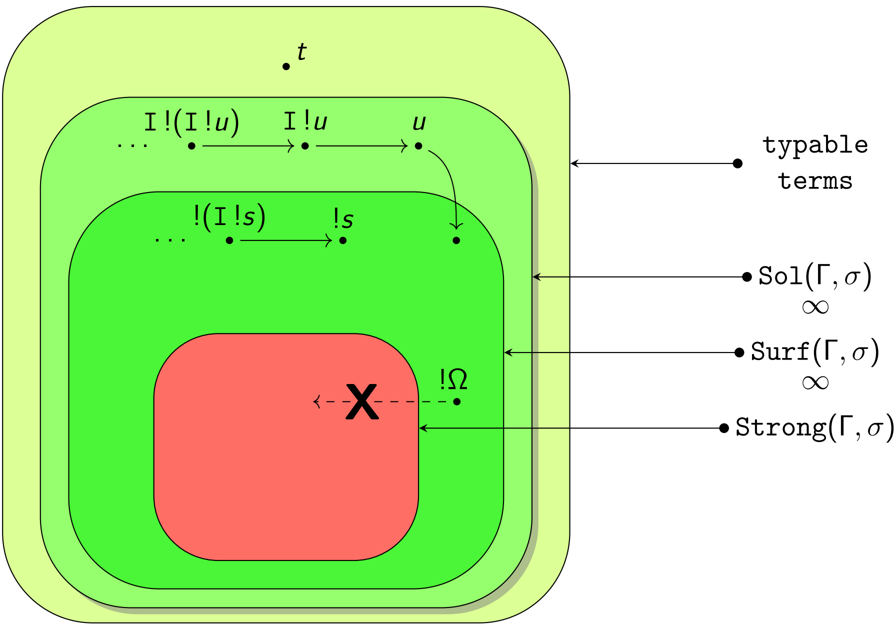
$$\frac{(\Gamma_i \vdash t : \sigma_i)_{i \in I}}{+_i \Gamma_i \vdash t : [\sigma_i]_{i \in I}}\ (\texttt{bg})$$

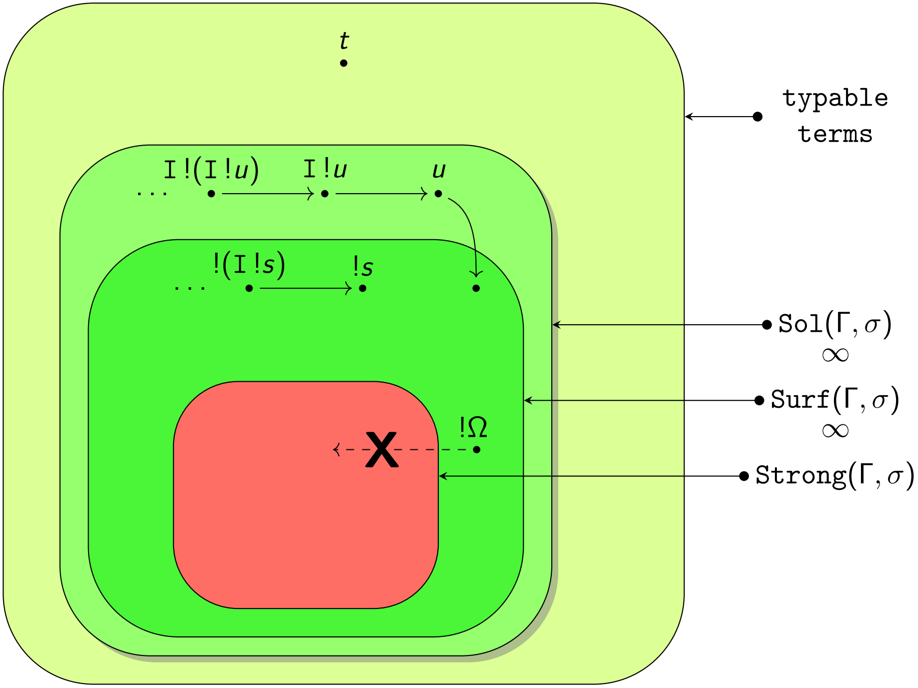
$$\dfrac{\dfrac{}{x{:}[[\tau]\Rightarrow[\,]\Rightarrow\sigma] \vdash x : [\tau] \Rightarrow [\,] \Rightarrow \sigma} \qquad \dfrac{\dfrac{}{y{:}[\tau_1] \vdash {\color{red}y} : \tau_1} \quad \dfrac{}{y{:}[\tau_2] \vdash {\color{red}y} : \tau_2}}{y{:}[\tau_1,\tau_2] \vdash\ !y : [\tau_1, \tau_2]}\,\texttt{bg}}{\dfrac{x{:}[[\tau]\Rightarrow[\,]\Rightarrow\sigma], y{:}[\tau] \vdash x\,!y : [\,] \Rightarrow \sigma \qquad \dfrac{}{\emptyset \vdash\ !\Omega : [\,]}\,\texttt{bg}}{x{:}[[\tau]\Rightarrow[\,]\Rightarrow\sigma], y{:}[\tau] \vdash x(!y)(!\Omega) : \sigma}}$$

$$\frac{(\Gamma_i \vdash t : \sigma_i)_{i\in I}}{+_i \Gamma_i \vdash t : [\sigma_i]_{i\in I}}\ (\texttt{bg})$$

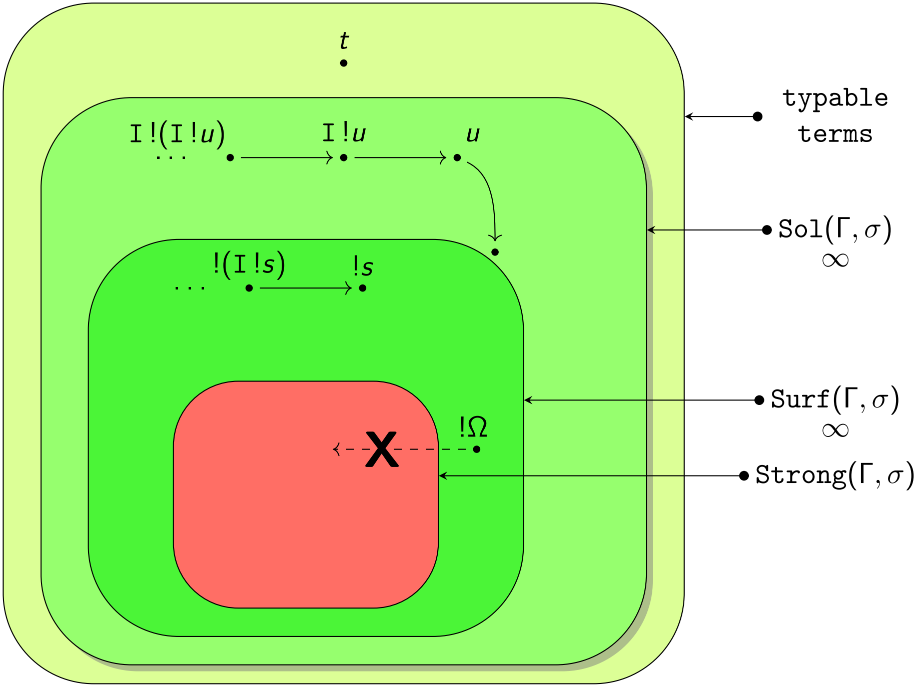
$$\dfrac{\dfrac{\dfrac{}{x{:}[[\tau]\Rightarrow[\,]\Rightarrow\sigma] \vdash x : [\tau] \Rightarrow [\,] \Rightarrow \sigma} \qquad \dfrac{\dfrac{}{y{:}[\tau_1] \vdash y : \tau_1} \quad \dfrac{}{y{:}[\tau_2] \vdash y : \tau_2}}{y{:}[\tau_1,\tau_2] \vdash\ !y : [\tau_1, \tau_2]}\,\texttt{bg}}{x{:}[[\tau]\Rightarrow[\,]\Rightarrow\sigma], y{:}[\tau] \vdash x\, !y : [\,] \Rightarrow \sigma} \qquad \dfrac{}{\emptyset \vdash\ !\Omega : [\,]}\,\texttt{bg}}{x{:}[[\tau]\Rightarrow[\,]\Rightarrow\sigma], y{:}[\tau] \vdash x(!y)(!\Omega) : \sigma}$$


Ω
t
terms
I !(I !u)
I !u
u
typable terms
!(I !s)
!s
Sol(Γ, σ)
∞
Surf(Γ, σ)
∞
!Ω
X
Strong(Γ, σ)

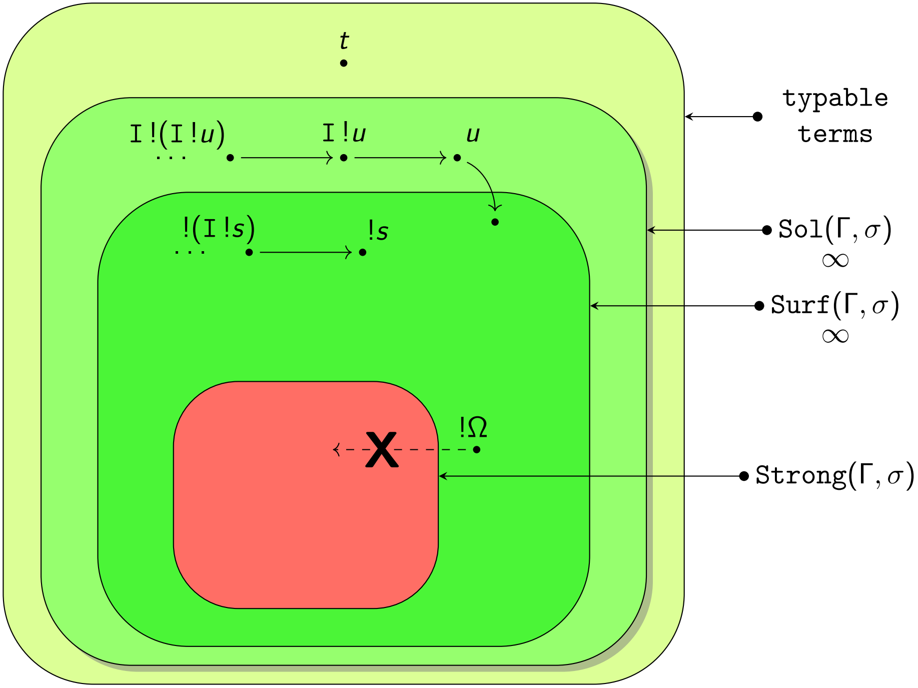

t
I !(I !u)
I !u
u
!(I !s)
!s
!Ω
X
typable terms
Sol(Γ, σ)
∞
Surf(Γ, σ)
∞
Strong(Γ, σ)



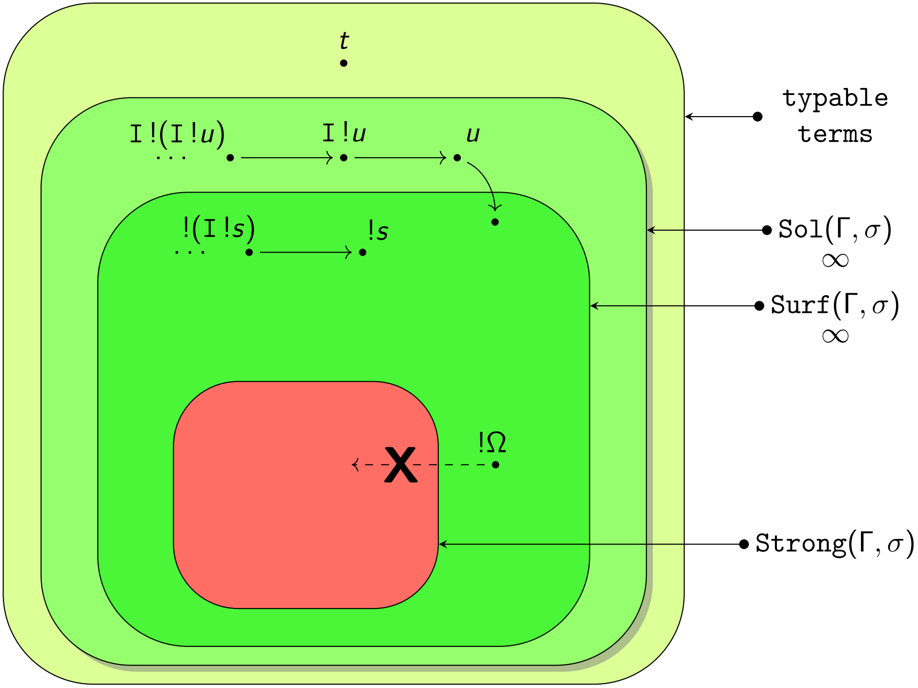
t
typable terms
$\mathtt{I}\,!(\mathtt{I}\,!u)$
$\mathtt{I}\,!u$
u
$!(\mathtt{I}\,!s)$
$!s$
$\mathtt{Sol}(\Gamma,\sigma)$
∞
$\mathtt{Surf}(\Gamma,\sigma)$
∞
$!\Omega$
X
$\mathtt{Strong}(\Gamma,\sigma)$



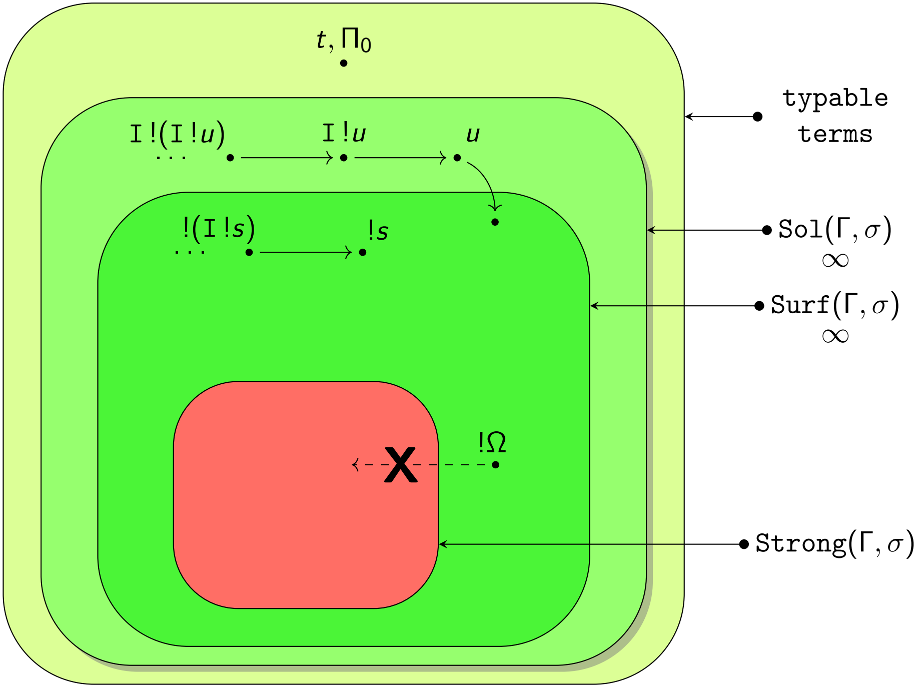
t
typable terms
$\mathtt{I}\,!(\mathtt{I}\,!u)$
$\mathtt{I}\,!u$
u
$\mathtt{Sol}(\Gamma, \sigma)$
∞
$!(\mathtt{I}\,!s)$
$!s$
$!\Omega$
X
$\mathtt{Surf}(\Gamma, \sigma)$
∞
$\mathtt{Strong}(\Gamma, \sigma)$



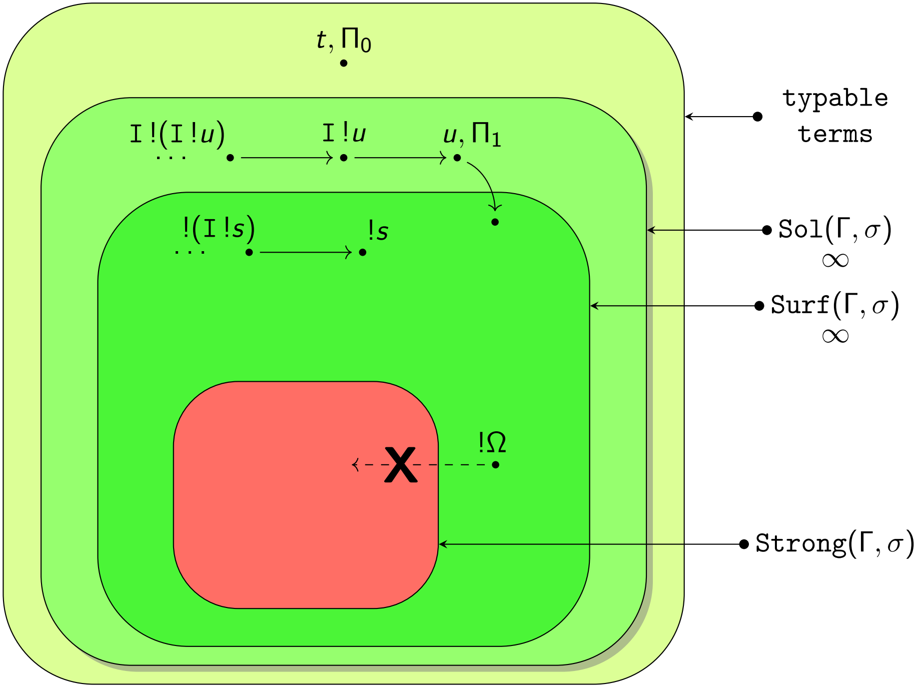
t
typable
terms
I !(I !u)
I !u
u
!(I !s)
!s
Sol(Γ, σ)
∞
Surf(Γ, σ)
∞
!Ω
X
Strong(Γ, σ)



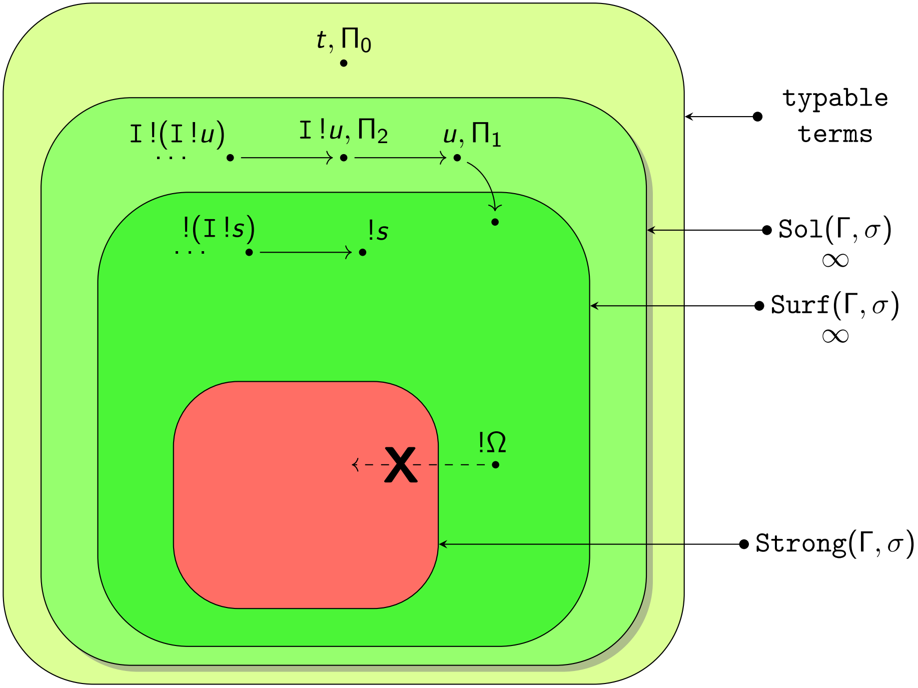
t
I !(I !u)
I !u
u
!(I !s)
!s
!Ω
X
typable
terms
Sol(Γ, σ)
∞
Surf(Γ, σ)
∞
Strong(Γ, σ)



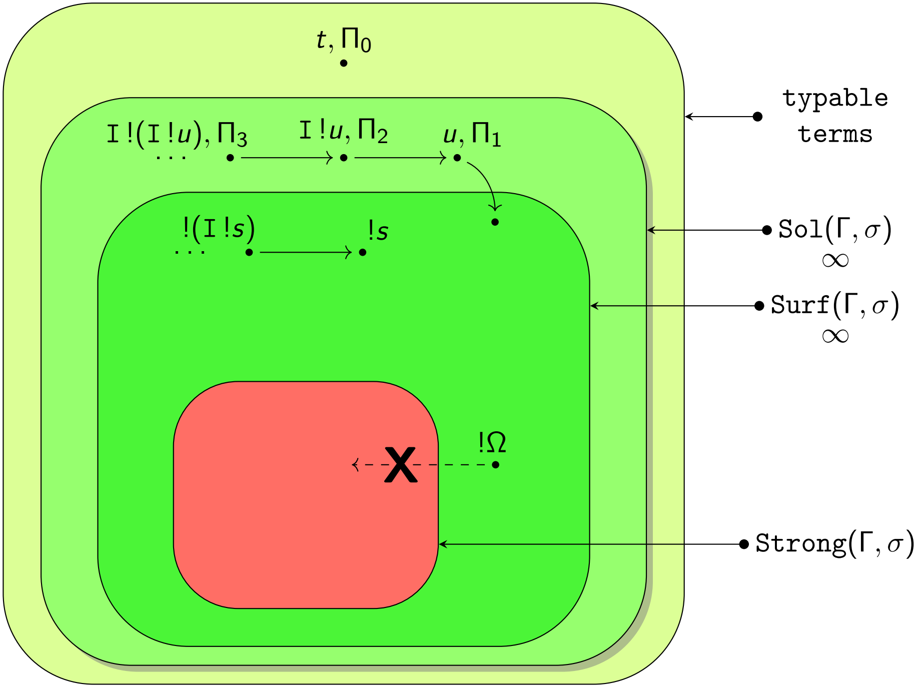
t, Π_0
$\mathtt{I}\,!(\mathtt{I}\,!u)$
$\mathtt{I}\,!u$
u
$!(\mathtt{I}\,!s)$
$!s$
$!\Omega$
typable terms
$\mathtt{Sol}(\Gamma, \sigma)$
∞
$\mathtt{Surf}(\Gamma, \sigma)$
∞
$\mathtt{Strong}(\Gamma, \sigma)$



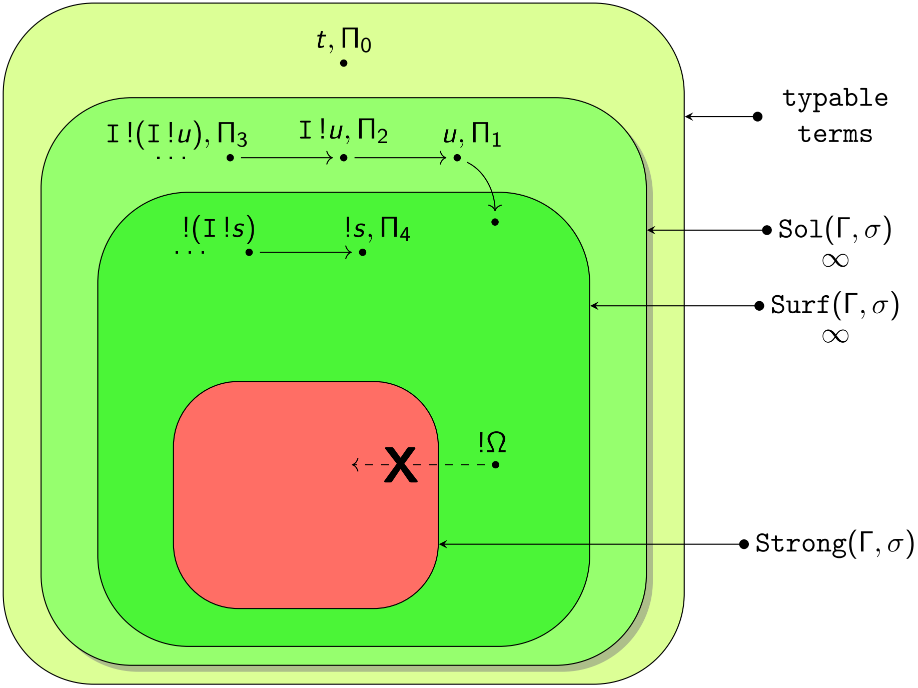
t, Π_0
typable terms
I !(I !u)
I !u
u, Π_1
!(I !s)
!s
$\mathtt{Sol}(\Gamma, \sigma)$
∞
$\mathtt{Surf}(\Gamma, \sigma)$
∞
!Ω
$\mathtt{Strong}(\Gamma, \sigma)$



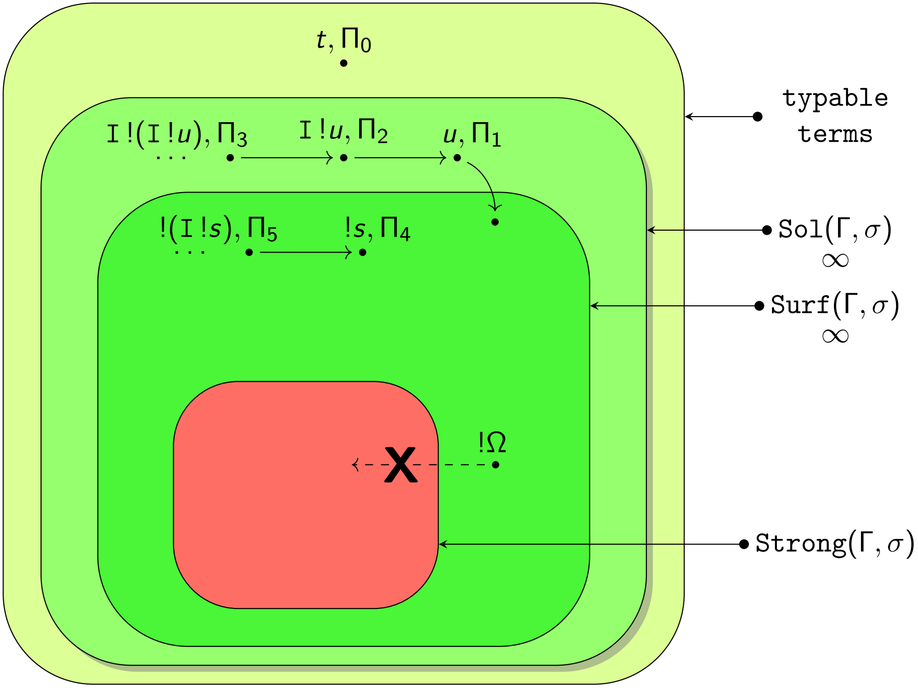
t, Π_0
$\mathtt{I}\,!(\mathtt{I}\,!u)$
$\mathtt{I}\,!u, \Pi_2$
u, Π_1
$!(\mathtt{I}\,!s)$
$!s$
$!\Omega$
typable terms
$\mathtt{Sol}(\Gamma, \sigma)$
∞
$\mathtt{Surf}(\Gamma, \sigma)$
∞
$\mathtt{Strong}(\Gamma, \sigma)$



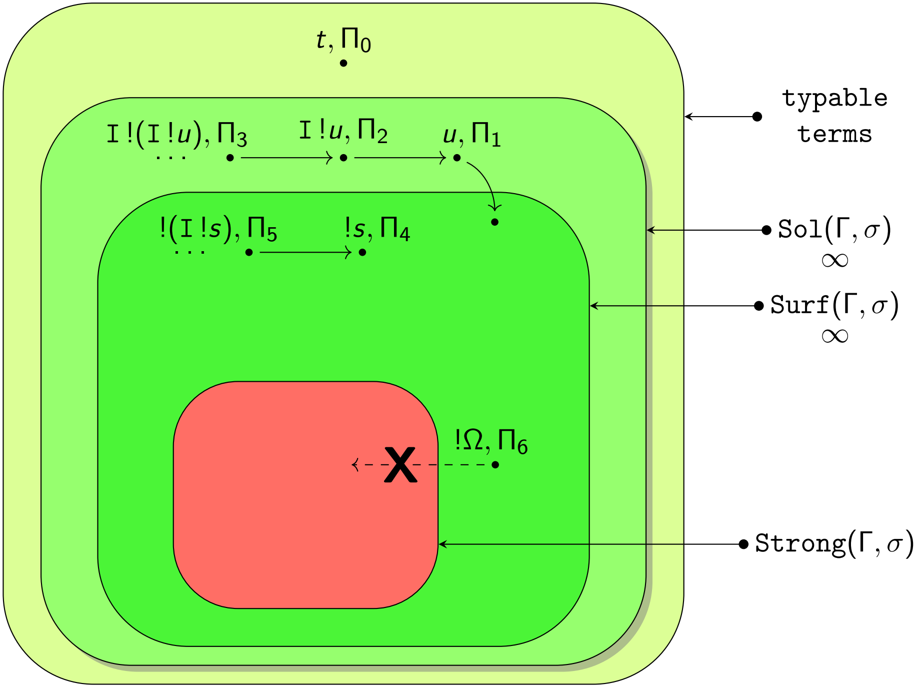
t, Π0
I !(I !u), Π3
I !u, Π2
u, Π1
!(I !s)
!s
!Ω
typable
terms
Sol(Γ, σ)
∞
Surf(Γ, σ)
∞
Strong(Γ, σ)



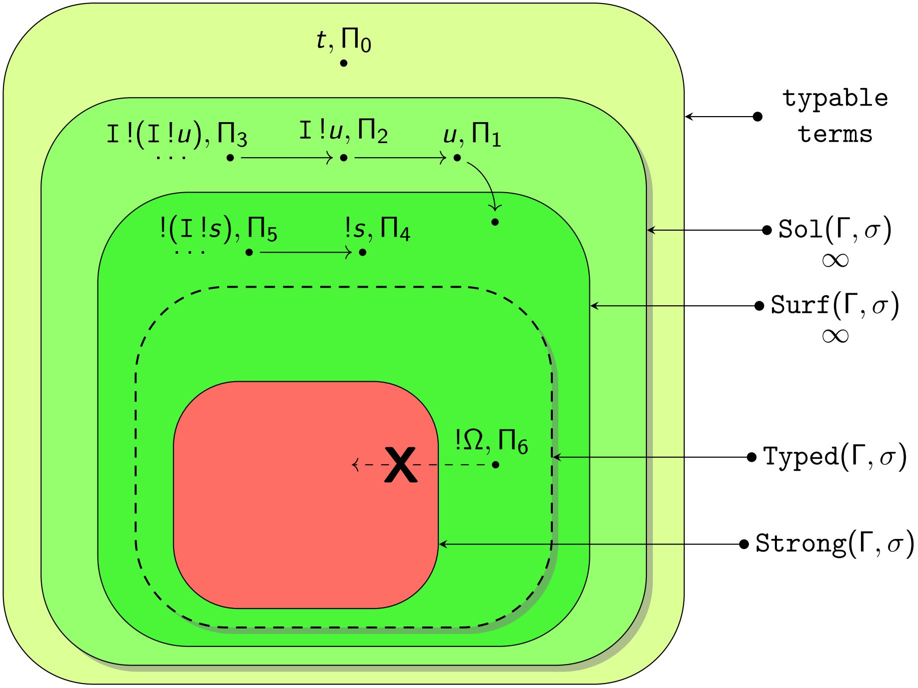
t, Π0
I !(I !u), Π3
I !u, Π2
u, Π1
!(I !s)
!s, Π4
!Ω
typable terms
Sol(Γ, σ)
∞
Surf(Γ, σ)
∞
Strong(Γ, σ)

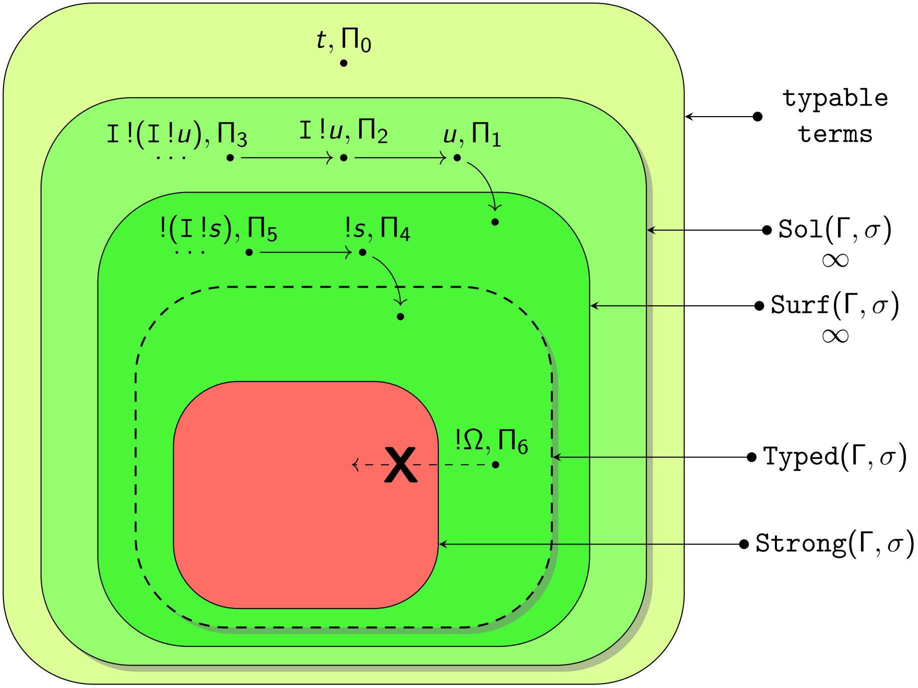


t, Π_0
$\mathtt{I}\,!(\mathtt{I}\,!u), \Pi_3$
$\mathtt{I}\,!u, \Pi_2$
u, Π_1
$!(\mathtt{I}\,!s), \Pi_5$
$!s, \Pi_4$
$!\Omega$
typable terms
$\mathtt{Sol}(\Gamma, \sigma)$
∞
$\mathtt{Surf}(\Gamma, \sigma)$
∞
$\mathtt{Strong}(\Gamma, \sigma)$

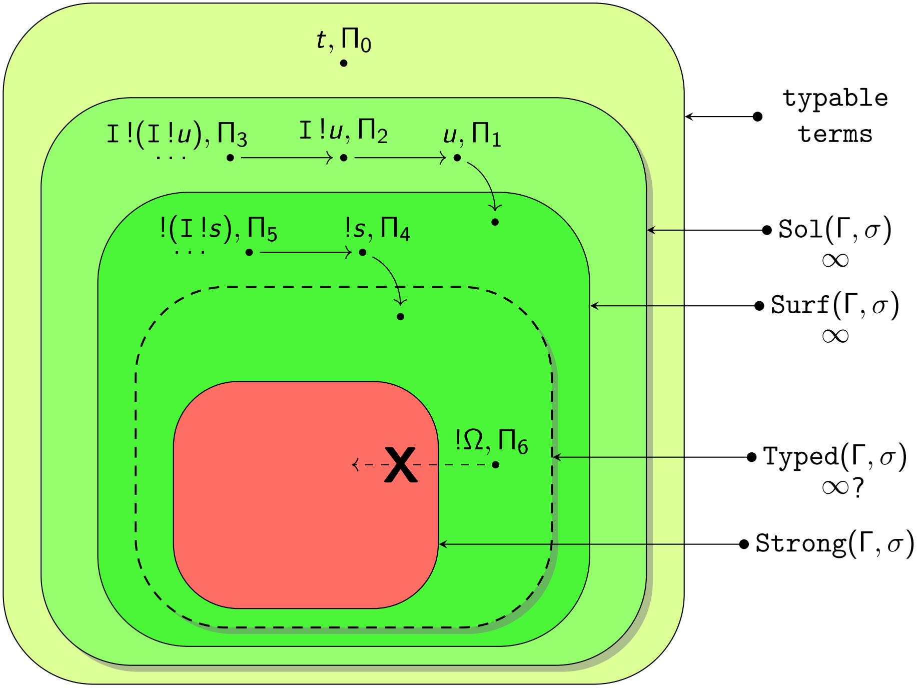


t, Π0
typable
terms
I !(I !u), Π3
I !u, Π2
u, Π1
...
!(I !s), Π5
!s, Π4
...
Sol(Γ, σ)
∞
Surf(Γ, σ)
∞
!Ω, Π6
X
Strong(Γ, σ)


t, Π_0
$\mathtt{I}\,!(\mathtt{I}\,!u), \Pi_3$
$\mathtt{I}\,!u, \Pi_2$
u, Π_1
$!(\mathtt{I}\,!s), \Pi_5$
$!s, \Pi_4$
$!\Omega, \Pi_6$
typable terms
$\mathtt{Sol}(\Gamma, \sigma)$
∞
$\mathtt{Surf}(\Gamma, \sigma)$
∞
$\mathtt{Typed}(\Gamma, \sigma)$
$\mathtt{Strong}(\Gamma, \sigma)$



t, Π0
I !(I !u), Π3
I !u, Π2
u, Π1
!(I !s), Π5
!s, Π4
!Ω, Π6
X
typable terms
Sol(Γ, σ)
∞
Surf(Γ, σ)
∞
Typed(Γ, σ)
Strong(Γ, σ)



t, Π_0
$\mathtt{I}\,!(\mathtt{I}\,!u), \Pi_3$
$\mathtt{I}\,!u, \Pi_2$
u, Π_1
$!(\mathtt{I}\,!s), \Pi_5$
$!s, \Pi_4$
$!\Omega, \Pi_6$
X
typable terms
$\mathtt{Sol}(\Gamma, \sigma)$
∞
$\mathtt{Surf}(\Gamma, \sigma)$
∞
$\mathtt{Typed}(\Gamma, \sigma)$
∞?
$\mathtt{Strong}(\Gamma, \sigma)$

$$\frac{(\Gamma_i \vdash t : \sigma_i)_{i \in I}}{+_i \Gamma_i \vdash t : [\sigma_i]_{i \in I}}\ (\texttt{bg})$$

$$\dfrac{\dfrac{}{x{:}[[\tau]\Rightarrow[\,]\Rightarrow\sigma] \vdash x : [\tau] \Rightarrow [\,] \Rightarrow \sigma} \quad \dfrac{\dfrac{}{y{:}[\tau_1] \vdash y : \tau_1} \quad \dfrac{}{y{:}[\tau_2] \vdash y : \tau_2}}{y{:}[\tau_1,\tau_2] \vdash\ !y : [\tau_1, \tau_2]}\,\texttt{bg}}{x{:}[[\tau]\Rightarrow[\,]\Rightarrow\sigma], y{:}[\tau] \vdash x\,!y : [\,] \Rightarrow \sigma} \quad \dfrac{}{\emptyset \vdash\ !\Omega : [\,]}\,\texttt{bg}$$

$$x{:}[[\tau]\Rightarrow[\,]\Rightarrow\sigma], y{:}[\tau] \vdash x(!y)(!\Omega) : \sigma$$

$$\frac{(\Gamma_i \vdash t : \sigma_i)_{i\in I}}{+_i\Gamma_i \vdash t : [\sigma_i]_{i\in I}}\ (\mathtt{bg})$$

$$\dfrac{\dfrac{\dfrac{}{x:[[\tau]\Rightarrow[\,]\Rightarrow\sigma] \vdash x : [\tau]\Rightarrow[\,]\Rightarrow\sigma} \quad \dfrac{\dfrac{}{y:[\tau_1]\vdash y:\tau_1} \quad \dfrac{}{y:[\tau_2]\vdash y:\tau_2}}{y:[\tau_1,\tau_2]\vdash\, !y : [\tau_1,\tau_2]}\,\mathtt{bg}}{x:[[\tau]\Rightarrow[\,]\Rightarrow\sigma], y:[\tau] \vdash x\,!y : [\,]\Rightarrow\sigma} \quad \dfrac{}{\emptyset \vdash\, !\Omega : [\,]}\,\mathtt{bg}}{x:[[\tau]\Rightarrow[\,]\Rightarrow\sigma], y:[\tau]\vdash x(!y)(!\Omega):\sigma}$$

$$\dfrac{}{\emptyset \vdash t : [\,]}\ (\texttt{bg})$$

$$\dfrac{\dfrac{\dfrac{}{x{:}[[\tau]\Rightarrow[\,]\Rightarrow\sigma] \vdash x : [\tau] \Rightarrow [\,] \Rightarrow \sigma} \qquad \dfrac{\dfrac{}{y{:}[\tau_1] \vdash y : \tau_1} \quad \dfrac{}{y{:}[\tau_2] \vdash y : \tau_2}}{y{:}[\tau_1,\tau_2] \vdash\, !y : [\tau_1, \tau_2]}\,\texttt{bg}}{x{:}[[\tau]\Rightarrow[\,]\Rightarrow\sigma], y{:}[\tau] \vdash x\,!y : [\,] \Rightarrow \sigma} \qquad \dfrac{}{\emptyset \vdash\, !\Omega : [\,]}\,\texttt{bg}}{x{:}[[\tau]\Rightarrow[\,]\Rightarrow\sigma], y{:}[\tau] \vdash x(!y)(!\Omega) : \sigma}$$

$$\frac{(\Gamma_i \vdash t : \sigma_i)_{i\in I}}{+_i \Gamma_i \vdash t : [\sigma_i]_{i\in I}}\ (\texttt{bg})$$

$$\dfrac{\dfrac{\dfrac{}{x{:}[[\tau]\Rightarrow[\,]\Rightarrow\sigma] \vdash x : [\tau]\Rightarrow[\,]\Rightarrow\sigma} \qquad \dfrac{\dfrac{}{y{:}[\tau_1]\vdash y:\tau_1}\quad\dfrac{}{y{:}[\tau_2]\vdash y:\tau_2}}{y{:}[\tau_1,\tau_2]\vdash\, !y:[\tau_1,\tau_2]}\,\texttt{bg}}{x{:}[[\tau]\Rightarrow[\,]\Rightarrow\sigma], y{:}[\tau]\vdash x\,!y : [\,]\Rightarrow\sigma} \qquad \dfrac{}{\emptyset\vdash\, !\Omega:[\,]}\,\texttt{bg}}{x{:}[[\tau]\Rightarrow[\,]\Rightarrow\sigma], y{:}[\tau] \vdash x(!y)(!\Omega) : \sigma}$$

Le λ!-calcul : Syntax and Operational Semantics

$$t, u \ ::= \ x \in \mathcal{V} \mid tu \mid \lambda x.u \mid !u \mid \mathtt{der}(u) \mid u[x := v] \mid \bot$$

Reduction :

$$\begin{array}{rcl} \mathtt{L}\langle \lambda x.t \rangle\, u & \mapsto_{dB} & \mathtt{L}\langle t[x := u] \rangle \\ t[x := \mathtt{L}\langle !u \rangle] & \mapsto_{s!} & \mathtt{L}\langle t\{x := u\} \rangle \\ \mathtt{der}(\mathtt{L}\langle !t \rangle) & \mapsto_{d!} & \mathtt{L}\langle t \rangle \end{array}$$

Contexts :

$$\begin{array}{lcl} \mathtt{L} & ::= & \diamond \mid \mathtt{L}[x := t] \\ \mathtt{S} & ::= & \diamond \mid \lambda x.\mathtt{S} \mid \mathtt{S}\ t \mid t\ \mathtt{S} \mid \mathtt{S}[x := t] \mid t[x := \mathtt{S}] \mid \mathtt{der}(\mathtt{S}) \\ \mathtt{F} & ::= & \diamond \mid \lambda x.\mathtt{F} \mid \mathtt{F}\ t \mid t\ \mathtt{F} \mid \mathtt{F}[x := t] \mid t[x := \mathtt{F}] \mid \mathtt{der}(\mathtt{F}) \mid !\mathtt{F} \end{array}$$

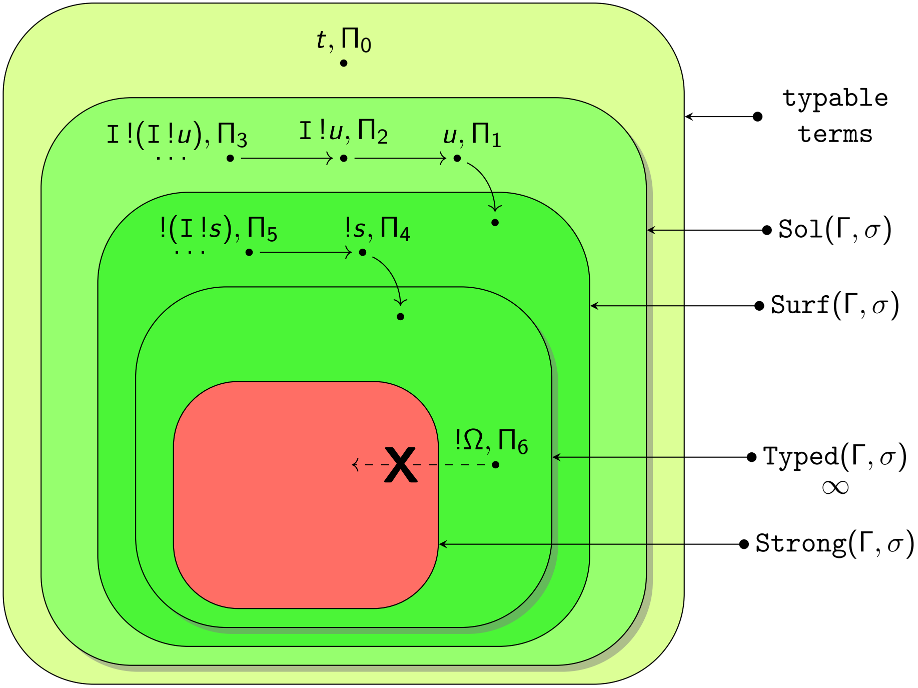
$$\frac{(\Gamma_i \vdash t : \sigma_i)_{i\in I}}{+_i \Gamma_i \vdash t : [\sigma_i]_{i\in I}}\ (\texttt{bg})$$

$$\frac{\dfrac{\dfrac{}{x{:}[[\tau]\Rightarrow[\,]\Rightarrow\sigma] \vdash x : [\tau] \Rightarrow [\,] \Rightarrow \sigma} \quad \dfrac{\dfrac{}{y{:}[\tau_1] \vdash y : \tau_1} \quad \dfrac{}{y{:}[\tau_2] \vdash y : \tau_2}}{y{:}[\tau_1,\tau_2] \vdash\ !y : [\tau_1, \tau_2]}\,\texttt{bg}}{x{:}[[\tau]\Rightarrow[\,]\Rightarrow\sigma], y{:}[\tau] \vdash x\,!y : [\,] \Rightarrow \sigma} \quad \dfrac{}{\emptyset \vdash\ !\bot : [\,]}\,\texttt{bg}}{x{:}[[\tau]\Rightarrow[\,]\Rightarrow\sigma], y{:}[\tau] \vdash x(!y)(!\Omega) : \sigma}$$

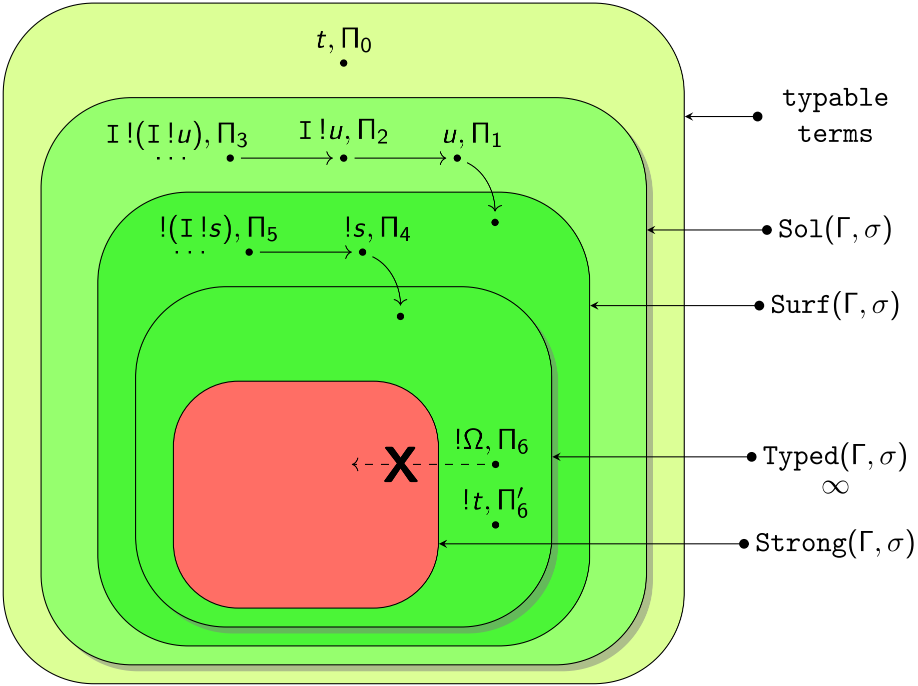
$$\frac{(\Gamma_i \vdash t : \sigma_i)_{i \in I}}{+_i \Gamma_i \vdash t : [\sigma_i]_{i \in I}} \ \texttt{(bg)}$$

$$\dfrac{\dfrac{}{x{:}[[\tau]\Rightarrow[\,]\Rightarrow\sigma] \vdash x : [\tau] \Rightarrow [\,] \Rightarrow \sigma} \quad \dfrac{\dfrac{}{y{:}[\tau_1] \vdash y : \tau_1} \quad \dfrac{}{y{:}[\tau_2] \vdash y : \tau_2}}{y{:}[\tau_1,\tau_2] \vdash\, !y : [\tau_1, \tau_2]}\,\texttt{bg}}{x{:}[[\tau]\Rightarrow[\,]\Rightarrow\sigma], y{:}[\tau] \vdash x\,!y : [\,] \Rightarrow \sigma} \quad \dfrac{}{\emptyset \vdash\, !\Omega : [\,]}\,\texttt{bg}$$

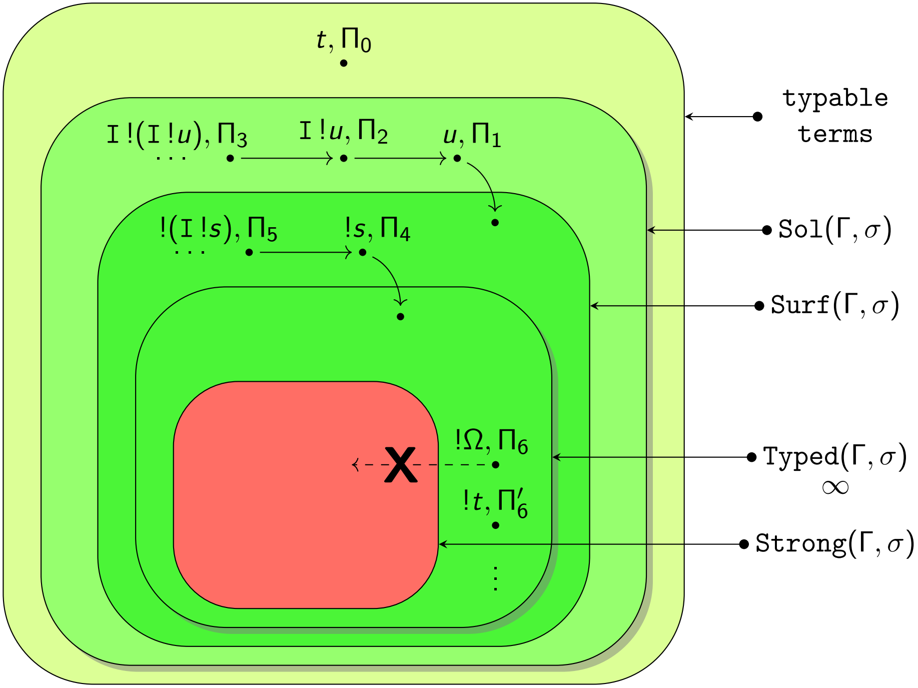
$$\overline{x{:}[[\tau]\Rightarrow[\,]\Rightarrow\sigma], y{:}[\tau] \vdash x(!y)(!\Omega) : \sigma}$$



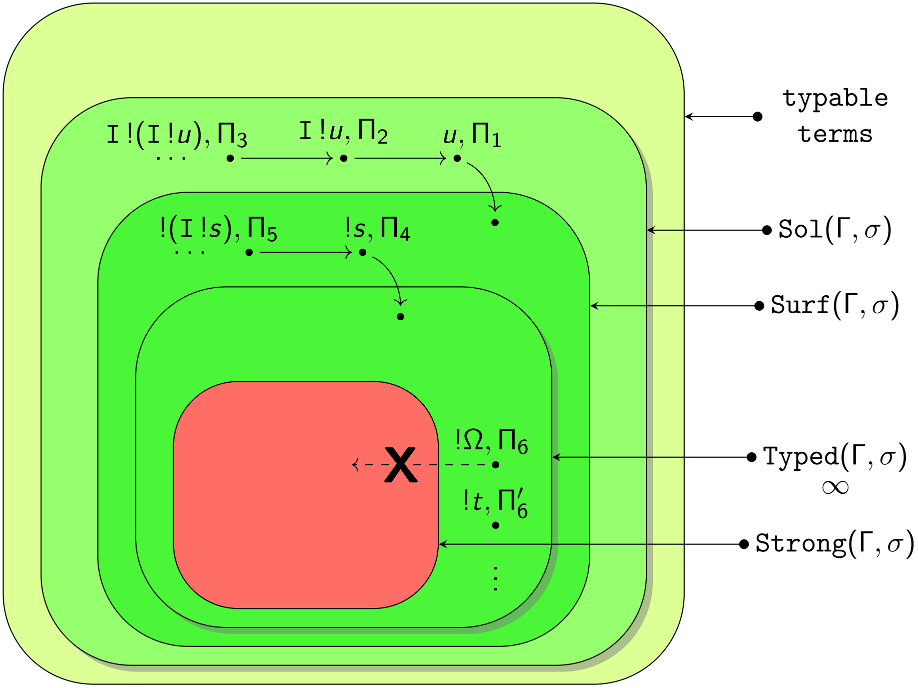
t, Π_0
typable terms
$\mathtt{I}\,!(\mathtt{I}\,!u), \Pi_3$
$\mathtt{I}\,!u, \Pi_2$
u, Π_1
$!(\mathtt{I}\,!s), \Pi_5$
$!s, \Pi_4$
$!\Omega, \Pi_6$
$\mathtt{Sol}(\Gamma, \sigma)$
$\mathtt{Surf}(\Gamma, \sigma)$
$\mathtt{Typed}(\Gamma, \sigma)$
∞
$\mathtt{Strong}(\Gamma, \sigma)$



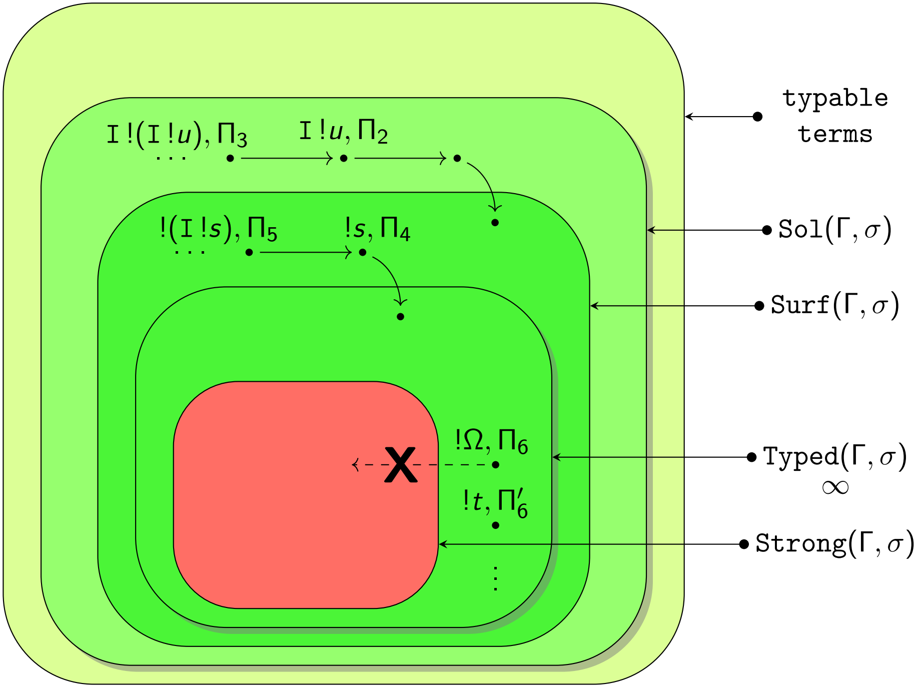
t, Π_0
typable terms
$\mathtt{I}\,!(\mathtt{I}\,!u), \Pi_3$
$\mathtt{I}\,!u, \Pi_2$
u, Π_1
$\mathtt{Sol}(\Gamma, \sigma)$
$!(\mathtt{I}\,!s), \Pi_5$
$!s, \Pi_4$
$\mathtt{Surf}(\Gamma, \sigma)$
$!\Omega, \Pi_6$
$\mathtt{Typed}(\Gamma, \sigma)$
∞
$!t, \Pi_6'$
$\mathtt{Strong}(\Gamma, \sigma)$



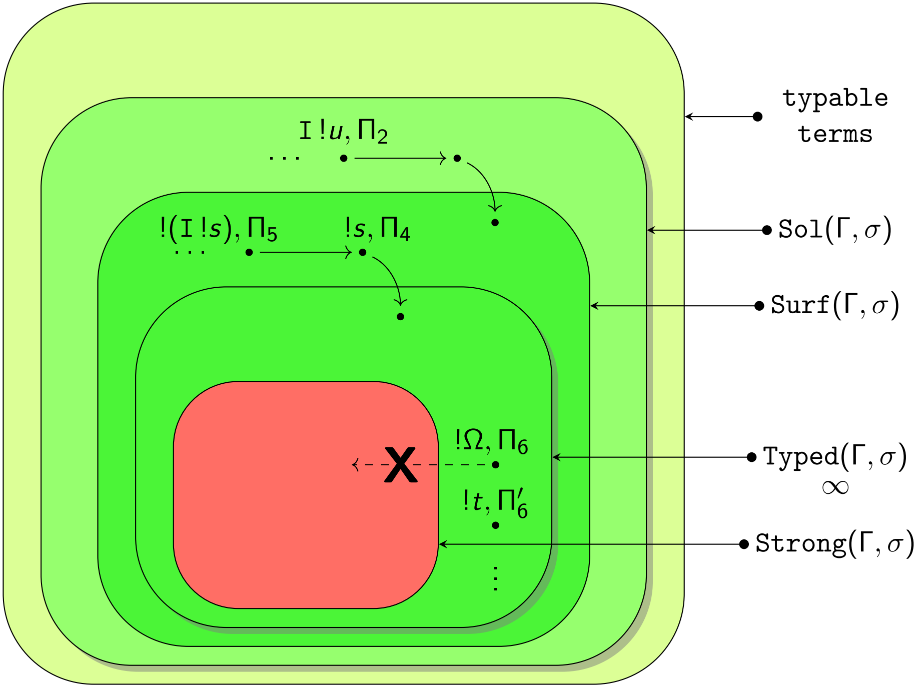
t, Π_0
$\mathtt{I}\,!(\mathtt{I}\,!u), \Pi_3$
$\mathtt{I}\,!u, \Pi_2$
u, Π_1
$!(\mathtt{I}\,!s), \Pi_5$
$!s, \Pi_4$
$!\Omega, \Pi_6$
$!t, \Pi_6'$
typable terms
$\mathtt{Sol}(\Gamma, \sigma)$
$\mathtt{Surf}(\Gamma, \sigma)$
$\mathtt{Typed}(\Gamma, \sigma)$
∞
$\mathtt{Strong}(\Gamma, \sigma)$



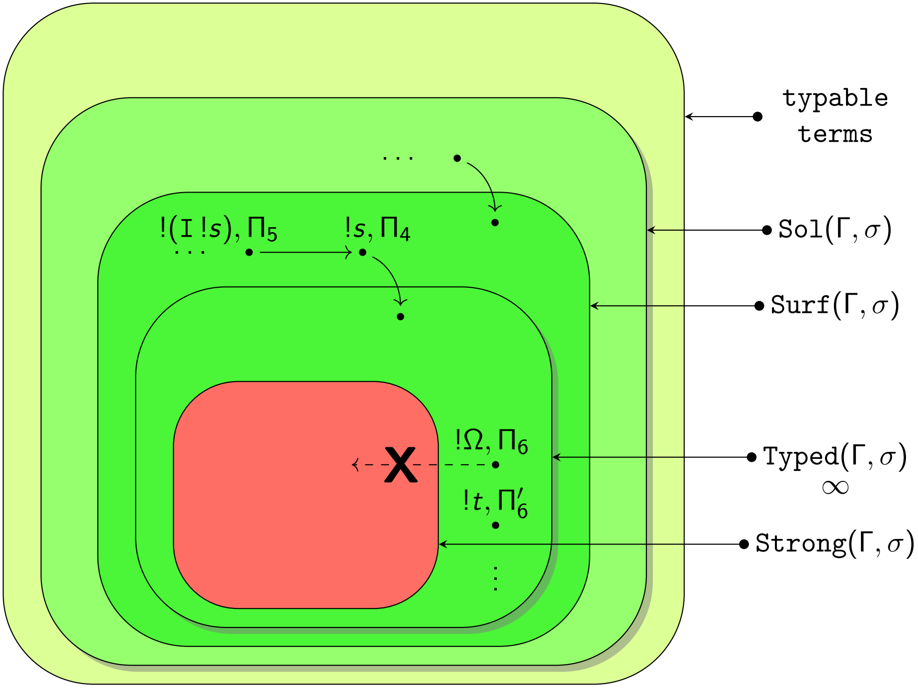
$\mathtt{I}\,!(\mathtt{I}\,!u), \Pi_3$
$\mathtt{I}\,!u, \Pi_2$
u, Π_1
$!(\mathtt{I}\,!s), \Pi_5$
$!s, \Pi_4$
$!\Omega, \Pi_6$
$!t, \Pi'_6$
typable terms
$\mathtt{Sol}(\Gamma, \sigma)$
$\mathtt{Surf}(\Gamma, \sigma)$
$\mathtt{Typed}(\Gamma, \sigma)$
∞
$\mathtt{Strong}(\Gamma, \sigma)$



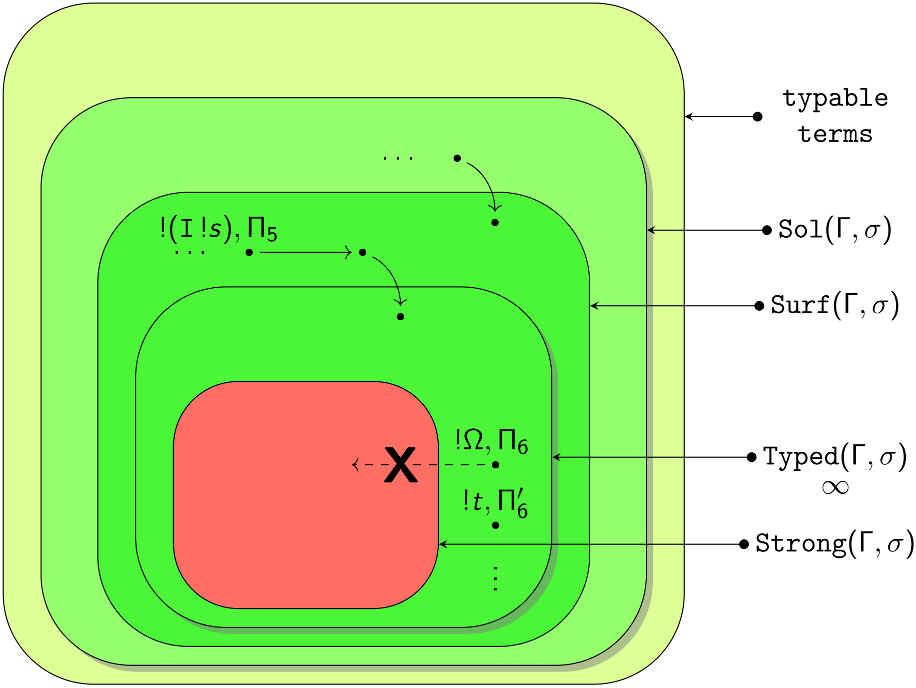
typable terms
Sol(Γ, σ)
Surf(Γ, σ)
Typed(Γ, σ)
∞
Strong(Γ, σ)
I !(I !u), Π3
I !u, Π2
!(I !s), Π5
!s, Π4
!Ω, Π6
!t, Π'6



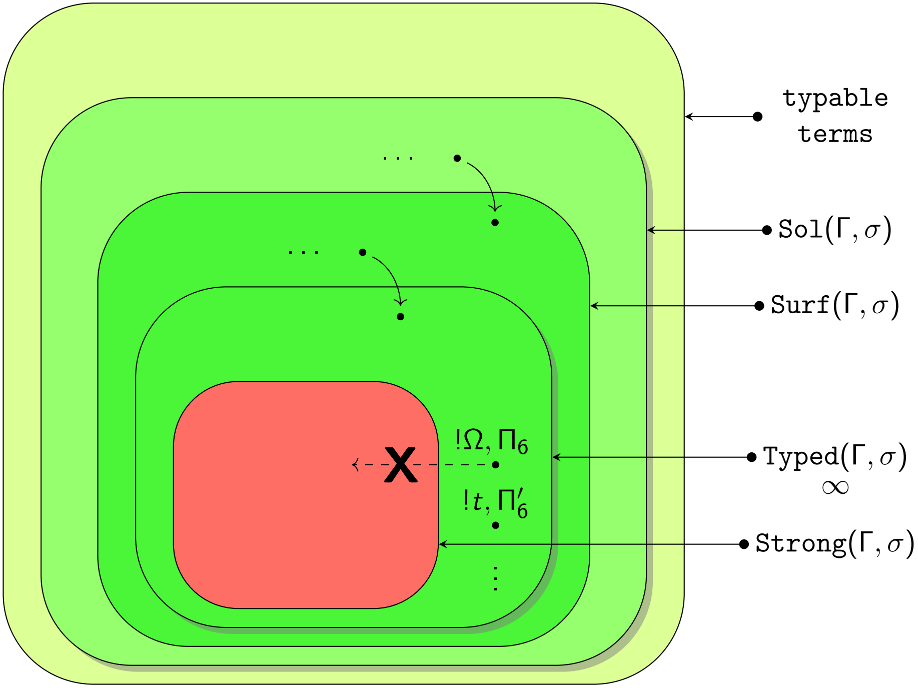
I !u, Π2
!(I !s), Π5
!s, Π4
!Ω, Π6
!t, Π′6
typable terms
Sol(Γ, σ)
Surf(Γ, σ)
Typed(Γ, σ)
∞
Strong(Γ, σ)



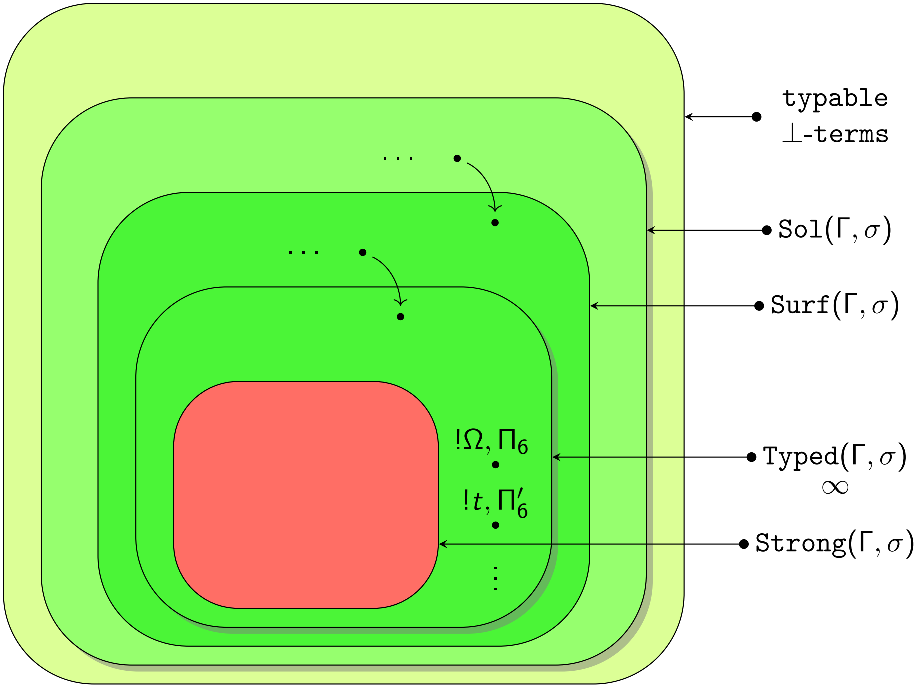
typable terms
Sol(Γ, σ)
Surf(Γ, σ)
Typed(Γ, σ)
∞
Strong(Γ, σ)
!(I !s), Π5
!s, Π4
!Ω, Π6
!t, Π′6



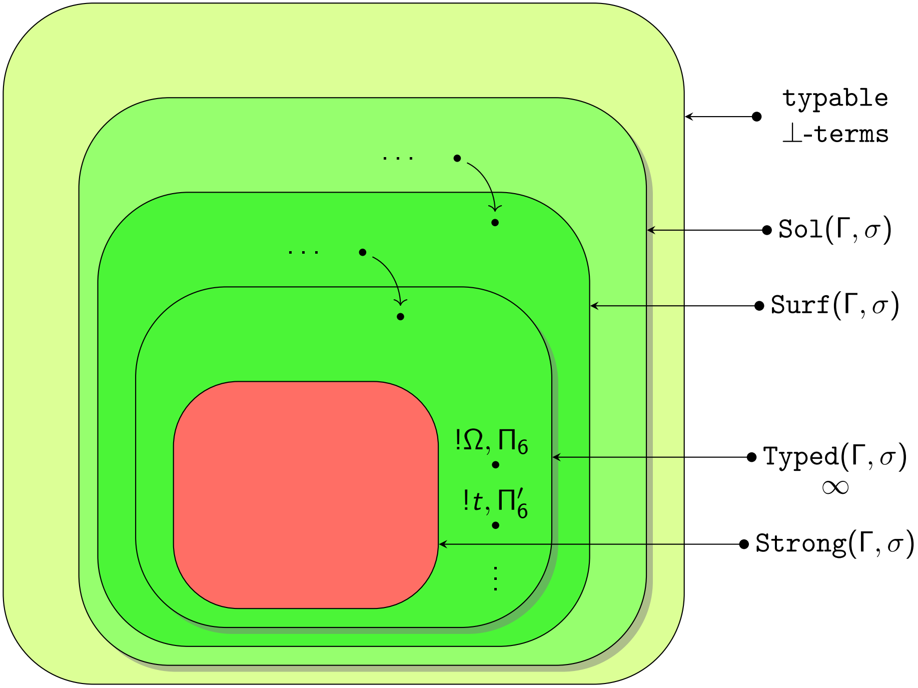
typable terms
Sol(Γ, σ)
Surf(Γ, σ)
Typed(Γ, σ)
∞
Strong(Γ, σ)
!(I !s), Π5
!Ω, Π6
!t, Π′6

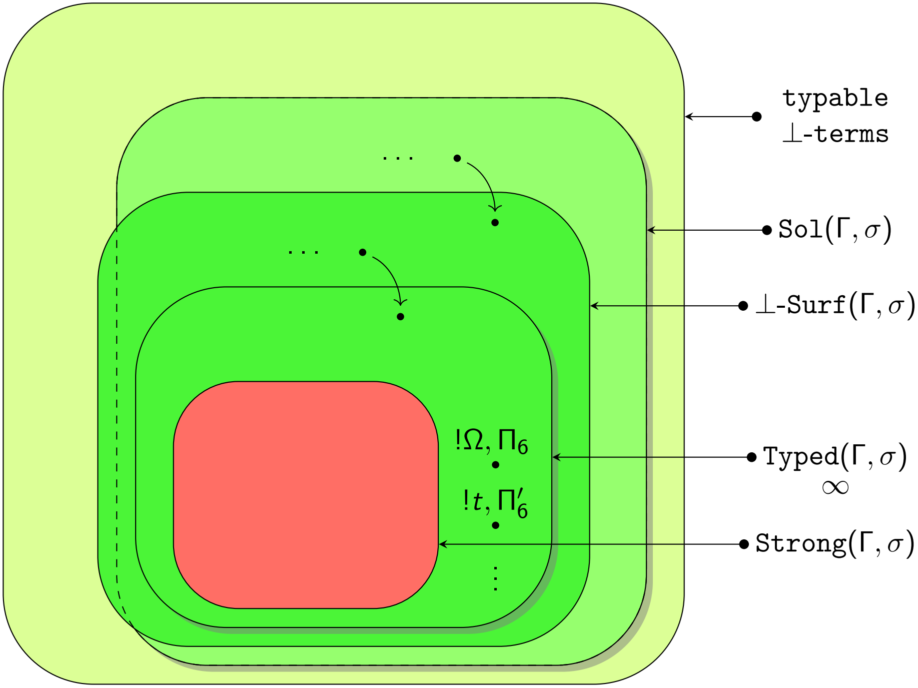


typable terms
Sol(Γ, σ)
Surf(Γ, σ)
Typed(Γ, σ)
∞
Strong(Γ, σ)
!Ω, Π₆
!t, Π′₆

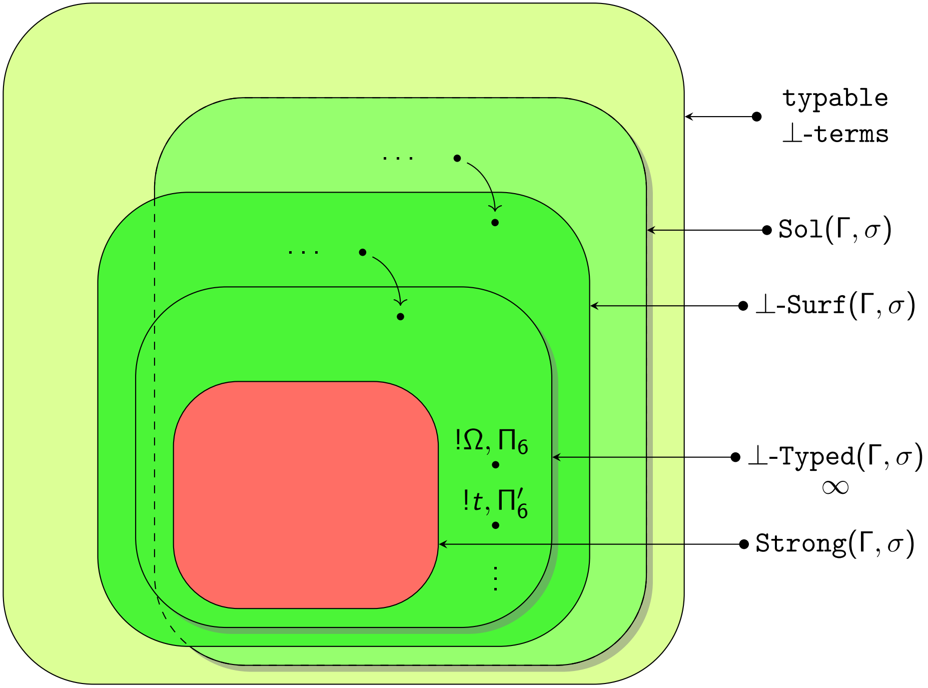


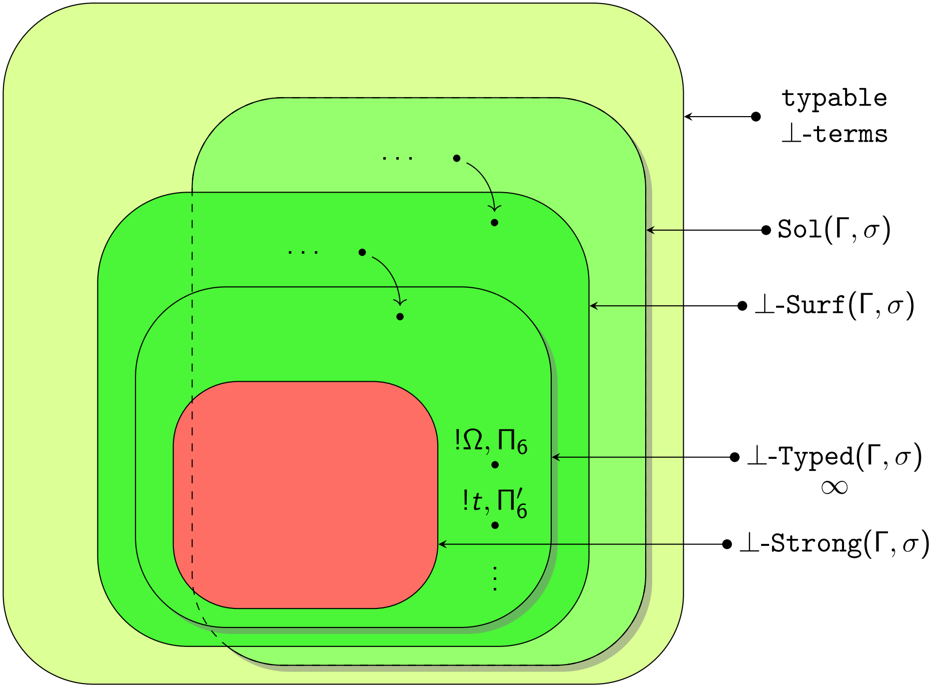
typable
⊥-terms
Sol(Γ, σ)
Surf(Γ, σ)
Typed(Γ, σ)
∞
Strong(Γ, σ)
!Ω, Π6
!t, Π′6


typable
⊥-terms
Sol(Γ, σ)
Surf(Γ, σ)
Typed(Γ, σ)
∞
Strong(Γ, σ)
!Ω, Π6
!t, Π′6

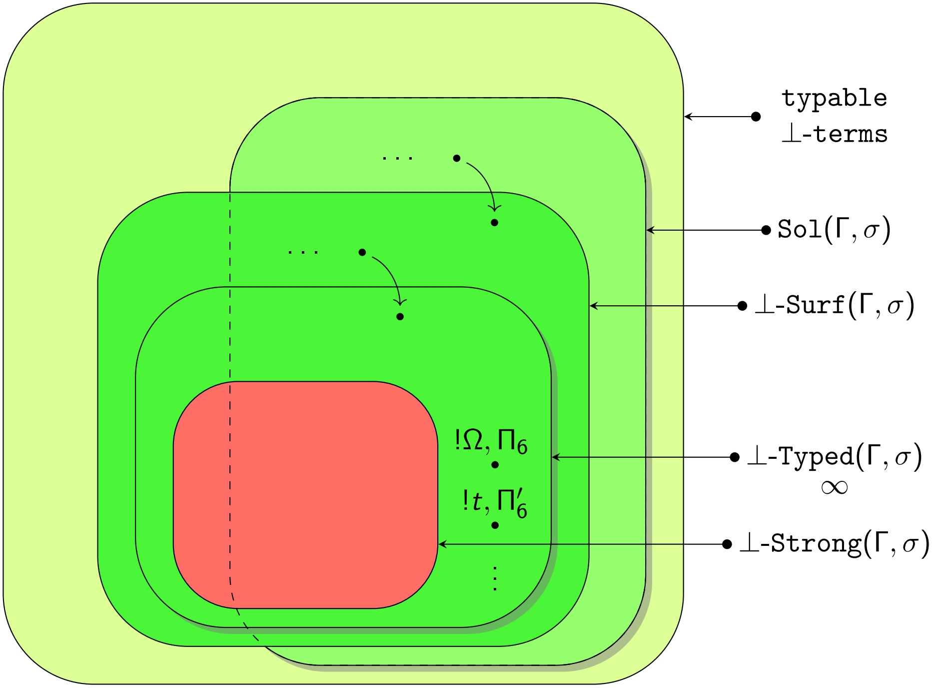


typable
⊥-terms
Sol(Γ, σ)
⊥-Surf(Γ, σ)
Typed(Γ, σ)
∞
Strong(Γ, σ)
!Ω, Π6
!t, Π′6

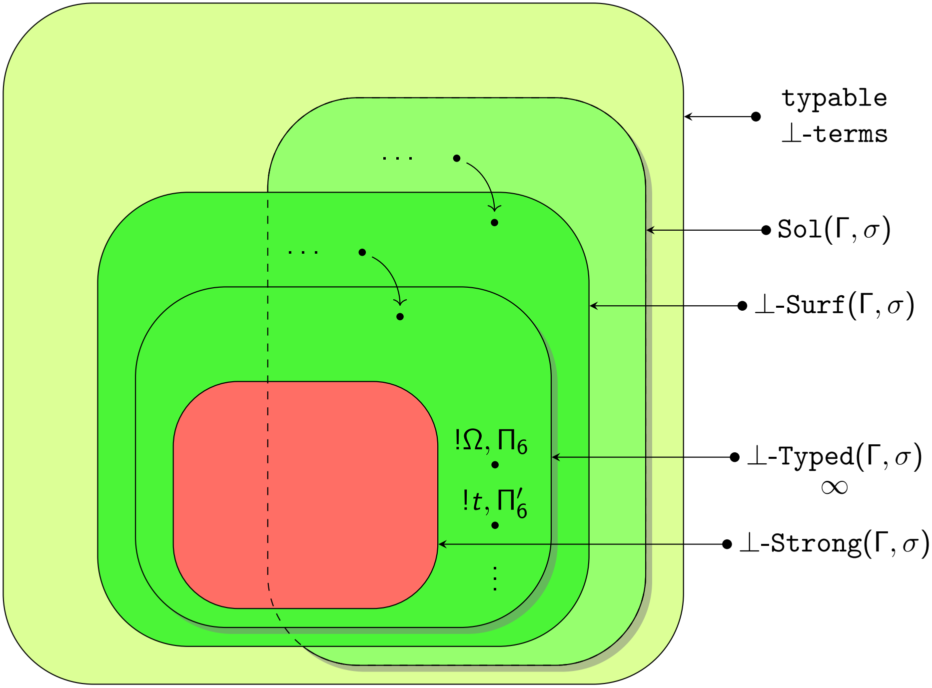

typable
⊥-terms
Sol(Γ, σ)
⊥-Surf(Γ, σ)
⊥-Typed(Γ, σ)
∞
Strong(Γ, σ)
!Ω, Π6
!t, Π'6



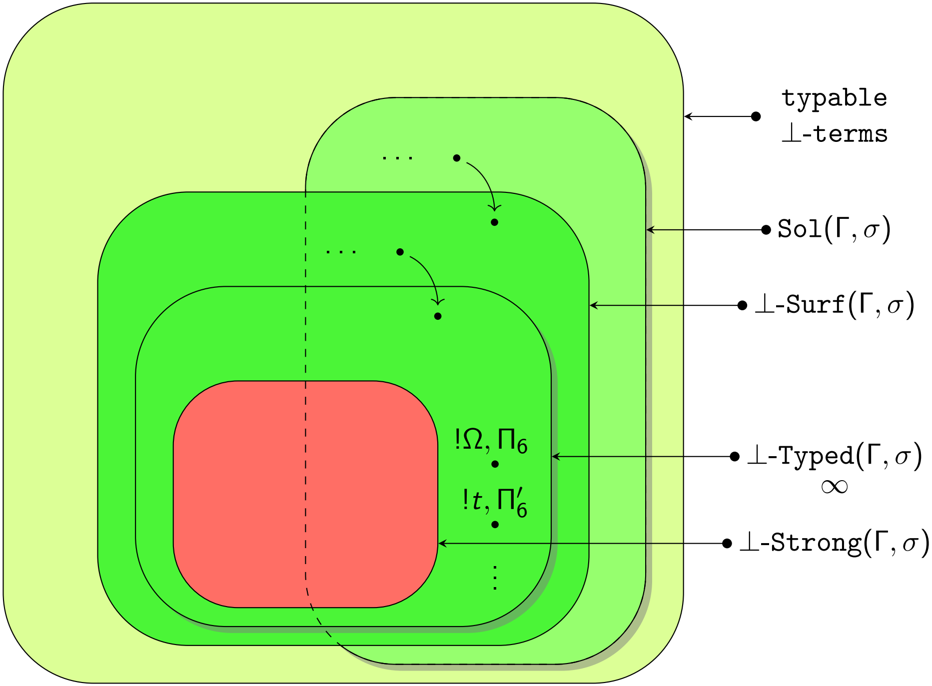
typable
⊥-terms
Sol(Γ, σ)
⊥-Surf(Γ, σ)
⊥-Typed(Γ, σ)
∞
⊥-Strong(Γ, σ)
!Ω, Π6
!t, Π'6



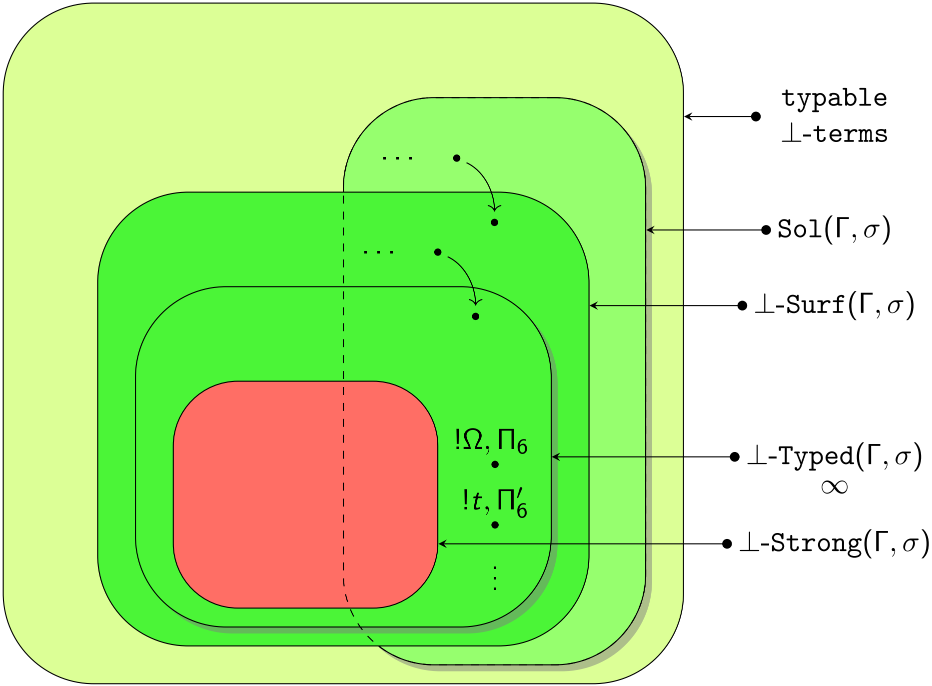
typable
⊥-terms
Sol(Γ, σ)
⊥-Surf(Γ, σ)
⊥-Typed(Γ, σ)
∞
⊥-Strong(Γ, σ)
!Ω, Π6
!t, Π'6



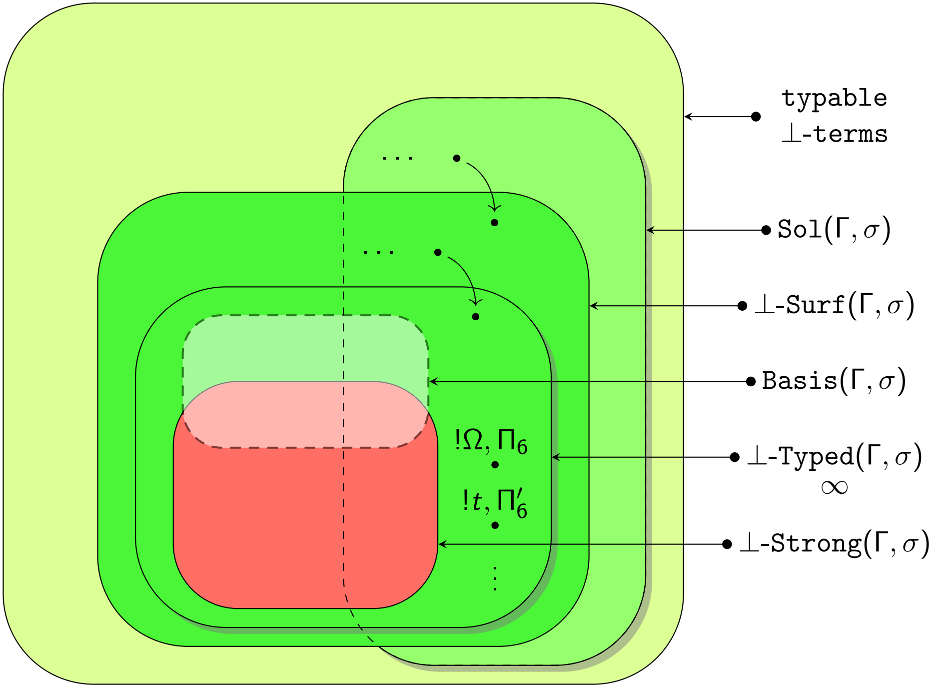
typable
⊥-terms
$\mathtt{Sol}(\Gamma, \sigma)$
$\perp\text{-}\mathtt{Surf}(\Gamma, \sigma)$
$\perp\text{-}\mathtt{Typed}(\Gamma, \sigma)$
∞
$\perp\text{-}\mathtt{Strong}(\Gamma, \sigma)$
$!\Omega, \Pi_6$
$!t, \Pi'_6$



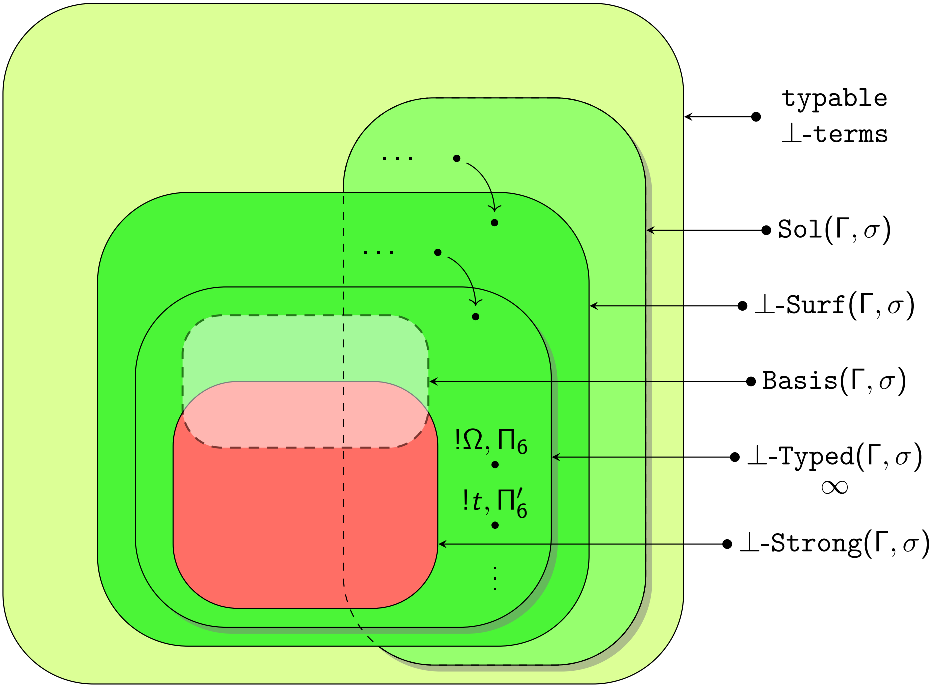
typable ⊥-terms
Sol(Γ, σ)
⊥-Surf(Γ, σ)
⊥-Typed(Γ, σ)
∞
⊥-Strong(Γ, σ)
!Ω, Π6
!t, Π′6

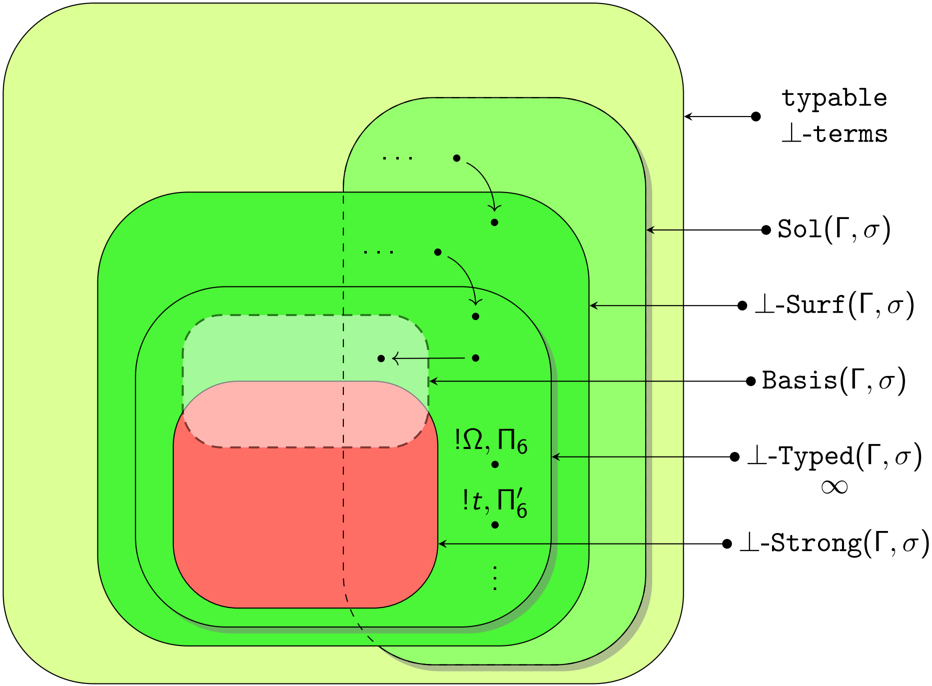


typable
⊥-terms
Sol(Γ, σ)
⊥-Surf(Γ, σ)
⊥-Typed(Γ, σ)
∞
⊥-Strong(Γ, σ)
!Ω, Π6
!t, Π′6

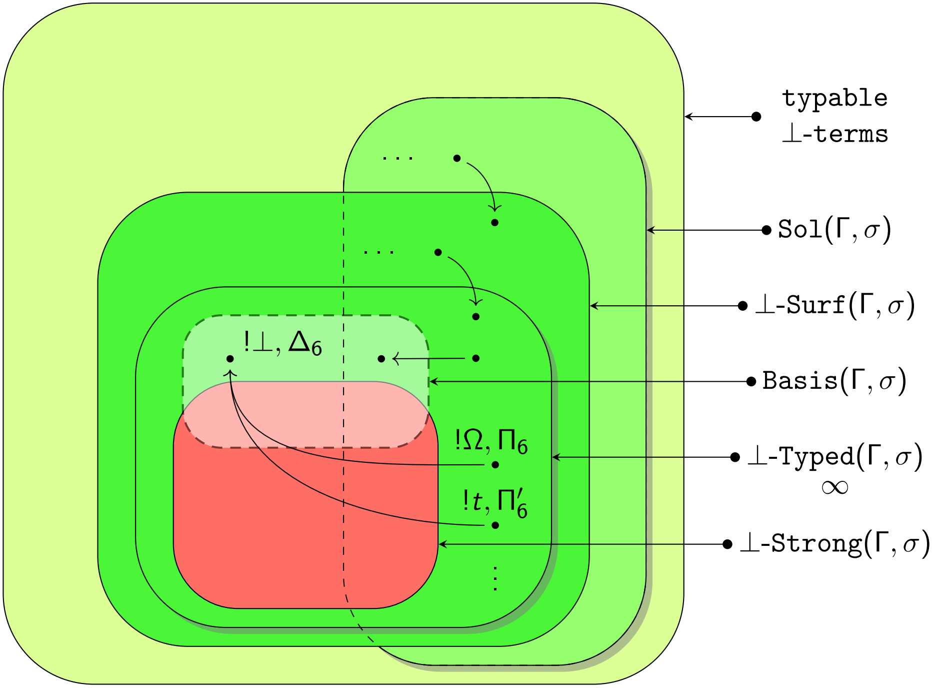


typable
⊥-terms
Sol(Γ, σ)
⊥-Surf(Γ, σ)
Basis(Γ, σ)
⊥-Typed(Γ, σ)
∞
⊥-Strong(Γ, σ)
!Ω, Π6
!t, Π′6

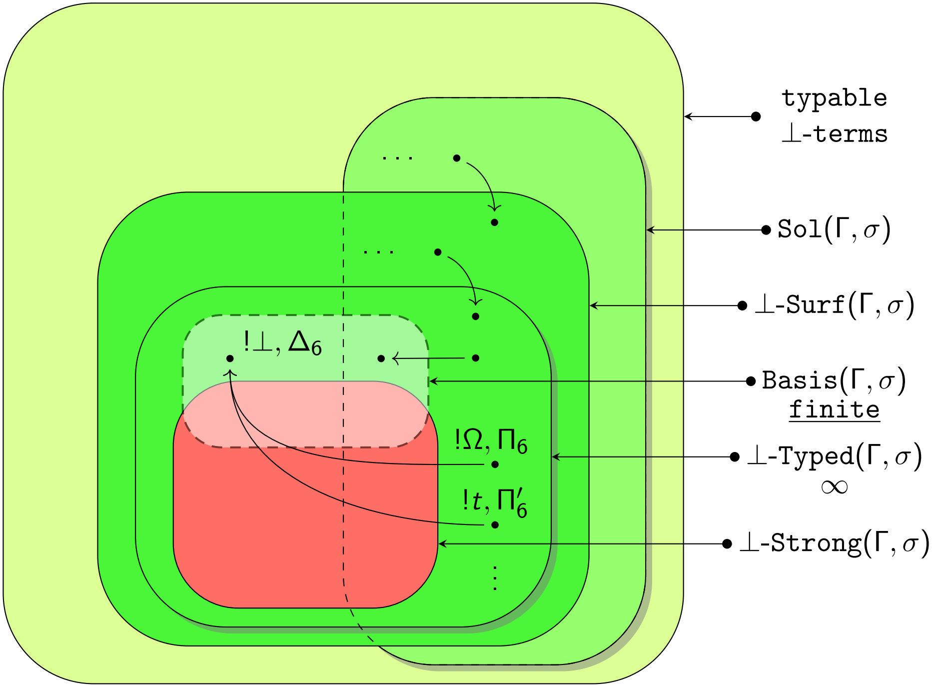

typable
⊥-terms
Sol(Γ, σ)
⊥-Surf(Γ, σ)
Basis(Γ, σ)
⊥-Typed(Γ, σ)
∞
⊥-Strong(Γ, σ)
!Ω, Π6
!t, Π′6



typable
⊥-terms
Sol(Γ, σ)
⊥-Surf(Γ, σ)
Basis(Γ, σ)
⊥-Typed(Γ, σ)
∞
⊥-Strong(Γ, σ)
!Ω, Π6
!t, Π′6



typable
⊥-terms
Sol(Γ, σ)
⊥-Surf(Γ, σ)
Basis(Γ, σ)
⊥-Typed(Γ, σ)
∞
⊥-Strong(Γ, σ)
!⊥, Δ6
!Ω, Π6
!t, Π′6



typable
⊥-terms
Sol(Γ, σ)
⊥-Surf(Γ, σ)
Basis(Γ, σ)
finite
⊥-Typed(Γ, σ)
∞
⊥-Strong(Γ, σ)
!⊥, Δ6
!Ω, Π6
!t, Π′6

The Full Algorithm

$$\frac{g \rightsquigarrow \mathsf{Var} \quad |}{x \Vdash_g H^{x:[\sigma]}(\emptyset;\sigma)}\ \text{VAR} \qquad \frac{\begin{matrix} g \rightsquigarrow \mathsf{Der}(g') \\ [\sigma] \Vvdash S(\tau,\diamond)\end{matrix} \quad \Big| \quad a \Vdash_{g'} H^{x:[\tau]}(\Gamma;[\sigma])}{\mathsf{der}(a) \Vdash_g H^{x:[\tau]}(\Gamma;\sigma)}\ \text{DR}$$

$$\frac{g \rightsquigarrow g' \quad | \quad a \Vdash_{g'} H^{x:[\tau]}(\Gamma;\sigma)}{a \Vdash_g H^{x:[\tau]}(\Gamma;\sigma)}\ \text{H-H} \qquad \frac{\begin{matrix} g \rightsquigarrow g' \\ \Gamma = \Gamma' + x:[\tau] \\ \sigma \Vvdash S(\tau,\diamond)\end{matrix} \quad \Big| \quad a \Vdash_{g'} H^{x:[\tau]}(\Gamma';\sigma)}{a \Vdash_g N(\Gamma;\sigma)}\ \text{N-H} \qquad \frac{g \rightsquigarrow g' \quad | \quad a \Vdash_{g'} N(\Gamma;\sigma)}{a \Vdash_g N(\Gamma;\sigma)}\ \text{N-N}$$

$$\frac{\begin{matrix} g \rightsquigarrow \mathsf{App}(g_a, g_b) \\ \Gamma = \Gamma_a + \Gamma_b \\ \mathcal{M} \Rightarrow \sigma \Vvdash S(\tau, \diamond \Rightarrow \sigma)\end{matrix} \quad \Big| \quad a \Vdash_{g_a} H^{x:[\tau]}(\Gamma_a; \mathcal{M} \Rightarrow \sigma) \quad b \Vdash_{g_b} N(\Gamma_b; \mathcal{M})}{ab \Vdash_g H^{x:[\tau]}(\Gamma;\sigma)}\ \text{APP}$$

$$\frac{n \in [\![0, \mathrm{sz}(\rho)]\!],\ \mathcal{M} \Vvdash S(\rho, [\diamond_1, \dots, \diamond_n]) \quad | \quad a \Vdash_{g_a} H^{x:[\tau]}(\Gamma_a, y:\mathcal{M};\sigma) \quad b \Vdash_{g_b} H^{x:[\rho]}(\Gamma_b;\mathcal{M})}{a[y\backslash b] \Vdash_g H^{x:[\tau]}(\Gamma;\sigma)}\ \text{ES-H}$$

$$\frac{\begin{matrix} g \rightsquigarrow \mathsf{Sub}(g_a, g_b) \\ \Gamma = \Gamma_a + \Gamma_b, \quad \text{fix } y \notin \mathrm{dom}(\Gamma) \cup \{x\} \\ n \in [\![1, \mathrm{sz}(\tau)]\!],\ [\rho_i]_{i \in [\![1,n]\!]} \Vvdash S(\tau, [\diamond_1, \dots, \diamond_n]) \\ j \in [\![1,n]\!],\ \sigma \Vvdash S(\rho_j, \diamond)\end{matrix} \quad \Big| \quad a \Vdash_{g_a} H^{y:[\rho_j]}(\Gamma_a, y:[\rho_i]_{i \in [\![1,n]\!] \backslash j};\sigma) \quad b \Vdash_{g_b} H^{x:[\tau]}(\Gamma_b; [\rho_i]_{i \in [\![1,n]\!]})}{a[y\backslash b] \Vdash_g H^{x:[\tau]}(\Gamma;\sigma)}\ \text{ES-CH}$$

$$\frac{\begin{matrix} g \rightsquigarrow \mathsf{Sub}(g_a, g_b) \\ \Gamma = \Gamma_a + \Gamma_b + z:[\tau], \quad \text{fix } y \notin \mathrm{dom}(\Gamma) \\ n \in [\![0, \mathrm{sz}(\tau)]\!],\ \mathcal{M} \Vvdash S(\tau, [\diamond_1, \dots, \diamond_n])\end{matrix} \quad \Big| \quad a \Vdash_{g_a} N(\Gamma_a, y:\mathcal{M};\sigma) \quad b \Vdash_{g_b} H^{z:[\tau]}(\Gamma_b;\mathcal{M})}{a[y\backslash b] \Vdash_g N(\Gamma;\sigma)}\ \text{ES-N}$$

The Full Algorithm

$$\dfrac{g \rightsquigarrow \mathsf{Var} \quad \mid}{x \Vdash_g H^{x:[\sigma]}(\emptyset;\sigma)}\ \textsf{VAR}
\qquad
\dfrac{\begin{array}{c} g \rightsquigarrow \mathsf{Der}(g') \\ [\sigma] \Vvdash S(\tau,\diamond) \end{array} \;\Big|\; a \Vdash_{g'} H^{x:[\tau]}(\Gamma;[\sigma])}{\mathsf{der}(a) \Vdash_g H^{x:[\tau]}(\Gamma;\sigma)}\ \textsf{DR}$$

$$\dfrac{\begin{array}{c} g \rightsquigarrow \mathsf{App}(g_a,g_b) \\ \Gamma = \Gamma_a + \Gamma_b \\ \mathcal{M} \Rightarrow \sigma \Vvdash S(\tau, \diamond \Rightarrow \sigma) \end{array} \;\Big|\; a \Vdash_{g_a} H^{x:[\tau]}(\Gamma_a;\mathcal{M} \Rightarrow \sigma) \quad b \Vdash_{g_b} N(\Gamma_b;\mathcal{M})}{ab \Vdash_g H^{x:[\tau]}(\Gamma;\sigma)}\ \textsf{APP}$$

$$\dfrac{g \rightsquigarrow g' \;\mid\; a \Vdash_{g'} H^{x:[\tau]}(\Gamma;\sigma)}{a \Vdash_g H^{x:[\tau]}(\Gamma;\sigma)}\ \textsf{H-H}
\qquad
\dfrac{\begin{array}{c} g \rightsquigarrow g' \\ \Gamma = \Gamma' + x : [\tau] \\ \sigma \Vvdash S(\tau,\diamond) \end{array} \;\Big|\; a \Vdash_{g'} H^{x:[\tau]}(\Gamma';\sigma)}{a \Vdash_g N(\Gamma;\sigma)}\ \textsf{N-H}
\qquad
\dfrac{g \rightsquigarrow g' \;\mid\; a \Vdash_{g'} N(\Gamma;\sigma)}{a \Vdash_g N(\Gamma;\sigma)}\ \textsf{N-N}$$

$$\dfrac{\begin{array}{c} g \rightsquigarrow \mathsf{Lam}(g') \\ \mathtt{fix}\ \ x \notin \mathsf{dom}(\Gamma) \end{array} \;\Big|\; a \Vdash_{g'} N(\Gamma, x : \mathcal{M};\sigma)}{\lambda x.a \Vdash_g N(\Gamma;\mathcal{M} \Rightarrow \sigma)}\ \textsf{ABS}
\qquad
\dfrac{\begin{array}{c} g \rightsquigarrow \mathsf{Bng}(g') \\ I \neq \emptyset \\ \Gamma = +_{i\in I}\Gamma_i \end{array} \;\Big|\; \left(a_i \Vdash_{g'} N(\Gamma_i;\tau_i)\right)_{i\in I} \quad \uparrow_{i\in I} a_i}{!\bigvee_{i\in I} a_i \Vdash_g N(\Gamma;[\tau_i]_{i\in I})}\ \textsf{BG}
\qquad
\dfrac{g \rightsquigarrow \mathsf{Bng}(\bot) \quad \mid}{!\bot \Vdash_g N(\emptyset;[\,])}\ \textsf{BG}_\bot$$

$$\dfrac{\begin{array}{c} g \rightsquigarrow \mathsf{Sub}(g_a,g_b) \\ \Gamma = \Gamma_a + \Gamma_b + z : [\rho], \qquad \mathtt{fix}\ \ y \notin \mathsf{dom}(\Gamma) \cup \{x\} \\ n \in \llbracket 0, \mathsf{sz}(\rho) \rrbracket,\ \mathcal{M} \Vvdash S(\rho,[\diamond_1,\ldots,\diamond_n]) \end{array} \;\Big|\; a \Vdash_{g_a} H^{x:[\tau]}(\Gamma_a, y : \mathcal{M};\sigma) \quad b \Vdash_{g_b} H^{z:[\rho]}(\Gamma_b;\mathcal{M})}{a[y \backslash b] \Vdash_g H^{x:[\tau]}(\Gamma;\sigma)}\ \textsf{ES-H}$$

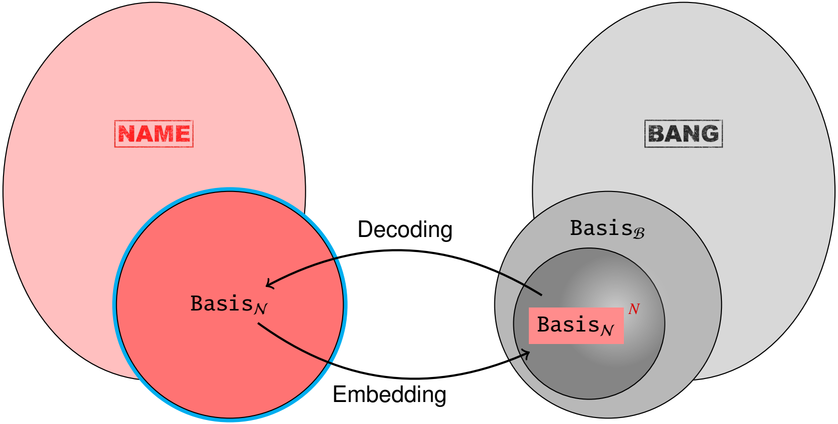
$$\dfrac{\begin{array}{c} g \rightsquigarrow \mathsf{Sub}(g_a,g_b) \\ \Gamma = \Gamma_a + \Gamma_b, \qquad \mathtt{fix}\ \ y \notin \mathsf{dom}(\Gamma) \cup \{x\} \\ n \in \llbracket 1, \mathsf{sz}(\tau) \rrbracket,\ [\rho_i]_{i\in\llbracket 1,n\rrbracket} \Vvdash S(\tau,[\diamond_1,\ldots,\diamond_n]) \\ j \in \llbracket 1, n \rrbracket,\ \sigma \Vvdash S(\rho_j,\diamond) \end{array} \;\Big|\; a \Vdash_{g_a} H^{y:[\rho_j]}(\Gamma_a, y : [\rho_i]_{i\in\llbracket 1,n\rrbracket\backslash j};\sigma) \quad b \Vdash_{g_b} H^{x:[\tau]}(\Gamma_b;[\rho_i]_{i\in\llbracket 1,n\rrbracket})}{a[y \backslash b] \Vdash_g H^{x:[\tau]}(\Gamma;\sigma)}\ \textsf{ES-CH}$$

$$\dfrac{\begin{array}{c} g \rightsquigarrow \mathsf{Sub}(g_a,g_b) \\ \Gamma = \Gamma_a + \Gamma_b + z : [\tau], \qquad \mathtt{fix}\ \ y \notin \mathsf{dom}(\Gamma) \\ n \in \llbracket 0, \mathsf{sz}(\tau) \rrbracket,\ \mathcal{M} \Vvdash S(\tau,[\diamond_1,\ldots,\diamond_n]) \end{array} \;\Big|\; a \Vdash_{g_a} N(\Gamma_a, y : \mathcal{M};\sigma) \quad b \Vdash_{g_b} H^{z:[\tau]}(\Gamma_b;\mathcal{M})}{a[y \backslash b] \Vdash_g N(\Gamma;\sigma)}\ \textsf{ES-N}$$

The `Basis` is preserved by the embedding:

Theorem

NAME $\quad t \in \mathtt{Basis}_{\mathcal{N}}(\Gamma, \sigma) \Leftrightarrow t^{N} \in \mathtt{Basis}_{\mathcal{B}}(\Gamma, \sigma) \quad$ BANG



$$t^{N} : \quad \textbf{NAME} \rightarrow \textbf{BANG}$$

$$
\begin{aligned}
x^{N} &:= x \\
(\lambda x.t)^{N} &:= \lambda x.\, t^{N} \\
(tu)^{N} &:= t^{N}\ !\ u^{N} \\
(t[x := u])^{N} &:= t^{N}[x := !\ u^{N}]
\end{aligned}
$$

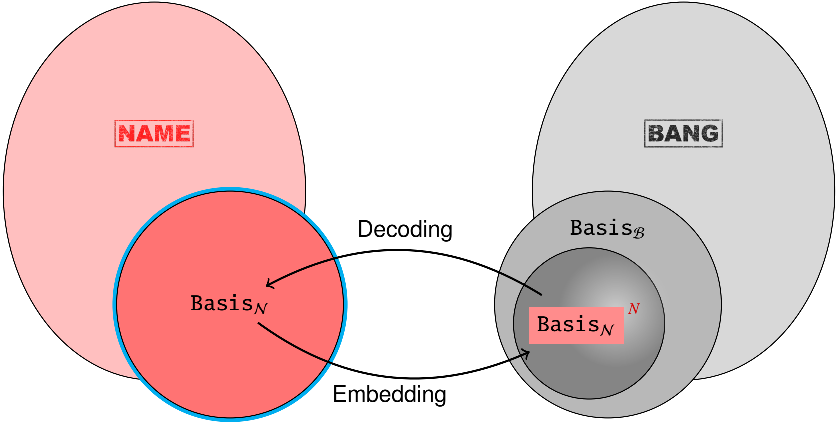
$$t^{V} : \quad \textbf{VALUE} \rightarrow \textbf{BANG}$$

$$
\begin{aligned}
x^{V} &:= !\ x \\
(\lambda x.t)^{V} &:= !\ \lambda x.\, t^{V} \\
(tu)^{V} &:= \mathtt{der}(t^{V})\ u^{V} \\
(t[x := u])^{V} &:= t^{V}[x := u^{V}]
\end{aligned}
$$

The `Basis` is preserved by the embedding:

Theorem

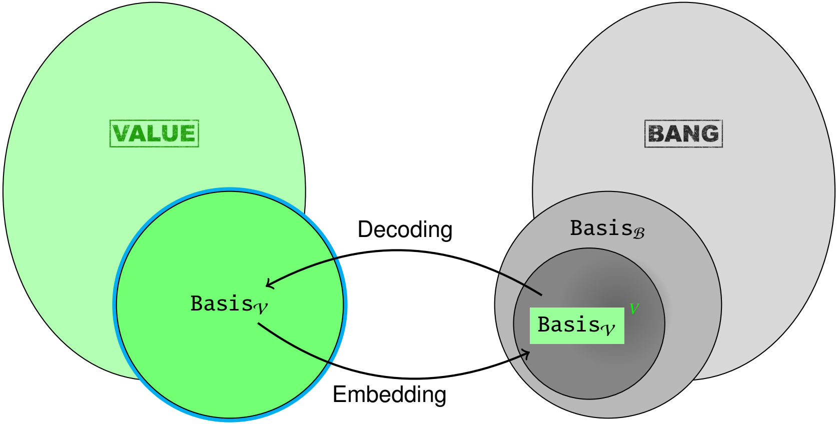
NAME $t \in \mathtt{Basis}_{\mathcal{N}}(\Gamma, \sigma) \Leftrightarrow t^{N} \in \mathtt{Basis}_{\mathcal{B}}(\Gamma, \sigma)$ BANG



The `Basis` is preserved by the embedding:

Theorem

VALUE $t \in \mathtt{Basis}_{\mathcal{V}}(\Gamma, \sigma) \Leftrightarrow t^{V} \in \mathtt{Basis}_{\mathcal{B}}(\Gamma, \sigma)$ BANG



$$t^{N} : \textbf{NAME} \to \textbf{BANG}$$

$$x^{N} := x$$

$$(\lambda x.t)^{N} := \lambda x.\, t^{N}$$

$$(tu)^{N} := t^{N} \,!\, u^{N}$$

$$(t[x := u])^{N} := t^{N}[x :=!\, u^{N}]$$

$$t^{V} : \textbf{VALUE} \to \textbf{BANG}$$

$$x^{V} := !\, x$$

$$(\lambda x.t)^{V} := !\, \lambda x.\, t^{V}$$

$$(tu)^{V} := \mathtt{der}(t^{V})\, u^{V}$$

$$(t[x := u])^{V} := t^{V}[x := u^{V}]$$

$\boxed{t}^{N}$: **NAME** $\rightarrow$ **BANG**

$$
\begin{aligned}
\boxed{x}^{N} &:= x \\
\boxed{\lambda x.t}^{N} &:= \lambda x.\boxed{t}^{N} \\
\boxed{tu}^{N} &:= \boxed{t}^{N}\,!\,\boxed{u}^{N} \\
\boxed{t[x := u]}^{N} &:= \boxed{t}^{N}[x :=!\,\boxed{u}^{N}]
\end{aligned}
$$

$\boxed{t}^{V}$: **VALUE** $\rightarrow$ **BANG**

$$
\begin{aligned}
\boxed{x}^{V} &:= !\,x \\
\boxed{\lambda x.t}^{V} &:= !\,\lambda x.!\,\boxed{t}^{V} \\
\boxed{tu}^{V} &:= \mathtt{der}(\boxed{t}^{V})\,\boxed{u}^{V} \\
\boxed{t[x := u]}^{V} &:= \boxed{t}^{V}[x := \boxed{u}^{V}]
\end{aligned}
$$

$$t^{N} : \text{NAME} \to \text{BANG}$$

$$\begin{aligned}
x^{N} &:= x \\
(\lambda x.t)^{N} &:= \lambda x.\, t^{N} \\
(tu)^{N} &:= t^{N}\, !\, u^{N} \\
(t[x := u])^{N} &:= t^{N}[x :=!\ u^{N}]
\end{aligned}$$

$$t^{V} : \text{VALUE} \to \text{BANG}$$

$$\begin{aligned}
x^{V} &:= !\, x \\
(\lambda x.t)^{V} &:= !\, \lambda x.!\, t^{V} \\
(tu)^{V} &:= \mathtt{der}(\mathtt{der}(t^{V})\, u^{V}) \\
(t[x := u])^{V} &:= t^{V}[x := u^{V}]
\end{aligned}$$

Properties of the Indirect NAME and VALUE Algorithm

Theorem

- *The inhabitation algorithm terminates.*
- *The algorithm is sound and complete (i.e. it exactly computes* $\texttt{Basis}\ (\Gamma, \sigma)$*).*

More Ambitious Third Goal

- Decidability by **finding all inhabitants** in the BANG IP.
- Decidability of the NAME and VALUE IP by **finding all inhabitants** from those of the BANG IP.
- Using generic properties so that other encodable models of computation can use these results.