

Integrality in the Matching-Jack conjecture and the Farahat-Higman algebra

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Generating series of permutations and matchings

1 [Representation theory of the symmetric group]

$$\sum_{\theta} t^{|\theta|} \frac{|\theta|!}{\dim(\theta)} s_{\theta}(\mathbf{p}) s_{\theta}(\mathbf{q}) s_{\theta}(\mathbf{r}) = \sum_{n \geq 0} t^n \sum_{\lambda, \mu, \nu \vdash n} \frac{\gamma_{\mu, \nu}^{\lambda}}{z_{\lambda}} p_{\lambda} q_{\mu} r_{\nu},$$

s_{θ} : the Schur function associated to the partition θ , expressed in the power-sum bases

$\mathbf{p} := (p_i)_{i \geq 1}$; $\mathbf{q} := (q_i)_{i \geq 1}$; $\mathbf{r} := (r_i)_{i \geq 1}$.

$z_{\lambda} := \frac{|\lambda|!}{c_{\lambda}}$.

$\gamma_{\mu, \nu}^{\lambda} := |\{(\sigma_1, \sigma_2) \text{ of type } (\mu, \nu) \text{ such that } \sigma_1 \cdot \sigma_2 = \sigma_{\lambda}\}|$, where σ_{λ} is a fixed permutation of type λ .

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Proof:

- ▶ $s_{\theta}(\mathbf{p}) = \sum_{\lambda \vdash |\theta|} \frac{\chi^{\theta}(\lambda)}{z_{\lambda}} p_{\lambda}$ χ^{θ} : characters of the symmetric group.
- ▶ $\gamma_{\mu, \nu}^{\lambda} = \sum_{\theta \vdash n} \frac{|\theta|!}{\dim(\theta) z_{\mu} z_{\nu}} \chi^{\theta}(\lambda) \chi^{\theta}(\mu) \chi^{\theta}(\nu)$

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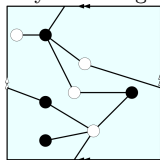
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The coefficients $\gamma_{\mu, \nu}^{\lambda}$ also count maps on **orientable** surfaces



A map on the torus

Generating series of permutations and matchings

- 2 Goulden-Jackson '96 [Representation Theory of the Gelfand pair $(\mathfrak{S}_{2n}, \mathfrak{B}_n)$]

$$\sum_{\theta} t^{|\theta|} \frac{\dim(2\theta)}{|2\theta|!} Z_{\theta}(\mathbf{p}) Z_{\theta}(\mathbf{q}) Z_{\theta}(\mathbf{r}) = \sum_{n \geq 0} t^n \sum_{\lambda, \mu, \nu \vdash n} \frac{\tilde{\gamma}_{\mu, \nu}^{\lambda}}{z_{\lambda} 2^{\ell(\lambda)}} p_{\lambda} q_{\mu} r_{\nu},$$

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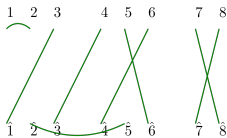
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a generalization of permutations

bipartite matchings \longleftrightarrow permutations

A matching of size 8.

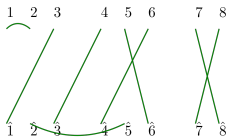
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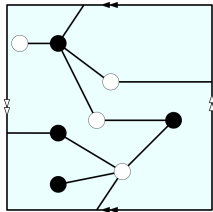


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The coefficients $\tilde{\gamma}_{\mu, \nu}^{\lambda}$ also count maps on general surfaces (orientable or not)



A map on the Klein bottle

Jack polynomials

We consider the following deformation of the Hall scalar product $\langle \cdot, \cdot \rangle_b$ defined on symmetric functions by

$$\langle p_\lambda, p_\mu \rangle_b = \delta_{\lambda\mu} z_\lambda (1+b)^{\ell(\lambda)}.$$

Definition

Jack polynomials of parameter $1+b$, denoted $J_\theta^{(b)}$ are defined as follows :

- 1 Triangularity and normalisation: if $\theta \vdash n$, then

$$J_\theta^{(b)} = \sum_{\mu \vdash n, \mu \leq \theta} u_{\theta\mu} m_\mu,$$

such that $u_{\theta[1^n]} = n!$.

(dominance order $\mu \leq \theta : \mu_1 + \mu_2 + \dots + \mu_i \leq \theta_1 + \theta_2 + \dots + \theta_i \forall i$)

- 2 Orthogonality: if $\theta \neq \xi$ then $\langle J_\theta^{(b)}, J_\xi^{(b)} \rangle_b = 0$.

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• For $b = 0 \rightarrow$ Schur functions $J_\theta^{(0)} = \frac{|\theta|!}{\dim(\theta)} s_\theta$.

• For $b = 1 \rightarrow$ Zonal polynomials $J_\theta^{(1)} = Z_\theta$.

The connection coefficients $c_{\mu,\nu}^\lambda$

$$\sum_{\theta \in \mathbb{Y}} \frac{t^{|\theta|}}{j_\theta^{(1+b)}} J_\theta^{(1+b)}(\mathbf{p}) J_\theta^{(1+b)}(\mathbf{q}) J_\theta^{(1+b)}(\mathbf{r}) = \sum_{n \geq 0} t^n \sum_{\lambda, \mu, \nu \vdash n} \frac{c_{\mu,\nu}^\lambda(b)}{z_\lambda (1+b)^{\ell(\lambda)}} p_\lambda q_\mu r_\nu,$$

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$b=0$

$$\begin{aligned} c_{\mu,\nu}^\lambda(0) &= |\{(\sigma_1, \sigma_2) \text{ of type } (\mu, \nu) \text{ such that } \sigma_1 \cdot \sigma_2 = \sigma_\lambda\}| \\ &= |\{\text{bipartite matchings } \delta \text{ of type } (\mu, \nu) \text{ with respect to } \lambda\}|. \end{aligned}$$

σ_λ : fixed permutation of type λ .

$b=1$

$$c_{\mu,\nu}^\lambda(1) = |\{\text{matchings } \delta \text{ of type } (\mu, \nu) \text{ with respect to } \lambda\}|.$$

Matching-Jack conjecture [Goulden and Jackson '96]

An "algebraic" formulation

The coefficients $c_{\mu,\nu}^\lambda$ are polynomial in the parameter b with non-negative integer coefficients.

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A combinatorial formulation

For every $\lambda \vdash n$ there exists a statistic ϑ_λ on matchings with non-negative integer values, such that:

- $\vartheta_\lambda(\delta) = 0$ iff δ is a bipartite matching.
- For every $\mu, \nu \vdash n$

$$c_{\mu,\nu}^\lambda(b) = \sum_{\substack{\text{matchings } \delta \text{ of type } (\mu, \nu) \\ \text{with respect to } \lambda}} b^{\vartheta_\lambda(\delta)}.$$

Partial results and main theorem

Definition of Jack polynomials + basic properties of power-sum functions: the coefficients $c_{\mu,\nu}^\lambda$ are rational functions in b .

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Theorem (Dołęga-Féray '15, Duke Math J.)

*The coefficients $c_{\mu,\nu}^\lambda$ are **polynomial** in b with rational coefficients. Moreover, $\deg(c_{\mu,\nu}^\lambda) \leq \text{rk}(\mu) + \text{rk}(\nu) - \text{rk}(\lambda)$.*

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Main theorem (BD '22)

*The coefficients $c_{\mu,\nu}^\lambda$ are polynomial in b with **integer** coefficients.*

+new proof of the polynomiality

Starting point of the proof: Matching-Jack conjecture for marginal coefficients $\bar{c}_{\mu,m}^\lambda$

Fix $\lambda, \mu \vdash n$ and $m \leq n$. We define $\bar{c}_{\mu,m}^\lambda := \sum_{\ell(\nu)=m} c_{\mu,\nu}^\lambda$.

Theorem (BD '21)

For every $\lambda \vdash n$ there exists a statistic ϑ_λ with non-negative integer values, such that:

- $\vartheta_\lambda(\delta) = 0$ iff δ is a bipartite matching.
- For every $\mu \vdash n$ and $m \leq n$

$$\bar{c}_{\mu,m}^\lambda(b) = \sum_{\substack{\text{matchings } \delta \text{ of marginal type } (\mu, m) \\ \text{with respect to } \lambda}} b^{\vartheta_\lambda(\delta)}$$

based on the work of Chapuy and Dołęga '20 on the b -conjecture

Scheme of the proof

Integrality for the marginal coefficients $\bar{c}_{\mu,m}^\lambda$

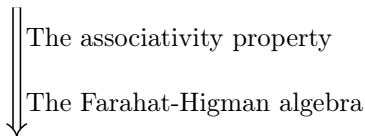
↓ The associativity property

Integrality for the coefficients $c_{\mu,\nu}^\lambda$

- 1 The associativity property: a system of linear equations relating $c_{\mu,\nu}^\lambda$ to $\bar{c}_{\mu,m}^\lambda$

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Integrality for the marginal coefficients $\bar{c}_{\mu,m}^\lambda$



Integrality for the coefficients $c_{\mu,\nu}^\lambda$

- 1 The associativity property: a system of linear equations relating $c_{\mu,\nu}^\lambda$ to $\bar{c}_{\mu,m}^\lambda$
- 2 The Farahat-Higman algebra: This linear system is invertible in \mathbb{Z} .

The associativity property and a system of linear equations

Jack polynomials orthogonality

$$\implies \sum_{\kappa \vdash n} c_{\mu, \kappa}^{\lambda} c_{\nu, \rho}^{\kappa} = \sum_{\theta \vdash n} c_{\theta, \rho}^{\lambda} c_{\mu, \nu}^{\theta} \quad \text{for } \lambda, \mu, \nu, \rho \vdash n \geq 1.$$

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Combinatorial interpretation for $b = 0$: Fix σ_{λ} of type λ . Two ways to enumerate the decompositions $\sigma_{\lambda} = \sigma_1 \cdot \sigma_2 \cdot \sigma_3$ of type (μ, ν, ρ) :

$$\sigma_{\lambda} = \sigma_1 \cdot \underbrace{(\sigma_2 \cdot \sigma_3)}_{\text{of type } \kappa} = \underbrace{(\sigma_1 \cdot \sigma_2)}_{\text{of type } \theta} \cdot \sigma_3$$

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Fix $m \leq n$. Taking the sum over ρ of length m we get:

$$\sum_{\kappa \vdash n} c_{\mu, \kappa}^{\lambda} \bar{c}_{\nu, m}^{\kappa} = \sum_{\theta \vdash n} \bar{c}_{\theta, m}^{\lambda} c_{\mu, \nu}^{\theta}, \quad \lambda, \mu, \nu \vdash n \text{ and } m \leq n.$$

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We prove by induction on the rank of κ that $c_{\mu, \kappa}^{\lambda}$ has integer coefficients for $\lambda, \mu \vdash n$:

- We fix a rank r and two partitions λ and μ .
- We choose (ν, m) in order to select partitions κ of rank $\leq r$.

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Recall:

$$\deg(\bar{c}_{\nu, m}^{\kappa}) \leq n - m + \text{rk}(\nu) - \text{rk}(\kappa).$$

The associativity property and a system of linear equations

We denote by $\mathcal{T}(n, r)$ the set of such pairs (ν, m) :

$$\mathcal{T}(n, r) := \{(\nu, m) \text{ such that } \text{rk}(\nu) + n - m = r \text{ and } \text{rk}(\nu) < r\}.$$

For $(\nu, m) \in \mathcal{T}(n, r)$:

$$\sum_{\text{rk}(\kappa)=r} c_{\mu, \kappa}^{\lambda} \underbrace{\bar{c}_{\nu, m}^{\kappa}}_{\substack{\rightarrow \\ \left\{ \begin{array}{l} \text{top connection coefficients} \\ \text{independent from } b \end{array} \right.}}$$
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\implies A linear system $\left\{ \begin{array}{l} c_{\mu, \kappa}^{\lambda} \text{ are the "unknowns"}. \\ \bar{c}_{\nu, m}^{\kappa} \text{ are the coefficients of the system.} \end{array} \right.$

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Step 2: We prove that this linear system is invertible in \mathbb{Z} using a new connection with the the Farahat-Higman algebra.

The Farahat Higman algebra

For $\nu \vdash n$

$$\mathcal{C}_\nu = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \text{type}(\sigma) = \nu}} \sigma \in Z(\mathbb{C}[\mathfrak{S}_n]).$$

$\{\mathcal{C}_\nu; \nu \vdash n\}$ form a basis of $Z(\mathbb{C}[\mathfrak{S}_n])$.

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Recall:

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We pass to the **graded** algebra \mathcal{Z}_n , spanned by $\{\mathcal{C}_\nu; \nu \vdash n\}$ and in which the multiplication is given by

$$\mathcal{C}_\nu * \mathcal{C}_\rho = \sum_{\substack{\kappa \vdash n \\ \text{rk}(\kappa) = \text{rk}(\nu) + \text{rk}(\rho)}} c_{\nu, \rho}^\kappa(0) \mathcal{C}_\kappa.$$

$\mathcal{Z}_n^{(r)}$: the vector space spanned by $\{\mathcal{C}_\nu; \nu \vdash n \text{ and } \text{rk}(\nu) = r\}$.

The Farahat-Higman algebra

Fact: The marginal coefficients $\bar{c}_{\nu, m}^{\kappa}$ encoding the linear system are **structure coefficients/change of basis coefficients** in \mathcal{Z}_n :

for $(\nu, m) \in \mathcal{T}(n, r)$ and κ of rank r

$$\bar{c}_{\nu, m}^{\kappa} = [\mathcal{C}_{\kappa}] \mathcal{C}_{\nu} * \left(\sum_{\ell(\rho)=m} \mathcal{C}_{\rho} \right)$$

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Theorem (BD '21)

*The family $\mathcal{C}_{\nu} * \left(\sum_{\ell(\rho)=m} \mathcal{C}_{\rho} \right)$ for $(\nu, m) \in \mathcal{T}(n, r)$ is a \mathbb{Z} -spanning family of $\mathcal{Z}_n^{(r)}$. By consequence, the system encoded by $(\bar{c}_{\nu, m}^{\kappa})$ is invertible in \mathbb{Z} .*

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- (Farahat-Higman) Stability by adding parts of size 1:

$$\bar{c}_{\nu, m}^{\kappa} = \bar{c}_{\nu \cup 1^n, m+n}^{\kappa \cup 1^n}, \text{ for } n \geq 1$$

\implies we pass to the projective limit $\mathcal{Z}_{\infty}^{(r)} := \varprojlim \mathcal{Z}_n^{(r)}$
(the graded Farahat-Higman algebra).

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Theorem (BD '21)

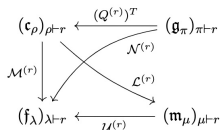
The family $\mathcal{C}_{\nu} * \left(\sum_{\ell(\rho)=m} \mathcal{C}_{\rho} \right)$ for $(\nu, m) \in \mathcal{T}(n, r)$ is a \mathbb{Z} -spanning family of $\mathcal{Z}_n^{(r)}$. By consequence, the system encoded by $(\bar{c}_{\nu, m}^{\kappa})$ is invertible in \mathbb{Z} .

- (Farahat-Higman) Stability by adding parts of size 1:

$$\bar{c}_{\nu, m}^{\kappa} = \bar{c}_{\nu \cup 1^n, m+n}^{\kappa \cup 1^n}, \text{ for } n \geq 1$$

\implies we pass to the projective limit $\mathcal{Z}_{\infty}^{(r)} := \varprojlim \mathcal{Z}_n^{(r)}$
(the graded Farahat-Higman algebra).

- Use two other bases of $\mathcal{Z}_n^{(r)}$ introduced by Farahat-Higman and Matsumoto-Novak.



Thank You!