Jack characters as generating series of bipartite maps and proof of Lassalle's conjecture

Houcine Ben Dali
Université de Lorraine, IECL, France
Université de Paris, IRIF, France

joint work with

Maciej Dołęga
Institute of Mathematics,
Polish Academy of Sciences, Poland.

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Symmetric functions

We consider the space of symmetric functions on an alphabet $\mathbf{x} := x_1, x_2, \dots$.

For $k \geq 0$, the power sum function p_k is defined

$$p_k(\mathbf{x}) := \sum_{i \geq 1} x_i^k,$$

and if $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_\ell]$ then

$$p_\lambda(\mathbf{x}) = p_{\lambda_1}(\mathbf{x}) p_{\lambda_2}(\mathbf{x}) \cdots p_{\lambda_\ell}(\mathbf{x}).$$

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The expansion of Schur functions on the power-sum basis is given by

$$s_\lambda(\mathbf{x}) = \sum_{\mu \vdash n} \frac{\chi^\lambda(\mu)}{z_\mu} p_\mu(\mathbf{x}) \qquad \text{where } z_\lambda := \frac{|\lambda|!}{|\mathcal{C}_\lambda|}.$$

Jack polynomials

Jack polynomials $J_\lambda^{(\alpha)}$ are symmetric functions which depend on a deformation parameter α.

- They can be obtained from Macdonald polynomials $J_\lambda^{(q,t)}$ by taking $q = t^\alpha$ and the limit $t \to 1$.
- When we take $\alpha = 1$ we obtain Schur functions.

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- generalizes a known interpretation of Schur functions in terms of pairs of permutations/oriented bipartite maps.
- answers a positivity conjecture of Lassalle 2008.
- gives an answer in some sense to a conjecture of Hanlon 1988.

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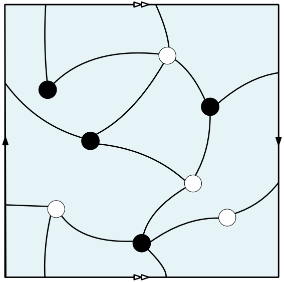
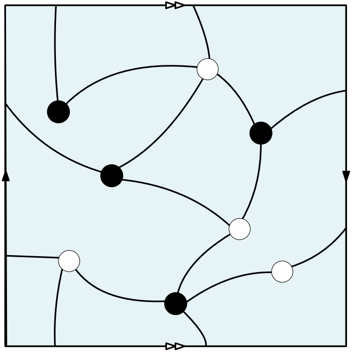
- A map is *oriented* if its all the connected components are embedded into orientable surfaces.
- A map is bipartite if its vertices are colored in white and black, and each white vertex has only black neighbors.



A non-oriented bipartite map on the Klein bottle.

Maps

- The face-type of a bipartite map M, denoted by $\diamond(M)$, is the partition given by the face degrees, divided by 2.

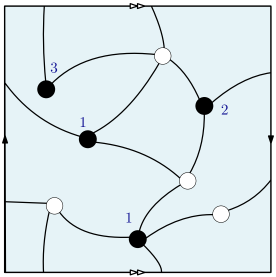


A non-oriented map of face-type $[4, 4, 2, 2]$.

Layered maps

Let k be a positive integer. A map M is k-layered if

- each black vertex has a label in $1, 2, \dots, k$.



A 3-layered map on the Klein bottle

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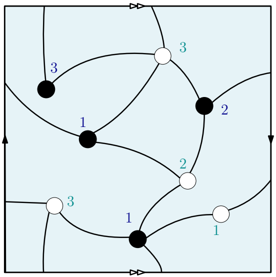

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Definition (Goulden–Jackson '96)

A statistic of non-orientability (on k-layered maps) is a statistic which associates to each k-layered map M a non-negative integer such that $\vartheta(M) = 0$ if and only if M is oriented.

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Maps will be counted with a weight $b^{\vartheta(M)}$, where $b := \alpha - 1$ is the shifted Jack parameter.

Jack polynomials in the power-sum basis

Theorem (BD–Dołęga '23)

There exists an explicit statistic of non-orientability ϑ, such that

$$J_\lambda^{(\alpha)} = (-1)^{|\lambda|} \sum_{\substack{\ell(\lambda)-layered \\ maps\ M}} \frac{p_{\diamond(M)} b^{\vartheta(M)}}{2^{|\mathcal{V}_\bullet(M)|-cc(M)} \alpha^{cc(M)}} \prod_{1 \leq i \leq \ell(\lambda)} \frac{(-\alpha\lambda_i)^{|\mathcal{V}_\circ^{(i)}(M)|}}{z_{\nu_\bullet^{(i)}(M)}},$$

- $p_{\diamond(M)}$ *is the power-sum function associated to the partition* $\diamond(M)$
- $|\mathcal{V}_\bullet(M)|$ *is the number of black vertices of* M.
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- a face-weight $p_{\diamond(M)}$
- a non-orientability weight $b^{\vartheta(M)}$
- a weight related to layers structure $(-\alpha\lambda_i)^{|\mathcal{V}_\circ^{(i)}(M)|}$

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Well known for $\alpha = 1$ (Young symmetrizers) and for $\alpha = 2$ (Féray–Śniady's 2010).

Jack characters (a dual approach)

Fix a partition μ.

$$\theta_\mu^{(\alpha)}(\lambda) := \begin{cases} 0, & \text{if } |\lambda| < |\mu|. \\ \binom{|\lambda|-|\mu|+m_1(\mu)}{m_1(\mu)} [p_{\mu,1^{|\lambda|-|\mu|}}] J_\lambda^{(\alpha)}, & \text{if } |\lambda| \geq |\mu|. \end{cases}$$

where $m_1(\mu)$ is the number of parts of size 1 in μ.

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There exists a statistic of non-orientability ϑ on layered maps, such that

$$\theta_{\mu}^{(\alpha)}(\lambda) = (-1)^{|\mu|} \sum_{\substack{\text{layered maps} M \\ \text{of face-type } \mu}} \frac{b^{\vartheta(M)}}{2^{|\mathcal{V}_{\bullet}(M)|-cc(M)}\alpha^{cc(M)}} \prod_{i \geq 1} \frac{(-\alpha\lambda_i)^{|\mathcal{V}_{\circ}^{(i)}(M)|}}{z_{\nu_{\bullet}^{(i)}(M)}}, \quad (1)$$

- For $\alpha = 1$: Stanley-Féray formula 2010.
- For $\alpha = 2$: Féray–Śniady formula for zonal characters 2010.

Idea of the proof

Known: There exists a unique α-shifted symmetric function $f_\mu(u_1, u_2, \dots)$ (i.e symmetric in the variables $u_1 - 1/\alpha, u_2 - 2/\alpha, \dots$) such that

$$\theta_\mu^{(\alpha)}(\lambda) = f_\mu(\lambda_1, \lambda_2, \dots, \lambda_\ell, 0, \dots) \text{ for every } \lambda.$$

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Theorem (Féray '19)

Fix a partition μ. The Jack character $\theta_\mu^{(\alpha)}$ is the unique α-shifted symmetric function of degree $|\mu|$ with top homogeneous part $\alpha^{|\mu|-\ell(\mu)}/z_\mu \cdot p_\mu$, such that $\theta_\mu^{(\alpha)}(\lambda) = 0$ for any partition $|\lambda| < |\mu|$.

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- We introduce the generating series of k-layered maps

$$F^{(k)}(t, \mathbf{p}, s_1, \dots, s_k) :=$$

$$\sum_{k-\text{layered maps } M} (-t)^{|\diamond(M)|} p_{\diamond(M)} \frac{b^{\vartheta(M)}}{2^{|\mathcal{V}_\bullet(M)|-cc(M)} \alpha^{cc(M)}} \prod_{1 \le i \le k} \frac{(-\alpha s_i)^{|\mathcal{V}_\circ^{(i)}(M)|}}{z_{\nu_\bullet^{(i)}(M)}}.$$

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We prove that this generating series satisfies the three conditions of the chracterization theorem.

Idea of the proof

- For a well-chosen statistic of non-orientability ϑ, this generating series can be constructed inductively using differential operators (Tutte decomposition):

$$F^{(k)}(t, \mathbf{p}, s_1, \ldots, s_k) = \exp\left(\sum_{n\geq 1} \frac{(-t)^n}{n} B_n(\mathbf{p}, -\alpha s_1)\right) \cdot F^{(k-1)}(t, \mathbf{p}, s_2, \ldots, s_k),$$

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where

$$B_n(\mathbf{p}, -\alpha s_1) := \Theta_Y \left(\Gamma_Y - \alpha s_1 Y_+\right)^n \frac{y_0}{1+b}$$

is an operator which adds a black vertex of degree n with label 1.
$Y := (y_0, y_1, y_2, \dots,)$ is a catalytic variable, and

$$\left.\begin{aligned} \Theta_Y &:= \textstyle\sum_{i\geq 1} p_i \frac{\partial}{\partial y_i}, \qquad Y_+ = \textstyle\sum_{i\geq 0} y_{i+1} \frac{\partial}{\partial y_i}, \\ \Gamma_Y &= (1+b) \cdot \textstyle\sum_{i,j\geq 1} y_{i+j} \frac{i\partial^2}{\partial p_i \partial y_{j-1}} \\ &\quad + \textstyle\sum_{i,j\geq 1} y_i p_j \frac{\partial}{\partial y_{i+j-1}} + b \cdot \textstyle\sum_{i\geq 1} y_{i+1} \frac{i\partial}{\partial y_i}. \end{aligned}\right\} \text{Chapuy–Dołęga operators.}$$

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A key step of the proof: Two commutation relations

$$[B_n(\mathbf{p}, u), B_m(\mathbf{p}, u)] = 0, \text{ for } n, m \geq 1,$$

$$\left[\sum_{n\geq 1} \frac{t^n}{n} B_n^{>}(\mathbf{p}, u), \sum_{n\geq 1} \frac{t^n}{n} B_n^{>}(\mathbf{p}, v)\right] = 0,$$

where

$$B_n^{>}(\mathbf{p}, u) := B_n(\mathbf{p}, u) - B_n(\mathbf{p}, 0).$$

Application 1: Creation operators for Jack polynomials

Theorem

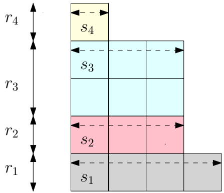
$$J^{(\alpha)}_{(\lambda_1,\lambda_2,\ldots,\lambda_\ell)} = \mathcal{B}^{(+)}_{\lambda_1} \cdot \mathcal{B}^{(+)}_{\lambda_2} \cdots \mathcal{B}^{(+)}_{\lambda_\ell} \cdot 1,$$

where

$$\mathcal{B}^{(+)}_n := [t^n] \exp\left(\sum_{n\geq 1} \frac{(-t)^n}{n} B_n(\mathbf{p}, -\alpha n)\right)$$

Application 2: Lassalle's conjecture 2008

Stanley's coordinates of a Young diagram



The Young diagram of the partition $[4, 3, 3, 3, 1]$, with $\mathbf{s} = (4, 3, 3, 1)$ and $\mathbf{r} = (1, 1, 2, 1)$ as Stanley coordinates.

Theorem (Lassalle's conjecture on Jack characters)

The normalized Jack character $(-1)^{|\mu|} z_\mu \theta_\mu^{(\alpha)}$ *is a polynomial in Stanley's coordinates* $r_1, r_2 \ldots, -s_1, -s_2 \ldots,$ *and* b *with non-negative integer coefficients.*

Proof:

- Non-negativity: the combinatorial interpretation in terms of maps.
- Integrality: (a different approach) the integrable system of Nazarov–Sklyanin.