JACK CHARACTERS AS GENERATING SERIES OF BIPARTITE MAPS AND PROOF OF LASSALLE'S CONJECTURE (A PRELIMINARY VERSION)

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Abstract. We give an explicit formula for the power-sum expansion of Jack polynomials. We deduce it from a more general formula that we provide here, that interprets Jack characters in terms of bipartite maps. Finally, we prove Lassalle's conjecture from 2008 on integrality and positivity of Jack characters in Stanley's coordinates.

1. Introduction

1.1. **Jack polynomials and Jack characters.** Jack polynomials $J_\lambda^{(\alpha)}$ are symmetric functions indexed by an integer partition λ and a deformation parameter α, and which have been introduced by Jack in [Jac71]. Jack polynomials interpolate, up to scaling factors, between Schur functions for $\alpha = 1$ and zonal polynomials for $\alpha = 2$. In his work [Sta89], Stanley initiated the combinatorial analysis of these symmetric functions with connection with various objects of algebraic combinatorics, such as partitions, tableaux, paths and maps [Mac95, GJ96a, KS97, DF16, Mol21, CD22].

Knop and Sahi have given in [KS97] a combinatorial interpretation for the coefficients of the Jack polynomial $J_\lambda^{(\alpha)}$ in the monomial basis in terms of tableaux of shape λ. Such a combinatorial expression does not exist for the expansion of the Jack polynomial $J_\lambda^{(\alpha)}$ in the power-sum basis for general partitions λ. In this paper, we give a combinatorial interpretation of Jack polynomials $J_\lambda^{(\alpha)}$ in power-sum basis in terms of maps.

Roughly, a map is a graph drawn on a locally orientable surface. The study of maps is a well developed area with strong connections with analytic combinatorics, mathematical physics and probability [BC86, LZ04, Cha11, Eyn16]. The relationship between generating series of maps and the theory of symmetric functions was first noticed via a character theoretic approach [JV90, GJ96b] and has then been developed to include other techniques such as matrix integrals and differential equations [La 09, DFS14, CD22].

Actually, we prove a more general result by giving a combinatorial interpretation for Jack characters: for a partition μ, the Jack character $\theta_\mu^{(\alpha)}$ is the function on Young diagrams defined by:

$$\theta_\mu^{(\alpha)}(\lambda) := \begin{cases} 0, & \text{if } |\lambda| < |\mu|. \\ \binom{|\lambda|-|\mu|+m_1(\mu)}{m_1(\mu)} [p_{\mu,1^{n-m}}] J_\lambda^{(\alpha)}, & \text{if } |\lambda| \geq |\mu|. \end{cases}$$

where $[p_\mu]$ is the extraction symbol with respect to the variable $\mathbf{p}$ and $m_1(\mu)$ is the number of parts equal to 1 in the partition μ.

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When $\alpha = 1$, Jack characters correspond to the characters of the symmetric group. These characters have a combinatorial interpretation in terms of permutations which has been conjectured by Stanley in [Sta06] and proved by Féray in [Fér10]. A nice combinatorial expression has also been established in [FŚ11b] for zonal characters, which correspond to $\theta^{(\alpha)}_\mu$ with $\alpha = 2$. In this paper, we give a combinatorial interpretation for $\theta^{(\alpha)}_\mu$ for general α in terms of maps. The expression we obtain interpolates between the expressions given in [Fér10] and [FŚ11b] for $\alpha = 1$ and $\alpha = 2$.

1.2. **Maps.** A *connected map* is a connected graph embedded into a surface such that all the connected components of the complement of the graph are simply connected (see [LZ04, Definition 1.3.7]). These connected components are called the *faces* of the map. We consider maps up to homeomorphisms of the surface. A connected map is *orientable* if the underlying surface is orientable. In this paper[1], a *map* is an unordered collection of connected maps. A map is orientable if each one of its connected components is orientable. Finally, the *size* of a map is its number of edges.

In this paper, all the maps considered are *bipartite*; *i.e* their vertices are colored in two colors, white and black, such that each edge connects two vertices of different colors. Note that in a bipartite map, all faces have even degree. We define then the *face-type* of a bipartite map M of size n, as the partition of n obtained by reordering the half degrees of the faces and we denote it $\nu_\diamond(M)$. We also denote its set of white and black vertices by $\mathcal{V}_\circ(M)$ and $\mathcal{V}_\bullet(M)$ respectively.

The following definition has been introduced by Goulden and Jackson and is commonly used in the context of the b-conjecture to define generating series which interpolate between generating series of orientable maps and generating series of non-orientable maps.

Definition 1.1 ([GJ96a]). *A statistic of non-orientability on bipartite maps is a statistic ϑ with non-negative integer values, such that $\vartheta(M) = 0$ if and only if M is orientable.*

In practice, a statistic of non-orientability is supposed to "measure" the non orientability of a map by counting the number of edges which contribute to its non orientability, following a given algorithm of decomposition of the map. Several examples of such statistics have been introduced in previous works [La 09, DFS14, Doł17, CD22].

1.3. **Layered maps.**

Definition 1.2. *Fix $k \geq 0$. We say that a bipartite map M is a k-layered if its black vertices (resp. white vertices) are partitioned into k sets (which may be empty), called the layers of the map; $\mathcal{V}_\circ(M) = \bigcup_{1\leq i\leq k} \mathcal{V}^{(i)}_\circ(M)$ (resp. $\mathcal{V}_\bullet(M) = \bigcup_{1\leq i\leq k} \mathcal{V}^{(i)}_\bullet(M)$), which satisfies the following condition: if v is a white vertex in a layer i, then all its neighbors are in layers $j \leq i$, and it has at least one neighbor in the layer i.*

For $1 \leq i \leq k$, we define the partition $\nu^{(i)}_\bullet(M)$ obtained by reordering the degree distribution of the black vertices in the layer i. A k-layered map is labelled if:

- *in each layer $1 \leq i \leq k$, the black vertices having the same degree j are numbered by $1, 2 \ldots \left(\nu^{(i)}_\bullet(M)\right)_j$.*

[1] This is not the standard definition of a map; usually a map is connected.

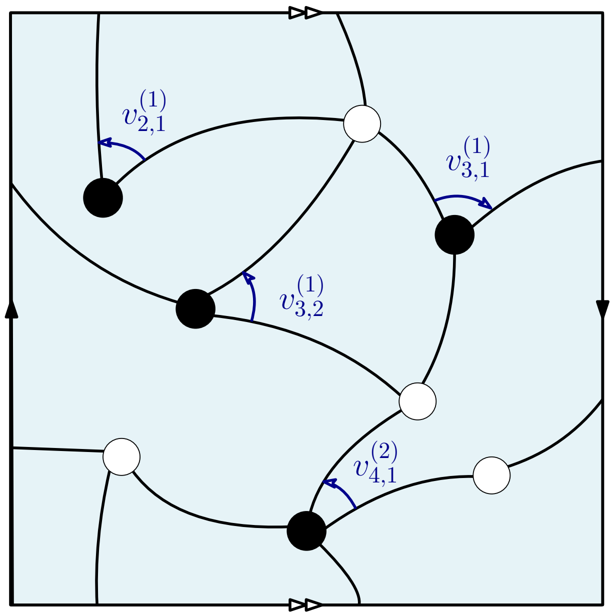


FIGURE 1. A 2-layered map on the Klein bottle, represented here by a square whose left side should be glued to the right one (with a twist) and the top side should be glued to the bottom one (without a twist), as indicated by the arrows. Moreover, $v_{j,k}^{(i)}$ denotes the black vertex of degree j numbered by k in the layer i.

- *each black vertex has a distinguished oriented corner.*

An example of a 2-layered map is given in Fig. 1. Note that a k-layered map can be seen as $(k+1)$-layered map with an empty layer $k+1$. We call then a *layered map* a k-layered map for some $k \geq 1$. We denote by $\mathcal{M}^{(k)}$ (resp. $\mathcal{M}^{(\infty)}$) the set of all labelled k-layered maps (resp. labelled layered maps). Similarly, we denote $\mathcal{M}_\mu^{(k)}$ and $\mathcal{M}_\mu^{(\infty)}$ those of face type μ.

Remark 1. This definition of layered maps is equivalent to the definition of maps equipped with particular functions which associate to each one of its vertices a positive integer. This definition has been introduced by Stanley in the context of permutations [Sta06] and generalized to maps in [DFS14, Section 1.6]. However, we prefer to present this definition as above since it will play a slightly different role in this paper.

1.4. **First main result.**

Theorem 1.3. *There exists a statistic of non-orientability ϑ on layered maps such that for any partitions μ and λ, we have*

$$\theta_\mu^{(\alpha)}(\lambda) = (-1)^{|\mu|} \sum_{M \in \mathcal{M}_\mu^{(\infty)}} \frac{b^{\vartheta_\rho(M)}}{2^{|\mathcal{V}_\bullet(M)|-cc(M)} \alpha^{cc(M)}} \prod_{i \geq 1} \frac{(-\alpha\lambda_i)^{|\mathcal{V}_\circ^{(i)}(M)|}}{z_{\nu_\bullet^{(i)}(M)}}, \tag{1}$$

where b is the parameter related to α by $b := \alpha - 1$ and $z_{\nu_\bullet^{(i)}(M)}$ is the classical normalization factor (see Section 2.1).

Note that the product here is finite since each layered map has a finite number of non empty layers.

Actually, we prove that this theorem holds for a family of statistics ϑ. The parameter b appears in various positivity conjectures related to Jack polynomials as the b-conjecture of Gulden and Jackson [GJ96a] and Lassalle's conjecture which will be discussed in Section 1.7. Although the two parameters α and b are related we prefer to keep both of them in the previous formula since they play different roles. In particular, one may notice that the quantity $1/\left(2^{(|\mathcal{V}_\bullet(M)|-cc(M)}\alpha^{cc(M)}\right)\prod_{1\leq i\leq k}\frac{(-\alpha\lambda_i)^{|\mathcal{V}_\circ^{(i)}(M)|}}{z_{\nu_\bullet^{(i)}(M)}}$ depends only on the underlying "k-layered graph" of M and is rather straightforward. This will not be the case for the quantity $b^{\vartheta(M)}$, called the *b-weight* of the map.

In the case $b=0$, and by definition of a statistic of non-orientability only bipartite maps appear in Eq. (1). Using Remark 1 and the correspondence between bipartite maps and permutations (see *e.g* [Cha18, Section 1.2]), one can show that when $b=0$, Eq. (1) corresponds to Féray-Stanley formula [Fér10, Thm 2]. Similarly, Eq. (1) coincides when $b=1$ with the expression given in [FŚ11b, Thm 1.2]. In addition to the cases $b=0$ and $b=1$, two variants of Eq. (1) have been obtained when the partition λ is of rectangular shape.

As a direct consequence of Theorem 1.3, we obtain the following interpretation of Jack polynomials in the basis of power-sum functions.

Theorem 1.4. *Let n be a positive integer and let λ be a partition of n. Then*

$$J_\lambda^{(\alpha)}=(-1)^n\sum_M p_{\nu_\diamond(M)}\frac{b^{\vartheta_\rho(M)}}{2^{|\mathcal{V}_\bullet(M)|-cc(M)}\alpha^{cc(M)}}\prod_{1\leq i\leq \ell(\lambda)}\frac{(-\alpha\lambda_i)^{|\mathcal{V}_\circ^{(i)}(M)|}}{z_{\nu_\bullet^{(i)}(M)}},$$

where the sum is taken over all $\ell(\lambda)$-layered maps M of size n.

1.5. **Féray's characterization of Jack characters.** Roughly, an α-shifted symmetric function $f(s_1,s_2,\dots)$ is a functions which is symmetric in the variables s_i-i/α (see Definition 2.2 for a formal definition). If f is an α-shifted symmetric function λ a partition, then we denote

$$f(\lambda):=f(\lambda_1,\lambda_2,\dots,\lambda_{\ell(\lambda)},0,\dots).$$

A shifted function is completely determined by its evaluation on Young diagrams $(f(\lambda))_{\lambda\in\mathbb{Y}}$. We obtain then an identification between shifted functions and functions on Young diagrams. In particular, Jack characters defined as functions on Young diagrams in Section 1.1 can be seen as shifted symmetric functions. The starting point of the proof of the main theorem is the following characterization the Jack characters $\theta_\mu^{(\alpha)}$ which is due to Valentin Féray (see [Śni15, Theorem A.2]). For completeness, we give in Section 2 the proof of this theorem.

Theorem 1.5. *Fix a partition μ. The Jack character $\theta_\mu^{(\alpha)}$ is the unique α-shifted symmetric function of degree $|\mu|$ with top homogeneous part $\alpha^{|\mu|-\ell(\mu)}/z_\mu\cdot p_\mu$, such that $\theta_\mu^{(\alpha)}(\lambda)=0$ for any partition $|\lambda|<|\mu|$.*

Our goal is to prove that the right-hand side of Eq. (1) satisfies the conditions of Theorem 1.5. To this purpose, we express the generating series of layered maps using differential operators.

1.6. **The generating series of k-layered maps.** Fix $k \geq 1$. We define the generating series of k-layered maps by:

$$F^{(k)}\left(t, \mathbf{p}, s_1, \ldots, s_k\right) := \sum_{M \in \mathcal{M}^{(k)}} (-t)^{|M|} p_{\nu_\diamond(M)} \frac{b^{\vartheta_\rho(M)}}{2^{|\mathcal{V}_\bullet(M)|-cc(M)} \alpha^{cc(M)}} \prod_{1 \leq i \leq k} \frac{(-\alpha s_i)^{|\mathcal{V}_\circ^{(i)}(M)|}}{z_{\nu_\bullet^{(i)}(M)}}. \tag{2}$$

Note that the normalizing factors which appear in this definition are related to the definition of labelling we use here,[2] see Section 1.3. This labelling allows to deal with maps with trivial automorphism groups and also plays an important role in the definition of statistic of non-orientability Section 4.1.

If λ is a partition such that $\ell(\lambda) \leq k$, then we denote

$$F^{(k)}(t, \mathbf{p}, \lambda) := F(t, \mathbf{p}, \lambda_1, \ldots, \lambda_k, 0 \ldots). \tag{3}$$

In Section 4.2, we describe an algorithm which allows to construct k-layered maps from $(k-1)$-layered maps. To encode such combinatorial operations, and following [CD22], we introduce a family of operators $C_\ell(t, \mathbf{p})$. Roughly, C_ℓ is the operator which allows to add a black vertex to a map, using ℓ new white vertices (see Section 4.3 for the definition of these operators). We also define the operator

$$B_\infty(t, \mathbf{p}, u) = \sum_{\ell \geq 0} u^\ell C_\ell(t, \mathbf{p}).$$

We have then the following recursive relation for the functions $F^{(k)}$.

Proposition 1.6. *The functions $F^{(k)}$ satisfy the following induction: $F^{(0)} = 1$ and for every $k \geq 1$*

$$F^{(k)}\left(t, \mathbf{p}, s_1, \ldots, s_k\right) = \exp\left(B_\infty(-t, \mathbf{p}, -\alpha s_1)\right) \cdot F^{(k-1)}\left(t, \mathbf{p}, s_2, \ldots, s_k\right). \tag{4}$$

We have then the following theorem.

Theorem 1.7 (Vanishing property)**.** *Let $k \geq 1$ and let λ be a partition of size n. Then the function $F^{(k)}(t, \mathbf{p}, \lambda)$ is a polynomial in t of degree less or equal than n. In other terms, if $\ell > n$ then*

$$[t^\ell] F^{(k)}(t, \mathbf{p}, \lambda) = 0.$$

Combinatorially, the vanishing property is equivalent to saying that in the series Eq. (2) defining $F(t, \mathbf{p}, \lambda)$, the total contribution of maps of size bigger than $|\lambda|$ is zero. In the cases $b = 0$ and $b = 1$, a combinatorial proof of this property was given in [FŚ11a] and [FŚ11b] respectively. Such proof does not seem to work for general b because of the presence of the b-weight. In this paper, we use the expression given in Proposition 1.6 for the function $F^{(k)}$ to prove Theorem 1.7.

On the other hand, we prove that the generating series of layers maps is a shifted symmetric function (see also Theorem 8.4).

[2] the factor $2^{|\mathcal{V}_\bullet(M)|-cc(M)}$ can be omitted by using a finer labelling.

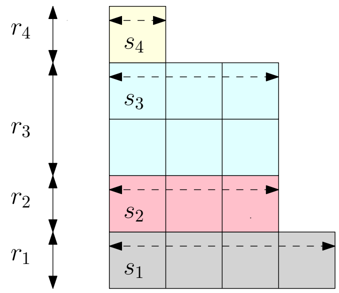


FIGURE 2. The Young diagram of the partition $[4, 3, 3, 3, 1]$ as the union of 4 rectangles, with $\mathbf{s} = (4, 3, 3, 1)$ and $\mathbf{r} = (1, 1, 2, 1)$ as multirectangular coordinates.

Theorem 1.8 (Shifted symmetry property)**.** *The function $F^{(k)}(t, \mathbf{p}, s_1, \dots, s_k)$ is α-shifted symmetric into the variables $s_1, s_2, \dots, s_k$.*

A key tool in the proof of this theorem, is the following commutation relations satisfied by the operators C_ℓ.

Theorem 1.9. *Let $m > 0$. Then*

$$[C_\ell, C_m] = \begin{cases} 0 & \text{if } \ell > 0, \\ (m+1)C_{m+1} & \text{if } \ell = 0. \end{cases}$$

1.7. **Lassalle's conjecture and second main result.** Stanley has introduced in [Sta04] the following definition of multirectangular coordinates.

Definition 1.10 ([Sta04])**.** *Let $k \geq 1$ and let $s_1 \geq s_2 \dots \geq s_k \geq 1$ and $r_1, \dots r_k$ be two sequences of non negative integers. We say that $(s_1, s_2, \dots, s_k)$ and $(r_1, \dots, r_k)$ are multirectangular coordinates for a partition λ and we denote $\lambda = \mathbf{s}^{\mathbf{r}}$, if λ is the union of k rectangles of sizes $s_i \times r_i$, or equivalently $\lambda = [s_1^{r_1} \dots s_k^{r_k}]$, see Fig. 2 for an example.*

Since we do not require that the sequence $\mathbf{s}$ should be strictly decreasing, the multi-rectangular coordinates are not unique in general.

If λ is a partition of multirectangular coordinates $(s_1, \dots, s_k)$ and $(r_1, \dots, r_k)$, we write for any partition μ,

$$\tilde{\theta}^{(\alpha)}_\mu(\mathbf{s}, \mathbf{r}) := \theta^{(\alpha)}_\mu(\lambda),$$

where $\mathbf{r} = (r_1, \dots, r_k, 0 \dots)$ and $\mathbf{s} = (s_1, \dots, s_k, 0, \dots)$. In the case $b = 0$, which corresponds to the irreducible characters of the symmetric group, Stanley found [Sta04] an explicit formula for $\tilde{\theta}^{(\alpha)}_\mu(\mathbf{s}, \mathbf{r})$ when λ is a rectangle, and he conjectured a formula for general $\mathbf{r}, \mathbf{s}$ that was soon after proved by Féray [Fér10]. An analogous formula was found in the case $b = 1$ [FŚ11b]. A conjecture that suggests positivity and integrality of Jack characters in multirectangular coordinates for arbitrary b was stated by Lassalle [Las08b], and it remained unproven for the last 15 years[3]. Our second main result is the proof of this conjecture.

[3]It has been proved in the particular case $\mathbf{s} = (s, 0, \dots)$ and $\mathbf{r} = (r, 0, \dots)$ in [BD22b].

Theorem 1.11. *The normalized Jack characters expressed in the Stanley coordinates* $(-1)^{|\mu|} z_\mu \tilde{\theta}^{(\alpha)}_\mu(\mathbf{s}, \mathbf{r})$ *are polynomials in the variables* $b, -s_1, -s_2, \dots, r_1, r_2, \dots$ *with non-negative integer coefficients, where* $b := \alpha - 1$.

Interestingly, the proof of Theorem 1.11 consists of two parts that are proved using very different techniques. In the first part we deduce positivity as a consequence of the combinatorial expression of Jack characters obtained in Theorem 1.3. In the second part, we obtain the integrality using integrable system of Nazarov–Sklyanin [NS13]. We relate their theory with Jack characters by proving an explicit combinatorial formula expressing certain basis of shifted symmetric functions in terms of normalized Jack characters. We conclude by showing that the transition matrix between these two bases is invertible over $\mathbb{Z}$. Consequently, we prove that Kerov polynomials for Jack characters have integer coefficients, which was an open problem posed by Lassalle in [Las09] (see Section 3 for details).

1.8. **Jack polynomials via differential operators.** Another application of the first main result is an inductive formula for the expansion of Jack polynomials in the power-sum basis using differential operators (see also Theorem 9.2).

Theorem 1.12. *Fix a partition* λ. *Let* μ *be the partition obtained from* λ *by removing the largest part;* $\mu = \lambda \backslash \lambda_1$. *Then,*

$$J^{(\alpha)}_\lambda = [t^{\lambda_1}] \exp\left(B_\infty(-t, \mathbf{p}, -\alpha\lambda_1)\right) \cdot J^{(\alpha)}_\mu.$$

One may notice that this formula is simpler than the one obtained by combining Theorem 1.4 and Proposition 1.6. In fact, it is obtained from these two results using some properties of the differential operators proved in Section 5 (see Section 9.2 for more details).

1.9. **Outline of the paper.** The paper is organised as follows. In Section 2 we introduce some notation related to partitions and symmetric functions and we give a proof for Theorem 1.5. In Section 3, we prove the integrality in Theorem 1.11. In Section 4 we explain the combinatorial decomposition of layered maps and we prove the differential expression of the function $F^{(k)}$ given in Proposition 1.6. The vanishing property of Theorem 1.7 will be proved in Section 5. In Section 6, we establish some preliminary commutation relations satisfied by the "catalytic" differential operators. Section 8 is dedicated to the proof of the shifted symmetry property of Theorem 1.8. In Section 9, we finish the proof of Theorem 1.3 and 1.12 and we prove the positivity in Theorem 1.11.

2. Notation and preliminaries

For the definitions and notation introduced in Sections 2.1 and 2.2 we refer to [Sta89, Mac95].

2.1. **Partitions.** A *partition* $\lambda = [\lambda_1, ..., \lambda_\ell]$ is a weakly decreasing sequence of positive integers $\lambda_1 \geq ... \geq \lambda_\ell > 0$. We denote by $\mathbb{Y}$ the set of all integer partitions. The integer ℓ is called the *length* of λ and is denoted $\ell(\lambda)$. The size of λ is the integer $|\lambda| := \lambda_1 + \lambda_2 + ... + \lambda_\ell$. If n is the *size* of λ, we say that λ is a partition of n and we write $\lambda \vdash n$. The integers

λ_1,...,λ_ℓ are called the *parts* of λ. For $i \geq 1$, we denote $m_i(\lambda)$ the number of parts of size i in λ. We set then

$$z_\lambda := \prod_{i\geq 1} m_i(\lambda)! i^{m_i(\lambda)}.$$

We denote by $\leq$ the *dominance partial* ordering on partitions, defined by

$$\mu \leq \lambda \iff |\mu| = |\lambda| \text{ and } \quad \mu_1 + ... + \mu_i \leq \lambda_1 + ... + \lambda_i \text{ for } i \geq 1.$$

We identify a partition λ with its *Young diagram*, defined by

$$\lambda := \{(i,j), 1 \leq i \leq \ell(\lambda), 1 \leq j \leq \lambda_i\}.$$

The conjugate partition of λ, denoted λ^t, is the partition associated to the Young diagram obtained by reflecting the diagram of λ with respect to the line $j = i$:

$$\lambda^t := \{(i,j), 1 \leq j \leq \ell(\lambda), 1 \leq i \leq \lambda_i\}.$$

Fix a box $\square := (i,j) \in \lambda$. Its *arm-length* is given by

$$a_\lambda(\square) := |\{(i,r) \in \lambda, r > j\}| = \lambda_i - j,$$

and its *leg-length* is given by

$$\ell_\lambda(\square) := |\{(r,j) \in \lambda, r > i\}| = (\lambda^t)_j - i.$$

Two α-deformations of the hook-length product were introduced in [Sta89];

$$\mathrm{hook}_\lambda^{(\alpha)} := \prod_{\square\in\lambda} \left(\alpha a_\lambda(\square) + \ell_\lambda(\square) + 1\right), \qquad \mathrm{hook}_\lambda'^{(\alpha)} := \prod_{\square\in\lambda} \left(\alpha(a_\lambda(\square) + 1) + \ell_\lambda(\square)\right).$$

Finally, we define the α-*content* of a box $\square := (i,j)$ by

$$c_\alpha(\square) := \alpha(j-1) - (i-1). \tag{5}$$

2.2. **Symmetric functions and Jack polynomials.** We fix an alphabet $\mathbf{x} := (x_1, x_2, ..)$. We denote by $\mathcal{S}$ the algebra of symmetric functions in $\mathbf{x}$ with coefficients in $\mathbb{Q}$. For every partition λ, we denote m_λ the monomial function and p_λ the power-sum function associated to the partition λ. We consider the associated alphabet of power-sum functions $\mathbf{p} := (p_1, p_2, ..)$.

Let $\mathcal{S}_\alpha$ be the algebra of symmetric functions with coefficients in $\mathbb{Q}(\alpha)$. We denote by $\langle ., . \rangle_\alpha$ the α-deformation of the Hall scalar product defined by

$$\langle p_\lambda, p_\mu \rangle_\alpha = z_\lambda \alpha^{\ell(\lambda)} \delta_{\lambda,\mu}, \text{ for any partitions } \lambda, \mu.$$

Macdonald [Mac95, Chapter VI.10] has proved that there exists a unique family of symmetric functions $(J_\lambda^{(\alpha)})_{\lambda\in\mathbb{Y}}$ in $\mathcal{S}_\alpha$ indexed by partitions, satisfying the following properties:

$$\begin{cases} \text{Orthogonality:} & \langle J_\lambda, J_\mu \rangle_\alpha = 0, \text{ for } \lambda \neq \mu, \\ \text{Triangularity:} & [m_\mu] J_\lambda = 0, \text{ unless } \mu \leq \lambda, \\ \text{Normalization:} & [m_{1^n}] J_\lambda = n!, \text{ for } \lambda \vdash n, \end{cases}$$

where $[m_\mu]J_\lambda$ denotes the coefficient of m_μ in J_λ, and 1^n is the partition with n parts equal to 1. These functions are known as the *Jack polynomials*. The squared norm of

Jack polynomials can be expressed in terms of the deformed hook-length products, see [Sta89, Theorem 5.8]:

$$j_\lambda^{(\alpha)} := \langle J_\lambda, J_\lambda \rangle_\alpha = \mathrm{hook}_\lambda^{(\alpha)} \, \mathrm{hook}_\lambda'^{(\alpha)} . \tag{6}$$

In this paper, Jack polynomials will always be expressed in the power-sum variables $\mathbf{p}$ rather than the alphabet $\mathbf{x}$ (this is possible since the power-sum functions form a basis of the symmetric functions algebra).

We have the following theorem due to Macdonald [Mac95, Chapter VI Eq. 10.25], which gives an expression of Jack polynomials when all power-sum variables p_i are equal to u.

Theorem 2.1 ([Mac95])**.** *For every $\lambda \in \mathcal{P}$, we have*

$$J_\lambda^{(\alpha)}(\underline{u}) = \prod_{\square \in \lambda} \left(u + c_\alpha(\square)\right),$$

where $\underline{u} := (u, u, ...)$.

2.3. **Lassalle's isomorphism and proof of Theorem 1.5.** We start by recalling some results on shifted symmetric functions from [Las08a]. Several of these results were based on the work of Knop and Sahi [KS96].

Definition 2.2 ([Las08a])**.** *We say that a polynomial in k variables $(s_1, \ldots, s_k)$ with coefficients in $\mathbb{Q}(\alpha)$ is α-shifted symmetric if it is symmetric in the variables $s_i - i/\alpha$. An α-symmetric function is a sequence $(f_k)_{k \geq 1}$ such that for every $k \geq 1$, the function f_k is an α-shifted symmetric polynomial in k variables and*

$$f_{k+1}(s_1, \ldots, s_k, 0) = f_k(s_1, \ldots, s_k). \tag{7}$$

We denote by $\mathcal{S}_\alpha^*$ the algebra of α-symmetric functions.

Before proving Theorem 1.5, we start by recalling some properties about Lassalle's isomorphism between symmetric functions and shifted symmetric functions. Let f be an α-shifted symmetric function in k parameters and λ a partition with $\ell(\lambda) \leq k$. Then we denote $f(\lambda) := f(\lambda_1, \ldots, \lambda_{\ell(\lambda)}, 0, \ldots)$. Lassalle [Las08a] has constructed an isomorphism $f \longmapsto f^\#$ between symmetric functions and α-shifted symmetric functions which satisfies the following properties

1. If f is homogeneous, then the top degree part of $f^\#$ is f.
2. For any partition ξ, the function $J_\xi^{(\alpha)\#}$ is the unique α-shifted symmetric function such that $J_\xi^{(\alpha)\#}(\lambda) = 0$ if λ does not contain ξ, but $J_\xi^{(\alpha)\#}(\xi) \neq 0$.

Moreover, the image of power-sum functions by this isomorophism are the Jack characters up to some scalar factors.

Lemma 2.3 ([Las08a])**.** *For any partition μ, we have $\alpha^{|\mu|-\ell(\mu)}/z_\mu \cdot p_\mu^\# = \theta_\mu^{(\alpha)}$.*

Proof. Both sides of the equation coincide on partitions λ such that $|\lambda| \geq |\mu|$ is proved in [Las08a, Proposition 2]. When $|\lambda| < |\mu|$, the character $\theta_\mu^{(\alpha)}$ vanishes by definition. On the other hand, the fact that $p_\mu^\#(\lambda) = 0$ is direct from the definition of Lassalle's isomorphism (see [Las08a, Eq. (3.1)]). ∎

Since power-sum functions $(p_\mu)_{\mu\in\mathbb{Y}}$ form a linear basis of $\mathcal{S}_\alpha$, we obtain from the last lemma that Jack characters $(\theta^{(\alpha)}_\mu)_{\mu\in\mathbb{Y}}$ form a basis of $\mathcal{S}^*_\alpha$. Similarly, $(J^{(\alpha)\#}_\mu)_{\mu\in\mathbb{Y}}$ is a basis for $\mathcal{S}^*_\alpha$.

We now prove Theorem 1.5.

Proof of Theorem 1.5. The fact that $\theta^{(\alpha)}_\mu(\lambda) = 0$ if $|\lambda| < |\mu|$ comes from the definition. Its top homogeneous part is obtained from property (1) and Lemma 2.3 above.

Uniqueness: Let G be an α-shifted symmetric function of degree less or equal than $|\mu|$ with the same top degree part as $\theta^{(\alpha)}_\mu$ and such that $G(\lambda) = 0$ for any $|\lambda| < |\mu|$. Set $G := F - \theta^{(\alpha)}_\mu$. Then G is an α-shifted symmetric function of degree at most $|\mu| - 1$ with

$$G(\lambda) = 0 \text{ for } |\lambda| < |\mu|. \tag{8}$$

We expand G in the $J^{(\alpha)\#}_\xi$ basis

$$G = \sum_\xi c_\xi J^{(\alpha)\#}_\xi. \tag{9}$$

As $\deg(G) \leq |\mu| - 1$, the sum can be restricted to partitions ξ of size at most $|\mu| - 1$. We will prove by contradiction that $G = 0$, *i.e* that $c_\xi = 0$ for all partitions ξ with $|\xi| \leq |\mu| - 1$. Assume this is not the case and consider a partition ξ_0 of minimal size such that $c_{\xi_0} \neq 0$. Then we evaluate Eq. (9) on the partition ξ_0.

$$G = \sum_\xi c_\xi J^{(\alpha)\#}_\xi.$$

But from Eq. (8) we have $G(\xi_0) = 0$ since $|\xi_0| < |\mu|$. On the other hand, $J^{(\alpha)\#}_\xi(\xi_0)$ if ξ_0 does not contain ξ (see property (2) above). Therefore the only non zero summand in the right hand side corresponds to $\xi = \xi_0$. This summand is indeed nonzero since by assumption $c_{\xi_0} \neq 0$ and we know that $J^{(\alpha)\#}_{\xi_0}(\xi_0) \neq 0$ from property (2). We conclude that the right-hand side is nonzero and we have reached a contradiction. Hence, $G = 0$ and the uniqueness is proved. □

3. Integrality in Lassalle's conjecture

Before we prove Theorem 1.3 we present the proof of Theorem 1.11. The positivity part follows directly from the combinatorial interpretation (in terms of layered maps) stated in Theorem 1.3, whose proof is technically involved and will occupy the most part of this paper. Integrality, however, requires new and different ideas. We prove it using different approach based on combinatorics of Nazarov–Sklyanin operators interpreted as lattice paths. These developments are independent of the other sections, and they are also of independent interest, as we show here by proving other problems stated in the literature as a byproduct.

3.1. **Nazarov–Sklyanin operators and α-polynomial functions.** Recall that $\theta^{(\alpha)}_\mu$ is a linear basis over the field of rational functions $\mathbb{Q}(\alpha)$ of the algebra $\mathcal{S}^*_\alpha$ of α-shifted symmetric functions. It turns that a strictly related algebra is of special interest. Let $\gamma := \sqrt{\alpha}^{-1} - \sqrt{\alpha}$ and define the *normalized Jack character*

$$\mathrm{Ch}^{(\alpha)}_\mu(\lambda) := \alpha^{\frac{\ell(\mu)-|\mu|}{2}} z_\mu \theta^{(\alpha)}_\mu(\lambda), \tag{10}$$

and the $\mathbb{Q}[\gamma]$-module $\mathcal{P}^*_\gamma$ spanned by $\left(\mathrm{Ch}^{(\alpha)}_\mu\right)_{\mu\in\mathbb{Y}}$. It turns that $\mathcal{P}^*_\gamma$ is in fact an algebra called the algebra of *α-polynomial functions*, see [DF16, Ś19, DS19] for the details, where it also becomes clear that this normalization for Jack characters is of a special interest due to the connections with random partitions. In fact, several important bases of $\mathcal{P}^*_\gamma$ grew up from this connection that we describe now.

3.1.1. *Kerov's transition measure.* Kerov associated with a Young diagrams λ certain probability measure μ_λ on $\mathbb{R}$ that is very useful for studying asymptotic behaviour of random Young diagrams. This *transition measure* is uniquely characterized by its Cauchy transform:

$$G_{\mu_\lambda}(z) = \int_{\mathbb{R}} \frac{d\mu_\lambda(x)}{z-x} = \frac{1}{z+\ell(\lambda)} \prod_{i=1}^{\ell(\lambda)} \frac{z+i-\lambda_i}{z+i-1-\lambda_i}. \tag{11}$$

In particular the ℓ-th moment $M_\ell(\lambda)$ of the transition measure μ_λ can be computed by applying a simple relation between the Cauchy transform expanded around infinity and the generating function of moments:

$$z^{-1} + \sum_{\ell\geq 1} M_\ell(\lambda) z^{-\ell-1} = G_{\mu_\lambda}(z).$$

Note that M_ℓ can be treated as functions by $M_\ell(\lambda) = M_\ell(\lambda_1,\dots,\lambda_{\ell(\lambda)})$, and define

$$M^{(\alpha)}_\ell(\lambda) := \alpha^{-\frac{\ell}{2}} M_\ell(\alpha\cdot\lambda). \tag{12}$$

It was proved in [DF16] that $M^{(\alpha)}_\ell$ is an algebraic basis of $\mathcal{P}^*_\gamma$.

Theorem 3.1. *The algebra $\mathcal{P}^*_\gamma$ is generated (over $\mathbb{Q}[\gamma]$) by $\left(M^{(\alpha)}_\ell\right)_{\ell\geq 2}$.*

The above theorem is a starting point for defining other interesting bases using other observables arising from classical and free probability. Besides the *moments*, we will use the *Boolean cumulants* $B^{(\alpha)}_\ell$, and the *free cumulants* $R^{(\alpha)}_\ell$. In our context it would be the most convenient to define them by the following recursive formulas that can be easily inverted over $\mathbb{Z}$, (see [DFŚ10, Proposition 2.2] and [CDM23, Proposition 2.4]).

Proposition 3.2. *For any integer $\ell \geq 2$,*

$$M^{(\alpha)}_\ell = \sum_{n\geq 1} \sum_{\substack{k_1,\dots,k_n\geq 2\\ k_1+\cdots k_n=\ell}} B^{(\alpha)}_{k_1}\cdots B^{(\alpha)}_{k_n}, \tag{13}$$

$$M^{(\alpha)}_\ell = \sum_{n\geq 1} \frac{(\ell)_{n-1}}{n!} \sum_{\substack{k_1,\dots,k_n\geq 2\\ k_1+\cdots k_n=\ell}} R^{(\alpha)}_{k_1}\cdots R^{(\alpha)}_{k_n}. \tag{14}$$

In particular we have the following theorem

Theorem 3.3. *The algebra $\mathcal{P}^*_\gamma$ is generated (over $\mathbb{Q}[\gamma]$) by $\left(X^{(\alpha)}_\ell\right)_{\ell\geq 2}$, where $X = M, B, R$.*

3.1.2. *Nazarov–Sklyanin operators.* Consider the (infinite) row vector $P = (P_{1,k})_{k\in\mathbb{N}_{\geq 1}}$ and dually the column vector $P^\dagger = (P^\dagger_{k,1})_{k\in\mathbb{N}_{\geq 1}}$, where $P_{1,k} := \sqrt{\alpha}^{-1}\cdot p_k$, and $P^\dagger_{k,1} := \sqrt{\alpha}\cdot k\cdot \frac{\partial}{\partial p_k} =: \sqrt{\alpha}^{-1}\cdot p_{-k}$, are regarded as operators on the algebra of symmetric functions $\mathcal{S}_\alpha$. Let $L = (L_{i,j})_{i,j\in\mathbb{N}_{\geq 0}}$ be the infinite matrix defined by $L_{i,j} := \sqrt{\alpha}^{-1}\cdot p_{j-i} - \delta_{i,j}\, i\gamma$, for all $i,j\in\mathbb{N}_{\geq 0}$, with the convention that $p_0 := 0$:

$$L = \begin{bmatrix} 0 & P_{1,1} & P_{1,2} & P_{1,3} & \cdots \\ P^\dagger_{1,1} & -\gamma & P_{1,1} & P_{1,2} & \ddots \\ P^\dagger_{2,1} & P^\dagger_{1,1} & -2\gamma & P_{1,1} & \ddots \\ P^\dagger_{3,1} & P^\dagger_{2,1} & P^\dagger_{1,1} & -3\gamma & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

Further, let $\tilde{L} = \left(L_{i,j}\right)_{i,j\in\mathbb{N}_{\geq 1}}$ be the submatrix of L indexed by positive integers. The main result of Nazarov–Sklyanin [NS13, Theorem 2] can be reformulated as follows:

Theorem 3.4. *The following equalities hold true for all $\ell \geq 0$:*

$$P\tilde{L}^\ell P^\dagger J^{(\alpha)}_\lambda(\mathbf{p}) = B^{(\alpha)}_{\ell+2}(\lambda)\cdot J^{(\alpha)}_\lambda(\mathbf{p}), \tag{15}$$

$$\left(L^\ell\right)_{0,0} J^{(\alpha)}_\lambda(\mathbf{p}) = M^{(\alpha)}_\ell(\lambda)\cdot J^{(\alpha)}_\lambda(\mathbf{p}). \tag{16}$$

Nazarov and Sklyanin stated their theorem differently, as they did not realize the connection with the transition measure, the fact which is crucial for us. This connection was first noticed by Moll [Mol15], and we refer the reader to the proof of Theorem 3.4 presented in [CDM23, Theorem 2.7].

3.2. **Integrality.** Theorem 3.3 implies that for every $\ell_1,\dots,\ell_k \geq 2$ and for every $X = B, M, R$ there exists a polynomial $X_{\ell_1,\dots,\ell_k}(x_0,x_1,x_2,\dots)$ with rational coefficients such that

$$X^{(\alpha)}_{\ell_1}\cdots X^{(\alpha)}_{\ell_k} = X_{\ell_1,\dots,\ell_k}(-\gamma, x_1, x_2,\dots)$$

with the convention that we identify the monomials with the normalized Jack characters $x_1^{m_1}\cdot x_2^{m_2}\cdots = \mathrm{Ch}^{(\alpha)}_{1^{m_1},2^{m_2},\dots}$. We prove that for $X = B, M$ the polynomials $X_{\ell_1,\dots,\ell_n}$ have non-negative integer coefficients and we provide their combinatorial interpretation in terms of Łukasiewicz ribbon paths introduced in [CDM23].

3.2.1. *Łukasiewicz ribbon paths.* Informally speaking, an *excursion* is a directed lattice path with steps of the form $(1,k), k\in\mathbb{Z}$, starting at $(0,0)$, finishing at $(\ell,0)$, and that stays in the first quadrant. More formally an excursion Γ of length ℓ is a sequence of points $\gamma = (w_0,\dots,w_\ell)$ on $(\mathbb{N}_{\geq 0})^2$ such that $w_j = (j, y_j)$, with $y_0 = y_\ell = 0$, and it is uniquely encoded by the sequence of its *steps* $e_j := w_j - w_{j-1} = (1, y_{j+1} - y_j)$. For a step $e = (1,y)$ its degree $\deg(e)$ is equal to y. Steps of degree 0 are called *horizontal steps*.

For a given excursion $\Gamma = (w_0, \dots, w_\ell)$, the set of points $\mathbf{S}(\gamma) := \{w_1, w_2, \dots, w_\ell\}$ (not counting the origin $w_0 = (0,0)$) naturally decomposes as

$$\mathbf{S}(\Gamma) = \bigcup_{n\in\mathbb{Z}} \mathbf{S}_n(\Gamma),$$

where $\mathbf{S}_n(\Gamma)$ is a set of points preceded by a step $(1,n)$. We also denote by $\mathbf{S}^i(\Gamma) \subset \mathbf{S}(\Gamma)$ the subset of points with second coordinate equal to i, i.e.

$$\mathbf{S}^i(\Gamma) := \{w_j = (j, y_j) \colon y_j = i\},$$

and additionally we denote $\mathbf{S}_0(\Gamma)$ by $\mathbf{S}_\to(\Gamma)$ to remind these points are procede by horizontal steps and we define $\mathbf{S}^i_\to(\Gamma) := \mathbf{S}_\to(\Gamma) \cap \mathbf{S}^i(\Gamma)$.

For an ordered tuple $\vec{\Gamma} = (\Gamma_1, \dots, \Gamma_k)$ of k excursions Γ_i, we will treat $\vec{\Gamma}$ itself as an excursion obtained by concatenating $\Gamma_1, \dots, \Gamma_k$, and we define $\mathbf{S}_n(\vec{\Gamma}), \mathbf{S}^i(\vec{\Gamma}), \mathbf{S}^i_\to(\vec{\Gamma})$, in the same way as before. We say that $p = (w_i, w_j)$ *is a pairing of degree* $n > 0$ if $w_i \in \mathbf{S}_{-n}(\vec{\Gamma}), w_j \in \mathbf{S}_n(\vec{\Gamma})$, and w_i appears before w_j in $\vec{\Gamma}$, i.e. $i < j$.

By definition, a *ribbon path on k sites of lengths $\ell_1, \dots, \ell_k$* is a pair $\vec{\mathbf{\Gamma}} = (\vec{\Gamma}, \mathbf{P}(\vec{\mathbf{\Gamma}}))$ consisting of an ordered tuple $\vec{\Gamma}$ of k excursions $\Gamma_1, \dots, \Gamma_k$ of lengths $\ell_1, \dots, \ell_k$, respectively, and a set $\mathbf{P}(\vec{\mathbf{\Gamma}})$ of disjoint pairings $p_1, \dots, p_q$ on $\vec{\Gamma}$. This notion of ribbon paths was introduced by Moll in [Mol21]. We denote by $\mathbf{P}_n(\vec{\mathbf{\Gamma}}) \subset \mathbf{P}(\vec{\mathbf{\Gamma}})$ the subset of pairings of degree n, and define $\mathbf{S}_n(\vec{\mathbf{\Gamma}}) := \mathbf{S}_n(\vec{\Gamma}) \setminus \mathbf{P}_{|n|}(\vec{\mathbf{\Gamma}})$ as the set obtained by removing the points belonging to $\mathbf{P}_n(\vec{\mathbf{\Gamma}})$ from $\mathbf{S}_n(\vec{\gamma})$. Also, let $\mathbf{S}^i_\to(\vec{\mathbf{\Gamma}}) := \mathbf{S}^i_\to(\vec{\Gamma})$. Then we have the decomposition[4]:

$$\mathbf{S}(\vec{\mathbf{\Gamma}}) = \bigcup_{i=0}^{\infty} \mathbf{S}^i_\to(\vec{\mathbf{\Gamma}}) \cup \bigcup_{n=1}^{\infty} \big(\mathbf{P}_n(\vec{\mathbf{\Gamma}}) \cup \mathbf{S}_{-n}(\vec{\mathbf{\Gamma}}) \cup \mathbf{S}_n(\vec{\mathbf{\Gamma}})\big).$$

We will denote $\mathbf{R}(\ell_1, \dots, \ell_k)$ the set of ribbon paths on k sites of lengths $\ell_1, \dots, \ell_k$.

The following theorem is a direct combinatorial interpretation of the operator $P\tilde{L}^{\ell_1-2}P^\dagger \cdots P\tilde{L}^{\ell_k-2}P^\dagger$ in terms of ribbon paths, and we leave its proof as a simple exercise (the full proof of its variant can be found in [Mol21]).

Theorem 3.5. *The following identity holds true:*

$$P\tilde{L}^{\ell_1-2}P^\dagger \cdots P\tilde{L}^{\ell_k-2}P^\dagger =$$

$$\sqrt{\alpha}^{-(\ell_1+\cdots+\ell_k)} \sum_{\substack{\vec{\mathbf{\Gamma}}\in\mathbf{R}(\ell_1,\dots,\ell_k),\\ |\mathbf{S}^0(\vec{\mathbf{\Gamma}})|=k}} \sqrt{\alpha}^{|\mathbf{S}_\to(\vec{\mathbf{\Gamma}})|} \prod_{n=1}^{\infty} (\alpha\cdot n)^{|\mathbf{P}_n(\vec{\mathbf{\Gamma}})|} \prod_{i=1}^{\infty} (-i\cdot\gamma)^{|\mathbf{S}^i_\to(\vec{\mathbf{\Gamma}})|} \cdot \prod_{j=1}^{\infty} p_j^{|\mathbf{S}_j(\vec{\mathbf{\Gamma}})|} \cdot \prod_{m=1}^{\infty} p_{-m}^{|\mathbf{S}_{-m}(\vec{\mathbf{\Gamma}})|}.$$

Finally, an excursion Γ which has only up steps of degree 1 is called a *Łukasiewicz path*. Similarly, a ribbon path $\vec{\mathbf{\Gamma}}$ whose non-paired up steps are only of degree 1 is called a *Łukasiewicz ribbon path*, i.e. $\mathbf{S}_k(\vec{\mathbf{\Gamma}}) = \emptyset, \forall\, k \geq 2$, if $\vec{\mathbf{\Gamma}}$ is a Łukasiewicz ribbon path. We will denote $\mathbf{L}(\ell_1, \dots, \ell_k)$ the set of Łukasiewicz ribbon paths on k sites of lengths $\ell_1, \dots, \ell_k$.

[4] In this paragraph, we abused the notation: the set of pairings $\mathbf{P}_n(\vec{\mathbf{\Gamma}})$ contains pairs of distinct points (w_i, w_j), but we implicitly treated such pairs as the 2-element sets $\{w_i, w_j\}$, for simplicity of notation.

Łukasiewicz paths are classical objects in combinatorics[5]. We show here that Łukasiewicz ribbon paths naturally arise in studying polynomials $X_{\ell_1,\dots,\ell_k}(x_0,x_1,\dots)$.

Theorem 3.6. *The following identities hold true:*

$$B^{(\alpha)}_{\ell_1}\cdots B^{(\alpha)}_{\ell_k} = \sum_{\substack{\vec{\mathbf{\Gamma}}\in\mathbf{L}(\ell_1,\dots,\ell_k),\\ |\mathbf{S}^0(\vec{\mathbf{\Gamma}})|=k}} \prod_{n=1}^{\infty} n^{|\mathbf{P}_n(\vec{\mathbf{\Gamma}})|}\prod_{i=1}^{\infty}(-i\cdot\gamma)^{|\mathbf{S}^i_{\to}(\vec{\mathbf{\Gamma}})|}\cdot \mathrm{Ch}^{(\alpha)}_{\mu(\vec{\mathbf{\Gamma}})}, \tag{17}$$

$$M^{(\alpha)}_{\ell_1}\cdots M^{(\alpha)}_{\ell_k} = \sum_{\vec{\mathbf{\Gamma}}\in\mathbf{L}(\ell_1,\dots,\ell_k)} \prod_{n=1}^{\infty} n^{|\mathbf{P}_n(\vec{\mathbf{\Gamma}})|}\prod_{i=1}^{\infty}(-i\cdot\gamma)^{|\mathbf{S}^i_{\to}(\vec{\mathbf{\Gamma}})|}\cdot \mathrm{Ch}^{(\alpha)}_{\mu(\vec{\mathbf{\Gamma}})}, \tag{18}$$

where $\mu(\vec{\mathbf{\Gamma}})$ is a partition given by $(1^{|\mathbf{S}_{-1}(\vec{\mathbf{\Gamma}})|}, 2^{|\mathbf{S}_{-2}(\vec{\mathbf{\Gamma}})|},\dots)$.

Proof. We start by explaining that equation (18) is a direct consequence of (17) and relation (13). Indeed, take Łukasiewicz ribbon path $\vec{\mathbf{\Gamma}}\in\mathbf{L}(\ell_1,\dots,\ell_k)$ and consider its points touching the x-axis $\mathbf{S}^0(\vec{\mathbf{\Gamma}})$. They must be of the form

$$\mathbf{S}^0(\vec{\mathbf{\Gamma}}) = \{\ell_1^1,\dots,\ell_1^1+\cdots+\ell_1^{n_1},\dots,\ell_1^1+\cdots+\ell_{k-1}^{n_{k-1}}+\ell_k^1,\dots,\ell_1^1+\cdots+\ell_k^{n_k}\},$$

where $\sum_{j=1}^{n_i}\ell_i^j=\ell_i$ for each $i=1,\dots,k$. In particular, we can consider $\vec{\mathbf{\Gamma}}$ as an element of $\mathbf{L}(\ell_1^1,\dots,\ell_1^{n_1},\dots,\ell_k^1,\dots,\ell_k^{n_k})$. Using this decomposition we can rewrite the RHS of (18) as

$$\sum_{n_1,\dots,n_k\geq 1}\ \sum_{\substack{\ell_1^1,\dots,\ell_1^{n_1}\geq 1,\\ \ell_1^1+\cdots+\ell_1^{n_1}=\ell_1}}\cdots\sum_{\substack{\ell_k^1,\dots,\ell_k^{n_k}\geq 1,\\ \ell_k^1+\cdots+\ell_k^{n_k}=\ell_k}}\ \sum_{\substack{\vec{\mathbf{\Gamma}}\in\mathbf{L}(\ell_1^1,\dots,\ell_1^{n_1},\dots,\ell_k^1,\dots,\ell_k^{n_k}),\\ |\mathbf{S}^0(\vec{\mathbf{\Gamma}})|=n_1+\cdots+n_k}}\prod_{n=1}^{\infty} n^{|\mathbf{P}_n(\vec{\mathbf{\Gamma}})|}\prod_{i=1}^{\infty}(-i\cdot\gamma)^{|\mathbf{S}^i_{\to}(\vec{\mathbf{\Gamma}})|}\cdot \mathrm{Ch}^{(\alpha)}_{\mu(\vec{\mathbf{\Gamma}})},$$

which, by (17), is equal to

$$\sum_{n_1,\dots,n_k\geq 1}\ \sum_{\substack{\ell_1^1,\dots,\ell_1^{n_1}\geq 1,\\ \ell_1^1+\cdots+\ell_1^{n_1}=\ell_1}}\cdots\sum_{\substack{\ell_k^1,\dots,\ell_k^{n_k}\geq 1,\\ \ell_k^1+\cdots+\ell_k^{n_k}=\ell_k}}\ \sum_{\substack{\vec{\mathbf{\Gamma}}\in\mathbf{L}(\ell_1^1,\dots,\ell_1^{n_1},\dots,\ell_k^1,\dots,\ell_k^{n_k}),\\ |\mathbf{S}^0(\vec{\mathbf{\Gamma}})|=n_1+\cdots+n_k}} B^{(\alpha)}_{\ell_1^1}\cdots B^{(\alpha)}_{\ell_1^{n_1}}\cdots B^{(\alpha)}_{\ell_k^1}\cdots B^{(\alpha)}_{\ell_k^{n_k}}.$$

Relation (13) finishes the proof of (18).

We now prove (17). Using the fact that $J^{(\alpha)}_\lambda\big|_{p_i=\delta_{i,1}}=1$ (see Theorem 2.1), we can use Theorem 3.4 to write $B^{(\alpha)}_{\ell_1}\cdots B^{(\alpha)}_{\ell_k}$ as

$$\left(P\tilde{L}^{\ell_1-2}P^\dagger\cdots P\tilde{L}^{\ell_k-2}P^\dagger J^{(\alpha)}_\lambda\right)\Big|_{p_i=\delta_{i,1}} = \sqrt{\alpha}^{-(\ell_1+\cdots+\ell_k)}\sum_{\substack{\vec{\mathbf{\Gamma}}\in\mathbf{R}(\ell_1,\dots,\ell_k),\\ |\mathbf{S}^0(\vec{\mathbf{\Gamma}})|=k}}\sqrt{\alpha}^{|\mathbf{S}_{\to}(\vec{\mathbf{\Gamma}})|}.$$

$$\cdot\prod_{n=1}^{\infty}(\alpha\cdot n)^{|\mathbf{P}_n(\vec{\mathbf{\Gamma}})|}\prod_{i=1}^{\infty}(-i\cdot\gamma)^{|\mathbf{S}^i_{\to}(\vec{\mathbf{\Gamma}})|}\cdot\left(\prod_{j=1}^{\infty}p_j^{|\mathbf{S}_j(\vec{\mathbf{\Gamma}})|}\cdot\prod_{m=1}^{\infty}p_{-m}^{|\mathbf{S}_{-m}(\vec{\mathbf{\Gamma}})|}J^{(\alpha)}_\lambda\right)\Big|_{p_i=\delta_{i,1}}. \tag{19}$$

[5]See [FS09] for an explanation of their name and more background.

Fix a ribbon path $\vec{\mathbf{\Gamma}} \in \mathbf{R}(\ell_1, \dots, \ell_k)$ and note that

$$\left(\prod_{j=1}^{\infty} p_j^{|\mathbf{S}_j(\vec{\mathbf{\Gamma}})|} \cdot \prod_{m=1}^{\infty} p_{-m}^{|\mathbf{S}_{-m}(\vec{\mathbf{\Gamma}})|} J_\lambda^{(\alpha)} \right) \Big|_{p_i = \delta_{i,1}} = 0 \tag{20}$$

whenever there exists $j > 1$ such that $\mathbf{S}_j(\vec{\mathbf{\Gamma}}) \neq \emptyset$. In other terms, if $\vec{\mathbf{\Gamma}}$ is not a Łukasiewicz ribbon path, then its contribution into (19) is zero, and we can replace the set $\mathbf{R}(\ell_1, \dots, \ell_k)$ in (19) by $\mathbf{L}(\ell_1, \dots, \ell_k)$. For a Łukasiewicz ribbon path (20) simplifies to:

$$\begin{aligned} &\left(p_1^{|\mathbf{S}_1(\vec{\mathbf{\Gamma}})|} \cdot \prod_{m=1}^{\infty} p_{-m}^{|\mathbf{S}_{-m}(\vec{\mathbf{\Gamma}})|} J_\lambda^{(\alpha)} \right) \Big|_{p_i = \delta_{i,1}} \\ &\qquad = \alpha^{\ell(\mu(\vec{\mathbf{\Gamma}}))} \binom{|\lambda| - |\mu(\vec{\mathbf{\Gamma}})| + |\mathbf{S}_{-1}(\vec{\mathbf{\Gamma}})|}{|\mathbf{S}_{-1}(\vec{\mathbf{\Gamma}})|} \cdot z_{\mu(\vec{\mathbf{\Gamma}})} \left[p_{1^{|\lambda| - |\mu(\vec{\mathbf{\Gamma}})|} \cup \mu(\vec{\mathbf{\Gamma}})} \right] J_\lambda^{(\alpha)} \\ &\qquad = \sqrt{\alpha}^{|\mu(\vec{\mathbf{\Gamma}})| + \ell(\mu(\vec{\mathbf{\Gamma}}))} \operatorname{Ch}^{(\alpha)}_{\mu(\vec{\mathbf{\Gamma}})}. \end{aligned} \tag{21}$$

Finally, notice that for any ribbon path $\vec{\mathbf{\Gamma}}$ of length ℓ one has

$$\sum_{n=1}^{\infty} n \big(|\mathbf{S}_{-n}(\vec{\mathbf{\Gamma}})| - |\mathbf{S}_n(\vec{\mathbf{\Gamma}})| \big) = 0, \qquad \sum_{n=1}^{\infty} \big(|\mathbf{S}_{-n}(\vec{\mathbf{\Gamma}})| + |\mathbf{S}_n(\vec{\mathbf{\Gamma}})| \big) = \ell - |\mathbf{S}_{\rightarrow}(\vec{\mathbf{\Gamma}})| - 2|\mathbf{P}_n(\vec{\mathbf{\Gamma}})|,$$

thus for any Łukasiewicz ribbon path $\vec{\mathbf{\Gamma}} \in \mathbf{L}(\ell_1, \dots, \ell_k)$

$$\begin{aligned} |\mu(\vec{\mathbf{\Gamma}})| + \ell(\mu(\vec{\mathbf{\Gamma}})) = \sum_{m \geq 1} (m+1) |\mathbf{S}_{-m}(\vec{\mathbf{\Gamma}})| = \sum_{m \geq 1} \big(|\mathbf{S}_{-m}(\vec{\mathbf{\Gamma}})| + |\mathbf{S}_m(\vec{\mathbf{\Gamma}})| \big) = \\ = \ell_1 + \dots + \ell_k - |\mathbf{S}_{\rightarrow}(\vec{\mathbf{\Gamma}})| - 2|\mathbf{P}_n(\vec{\mathbf{\Gamma}})|. \end{aligned}$$

In particular the RHS of (21) is equal to $\sqrt{\alpha}^{\ell_1 + \dots + \ell_k - |\mathbf{S}_{\rightarrow}(\vec{\mathbf{\Gamma}})| - 2|\mathbf{P}_n(\vec{\mathbf{\Gamma}})|} \operatorname{Ch}^{(\alpha)}_{\mu(\vec{\mathbf{\Gamma}})}$ for a Łukasiewicz ribbon path $\vec{\mathbf{\Gamma}} \in \mathbf{L}(\ell_1, \dots, \ell_k)$. Plugging this into (19) yields precisely the desired identity (17), which finishes the proof. □

3.2.2. *Consequences.* Before we conclude integrality of Lassalle's conjecture let us point several applications of Theorem 3.6.

In the special case $\alpha = 1$ a positivity between Boolean cumulants was conjectured by Rattan in Śniady in [RŚ08], and was proved very recently by Koshida [Kos21] by use of Khovanov's Heisenberg category. Koshida, however, was not able to find an interpretation of the positivity (his proof was a complicated induction) and left it as an open problem. Theorem 3.6 provides an explicit combinatorial interpretation for general γ; in particular in the special case $\alpha = 1$ it gives an alternative proof to the work of Koshida, and it completes his studies.

Furthermore, it implies the following theorem.

Theorem 3.7. *The following $\mathbb{Z}[\gamma]$-algebras are all equal:*

$$\mathbb{Z}[\gamma][\operatorname{Ch}^{(\alpha)}_\mu \colon \mu \in \mathbb{Y}] = \mathbb{Z}[\gamma, M_2^{(\alpha)}, M_3^{(\alpha)}, \dots] = \mathbb{Z}[\gamma, R_2^{(\alpha)}, R_3^{(\alpha)}, \dots] = \mathbb{Z}[\gamma, B_2^{(\alpha)}, B_3^{(\alpha)}, \dots].$$

This theorem gives the biggest progress so far towards another conjecture of Lassalle from [Las09] that postulates that $\mathrm{Ch}_\mu^{(\alpha)}$ is a polynomial in $\gamma, R_2^{(\alpha)}, R_3^{(\alpha)}, \dots$ with positive integer coefficients (this formulation, which is a more precise version of the original Lassalle's conjecture was formulated in [DF16]). Rationality of the coefficients of this polynomial (called *Kerov polynomial* for Jack characters) was proven in [DF16], and the top-degree part of the Kerov polynomial was found by Śniady in [Ś19]; these properties have found applications for studying random Young diagrams [DF16, DS19, CDM23].

Proof of Theorem 3.7. The last two equalities are well-known to the experts and follow from Proposition 3.2 and the fact that the equations (13) and (14) are invertible over $\mathbb{Z}$.

In order to prove the first equality, consider the lexicographic order on the set of partitions μ of size n and extend it to the set of partitions of size at most n by defining $\nu \leq \mu$ if $|\nu| < |\mu|$, or $|\nu| = |\mu|$ and $\nu \leq_{\text{lex}} \mu$. Notice then the following fact: for any partition μ of length ℓ we have

$$M^{(\alpha)}_{\mu_1+1} \cdots M^{(\alpha)}_{\mu_\ell+1} = \mathrm{Ch}_\mu^{(\alpha)} + \sum_{\rho<\mu} a^\mu_\rho(\gamma)\,\mathrm{Ch}_\rho^{(\alpha)},$$

where $a^\mu_\rho(\gamma) \in \mathbb{Z}[\gamma]$.Indeed, (18) implies that the contribution of $\mathrm{Ch}_\rho^{(\alpha)}$ comes from $\vec{\mathbf{\Gamma}} \in \mathbf{L}(\mu_1+1, \dots, \mu_{\ell(\mu)}+1)$ with $|\mu(\vec{\mathbf{\Gamma}})| = \rho$. Therefore

$$|\mu(\vec{\mathbf{\Gamma}})| = |\mu| + \ell(\mu) - \sum_{n\geq 1} |\mathbf{S}_n(\vec{\mathbf{\Gamma}})| - |\mathbf{S}_\rightarrow(\vec{\mathbf{\Gamma}})| - 2|\mathbf{P}_n(\vec{\mathbf{\Gamma}})| \leq |\mu|,$$

where the last inequality is an equality if and only if $|\mathbf{S}_\rightarrow(\vec{\mathbf{\Gamma}})| = 2|\mathbf{P}_n(\vec{\mathbf{\Gamma}})|$ and $\sum_{n\geq 1} |\mathbf{S}_n(\vec{\mathbf{\Gamma}})| = \ell(\mu)$. There is a unique $\vec{\mathbf{\Gamma}} \in \mathbf{L}(\mu_1+1, \dots, \mu_{\ell(\mu)}+1)$ that satisfies these conditions: $\Gamma = (\Gamma_1, \dots, \Gamma_{\ell(\mu)})$, where Γ_i is given by an up-step of degree μ_i followed by μ_i down-steps. In particular $\mu(\vec{\mathbf{\Gamma}}) = \mu$, which implies that the matrix $(a^\mu_\rho(\gamma))_{\rho,\mu}$ is uni-triangular over $\mathbb{Z}[\gamma]$, thus it is invertible. This finishes the proof. ☐

We have the following corollary

Corollary 3.8. *The normalized Jack characters expressed in the Stanley coordinates* $(-1)^{|\mu|} z_\mu \tilde{\theta}^{(\alpha)}_\mu(\mathbf{s}, \mathbf{r})$ *are polynomials in the variables* $b, -s_1, -s_2, \dots, r_1, r_2, \dots$ *with integer coefficients, where* $b := \alpha - 1$.

Proof. Strictly from the definition (11) of the moments one has

$$\sqrt{\alpha}^{\ell} \cdot M_\ell^{(\alpha)}(\mathbf{s}^{\mathbf{r}}) = [z^{-\ell-1}] \prod_{i=1}^{k} \frac{1}{z + r_1 + \cdots + r_k} \prod_{i=1}^{k} \frac{z - \big(\alpha \cdot s_i - (r_1 + \cdots + r_{i-1})\big)}{z - \big(\alpha \cdot s_i - (r_1 + \cdots + r_i)\big)},$$

where we extract the coefficient of $z^{-\ell-1}$ in the above rational function treated as a formal power series in z^{-1}. In particular it is clear that $\sqrt{\alpha}^{\ell} M_\ell^{(\alpha)}$ is a polynomial in $b, -s_1, -s_2, \dots, r_1, r_2, \dots$ with integer coefficients. Then the result follows from the definition of $\gamma := b \cdot \sqrt{\alpha}^{-1}$, the definition of the normalized Jack characters (10), and Theorem 3.7. ☐

4. The combinatorial model and differential equations

4.1. **Statistics of non-orientability.** The purpose of this subsection is to define a family of statistic of non-orientability (see Definition 1.1) on layered maps. We start by some general definitions related to non-orientable maps.

Let M be a bipartite map and let c_1 and c_2 be two corners of M of different colors. Then we have two ways to add an edge to M between these two corners. We denote by e_1 and e_2 these edges. We say that the pair (e_1, e_2) is a *pair of twisted edges* on the map M and we say that e_2 is obtained by twisting e_1 (see Fig. 3). Note that if M is connected and orientable, then exactly one of the maps $M \cup \{e_1\}$ and $M \cup \{e_2\}$ is orientable. For a given map with a distinguished edge (M, e), we denote $(\widetilde{M}, \tilde{e})$ the map obtained by twisting the edge e.

We recall that b is the parameter related to the Jack parameter α by $b = \alpha - 1$. We now give the definition of a measure of non-orientability due to La Croix.

Definition 4.1. [La 09, Definition 4.1] *We call a measure of non-orientability* (MON) *a function ρ defined on the set of connected maps (M, e) with a distinguished edge, with values in $\{1, b\}$, satisfying the following conditions:*

- *if e connects two corners of the same face of $M \backslash \{e\}$, and the number of the faces increases by 1 by adding the edge e on the map $M \backslash \{e\}$, then $\rho(M, e) = 1$. In this case we say that e is a diagonal.*
- *if e connects two corners in the same face $M \backslash \{e\}$, and the number of the faces of $M \backslash \{e\}$ is equal to the number of faces of M, then $\rho(M, e) = b$. In this case we say that e is a twist.*
- *if e connects two corners of two different faces lying in the same connected component of $M \backslash \{e\}$, then ρ satisfies $\rho(M, e) + \rho(\widetilde{M}, \tilde{e}) = 1 + b$. In this case we say that e is a handle. Moreover, if M is orientable then $\rho(M, e) = 1$.*
- *if e connects two faces lying in two different connected components, then $\rho(M, e) = 1$. In this case, we say that e is a bridge.*

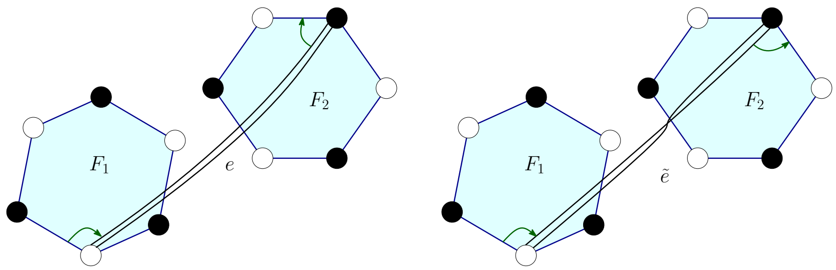
Let M be a bipartite map and let $e_1, e_2, \dots, e_d$ be d distinct edges of M. For $1 \leq i \leq d$, we denote M_j be the map obtained by deleting the edges e_1, e_2,..., e_j from M. We define $\rho(M, e_1, e_2, \dots, e_d)$ as the weight obtained by deleting the edges e_j successively:

$$\rho(M, e_1, e_2, \dots, e_d) := \prod_{1 \leq j \leq d} \rho(M_j, e_j).$$

Given a MON ρ, one can associate a b-weight $\rho(M)$ to a layered map M by choosing an order on the edges of M. We start by defining an order on black vertices.

We fix an integer $k \geq 1$ and a labelled k-layered map M. To each black vertex v of a M we associate the triplet of positive integers (i, n, j), where i is the layer containing the vertex, n is its degree and j is the number given to v by the labelling of the map. By definition, the triplets associated to two distinct black vertices are different. We define then a linear order $\prec_M$ on the black vertices of M given by the lexicographic order on the triplets $(-i, n, j)$. In other terms, the maximal black vertex with respect to $\prec_M$ is the vertex contained in the layer of smallest index, and having maximal degree and maximal

(A) A pair of twisted edges $(e, \tilde{e})$ between two corners of the same face; e is a diagonal while $\tilde{e}$ is a twist.



(B) A pair of twisted edges $(e, \tilde{e})$ between two corners of different faces F_1 and F_2. If F_1 and F_2 are on the same connected component then the edges e and $\tilde{e}$ are handles, otherwise they are bridges.

FIGURE 3. The different ways of adding an edge on a map. The added edge is represented each time by a band, that can be twisted at most once. The arrows indicate how the added edge connects the two respective corners.

label. Note that when we delete the maximal vertex from a k-layered map and all the edges incident to it, the map obtained is also k-layered.

Remark 2. Note that if M is a labelled layered map, and N is one of its connected component then N inherits a structure of labelled layered map for which the black vertices roots are unchanged and the order $\prec_N$ is the order induced by $\prec_m$.

We now introduce a statistic of non-orientability on layered labelled maps which will have a key role in this paper.

Definition 4.2 (Statistic of non-orientability on k-layered maps)**.** *Let ρ be a* MON *and let M be a connected labelled k-layered map of size n. We define the b-weight $\rho(M)$ of M, as the weight obtained by decomposing M recursively as following. If M is the empty map then $\rho(M) = 1$. Otherwise, let v be the maximal black vertex of M with respect to $\prec_M$, and let c be its root and d its degree. We denote by $e_1, \dots, e_d$ the edges incident to v as they appear when we turn around v starting from the root c. Let M' be the map obtained from M by deleting v and all the edges incident to it. Then, by definition,*

$$\rho(M) := \rho(M, e_1, \dots, e_d) \cdot \rho(M').$$

We extend this definition to disconnected maps by multiplicativity; if M is a labelled k-layered map with m connected components $M_1, \dots, M_m$, then $\rho(M) := \prod_{1 \le i \le m} \rho(M_i)$.

Finally, we define the statistic ϑ_ρ on labelled layered maps with values in non-negative integers, defined for every M by $\rho(M) = b^{\vartheta_\rho(M)}$.

It follows from the definitions that ϑ_ρ is a statistic of non-orientability over labelled bipartite maps.

Remark 3. The statistic ϑ_ρ is a variant of the statistics introduced in [La 09, Doł17, CD22], except that the decomposition algorithm used here is different and depends not only on the structure of the map but also on its labelling.

4.2. **Catalytic operators in Y and differential equations.** In this subsection, we fix an integer $k \geq 1$, and k variables $s_1, s_2, \dots, s_k$. We consider the algebra

$$\mathcal{P} := \mathrm{Span}_{\mathbb{Q}(b)}\{p_\lambda\}_{\lambda\in\mathbb{Y}}.$$

We also consider an alphabet $Y := \{y_0, y_1, \dots\}$ and the space[6] $\mathcal{P}_Y$

$$\mathcal{P}_Y := \mathrm{Span}_{\mathbb{Q}(b)}\left\{y_i p_\lambda\right\}_{i\in\mathbb{N},\lambda\in\mathbb{Y}}.$$

We fix a MON ρ. Let M be a k-layered map. We define its *marking* $\kappa(M)$ by

$$\kappa(M) := \frac{\rho(M)}{2^{|\mathcal{V}_\bullet(M)|-cc(M)}\alpha^{cc(M)}} p_{\nu_\diamond(M)} \prod_{1\leq i\leq k} (-\alpha s_i)^{|\mathcal{V}_\circ^{(i)}(M)|} \in \mathcal{P},$$

where $\nu_\diamond(M)$ is the face-type of M defined in Section 1.2.

Definition 4.3 ([CD22])**.** *We say that a map M is rooted if it has a distinguished oriented black corner c, called the root of the map. Let (M, c) be a rooted labelled k-layered map. We associate to (M, c) the marking $\kappa(M, c)$ defined by:*

$$\kappa(M,c) := \frac{\rho(M)}{2^{|\mathcal{V}_\bullet(M)|-cc(M)}\alpha^{cc(M)}} y_{\deg(f_c)} \prod_{f\neq f_c} p_{\deg(f)} \prod_{1\leq i\leq k} (-\alpha s_i)^{|\mathcal{V}_\circ^{(i)}(M)|} \in \mathcal{P}_Y,$$

where $\deg(f_c)$ is the degree of the root face, and the first product runs over the faces of M different from the root face.

We use the catalytic operators introduced in [CD22]:

$$Y_+ = \sum_{i\geq 0} y_{i+1}\frac{\partial}{\partial y_i}, \tag{22}$$

$$\Gamma_Y = (1+b)\cdot\sum_{i,j\geq 1} y_{i+j}\frac{i\partial^2}{\partial p_i \partial y_{j-1}} + \sum_{i,j\geq 1} y_i p_j \frac{\partial}{\partial y_{i+j-1}} + b\cdot\sum_{i\geq 1} y_{i+1}\frac{i\partial}{\partial y_i}. \tag{23}$$

The following proposition gives a combinatorial interpretation for the operator Γ_Y, which corresponds to the particular case $k = 1$ in [CD22, Proposition 4.4]. We give here the main arguments of the proof.

Proposition 4.4 ([CD22])**.** *Let (M, c) be a k-layered rooted map. Then for every MON ρ, we have*

$$\Gamma_Y \kappa(M,c) = \sum_e \kappa(M\cup\{e\}, c),$$

where the sum is taken over all ways to add an edge e incident to the root corner c and a white corner c' (without adding a new white vertex).

Proof. We distinguish three cases.

[6]This notation is slightly different from the one used in [CD22].

- The two corners c and c' lie in distinct faces of respective sizes i and j. When we add the edge e we form a face of size $i+j+1$. Let us show that this case corresponds to the first term of the operator Γ_Y. Let $\tilde{e}$ denote the edge obtained by twisting e, see Section 4.1.

  If the two faces lie in the same connected component of M, then e is a handle and by definition of a MON
$$\rho(M \cup \{e\}, e) + \rho(M \cup \{\tilde{e}\}, \tilde{e}) = 1 + b,$$
  and this explains the factor $1+b$ in the first terms of Γ_Y.

  If the two faces lie in two different connected components, then e is a bridge and
$$\rho(M \cup \{e\}, e) = \rho(M \cup \{\tilde{e}\}, \tilde{e}).$$
  In this case, the factor $1+b=\alpha$ in the first term of Γ_Y is related to the fact that the number of connected components of M decreases by 1.
- The two corners c and c' lie in the same face of degree $i+j-1$ and the added edge e is a diagonal, which splits the face into two faces of respective degrees i and j. Then $\rho(M \cup \{e\}, e) = 1$.
- The two corners c and c' lie in the same face of degree i and the added edge e is a twist. By adding e we form a face of degree $i+1$ and $\rho(M \cup \{e\}, e) = b$. ☐

4.3. **Proof of Proposition 1.6.** We consider the operator Θ_Y, from $\mathcal{P}_Y$ to $\mathcal{P}$, defined by
$$\Theta_Y := \sum_{i \geq 1} p_i \frac{\partial}{\partial y_i}.$$
Applied on the marking of a rooted map, Θ_Y allows to forget the root and to obtain the marking of an unrooted map. In other words, if (M,c) is a rooted map, then
$$\Theta_Y \cdot \kappa(M,c) = \kappa(M),$$
where M is the map obtained by forgetting the root c in (M,c). For $n \geq 0$, the operator B_n is defined by
$$B_n(\mathbf{p}, u) := \Theta_Y \left(\Gamma_Y + uY_+\right)^n \frac{y_0}{1+b}.$$

Proposition 4.5. *Let M be a k-layered bipartite map and let $n \geq 1$. Then*
$$B_n(\mathbf{p}, -\alpha s_1) \cdot \kappa(M) = \sum_{M'} \kappa(M'),$$
where the sum is taken over all k-layered maps obtained by adding a black vertex of degree n to the map M in the layer 1 (using possibly new white vertices which are necessarily in the layer 1).

Proof. We start by adding an isolated black vertex v of degree 0 to M, this corresponds to the multiplication by y_0/α (the division by α is due to the fact that we increase the number of connected components of the map by 1). We have two ways to add an edge incident to v:

- We add an edge to an existing black corner in the map, this corresponds to the term Γ_Y (see Proposition 4.4).

- We add an edge connected to a new white vertex. Since the weight of a white vertex in the layer 1 is $-\alpha s_1$, this corresponds to $(-\alpha s_1)Y_+$.

Finally, we apply Θ_Y to forget the root of the map. □

For each $\ell, k \geq 0$, we define $C_{\ell,k}(\mathbf{p})$ by

$$C_{\ell,k}(\mathbf{p}) := [u^\ell]B_{k+\ell}(\mathbf{p}, u).$$

From Proposition 4.5 we get that $C_{\ell,k}$ acts on the marking of a bipartite map by adding a black vertex of degree $\ell + k$ with ℓ new white neighbors. The operators C_ℓ are then defined for $\ell \geq 0$ as the marginal sums

$$C_\ell(t, \mathbf{p}) := \sum_{k\geq 1} \frac{t^{\ell+k}}{\ell + k} C_{\ell,k}(\mathbf{p}) + \mathbb{1}_{\ell>0} \frac{t^\ell}{\ell} C_{\ell,0}(\mathbf{p}).$$

Finally, we set

$$B_\infty(t, \mathbf{p}, u) = \sum_{n\geq 0} \frac{t^n}{n} B_n(\mathbf{p}, u) = \sum_{\ell\geq 0} u^\ell C_\ell(t, \mathbf{p}).$$

The following commutation result will be useful in the proof of Proposition 1.6. The proof is postponed to Section 5.

Proposition 4.6. *Let $m, \ell \geq 1$ and let u be a formal parameter. Then,*

$$[B_\ell(\mathbf{p}, u), B_m(\mathbf{p}, u)] = 0.$$

We deduce the following construction of the function $F^{(k)}$ with the differential operators.

Proof of Proposition 1.6. Let $k \geq 1$. From the definitions

$$F^{(k-1)}(t, \mathbf{p}, s_2 \ldots, s_k) = \sum_{M\in\mathcal{M}_{k-1}} (-t)^{|M|} p_{\nu_\diamond(M)} \frac{\rho(M)}{2^{|\mathcal{V}_\bullet(M)|-cc(M)}\alpha^{cc(M)}} \prod_{1\leq i\leq k} \frac{(-\alpha s_{i+1})^{|\mathcal{V}_\circ^{(i)}(M)|}}{z_{\nu_\bullet^{(i)}(M)}}.$$

We can rewrite this sum over k-layered maps with empty layer 1, by reindexing the layer j by $j + 1$ for $1 \leq j \leq k - 1$. Hence,

$$F^{(k-1)}(t, \mathbf{p}, s_2 \ldots, s_k) = \sum_M (-t)^{|M|} \kappa(M) \prod_{1\leq i\leq k-1} \frac{1}{z_{\nu_\bullet^{(i)}(M)}},$$

where the sum is taken over all labelled k-layered maps with empty layer 1. Fix a such a map M. To obtain a labelled k-layered M' from M, we should add the layer 1 (possibly empty). We proceed as follows.

- We start by fixing a non-negative integer d and a partition μ of length d (this partition is empty if $d = 0$).
- We add successively for each $1 \leq i \leq d$ a black vertex v_i of degree μ_i, using eventually new white vertices, such that all the added vertices are in the layer 1 of the map.
- The edges $e_1, \ldots, e_{\mu_i}$ incident to a vertex vertex v_i are added successively in a cyclic order around the vertex. The vertex root is chosen such that if we travel around v_i starting from the root corner we see the edges in the following order $e_{\mu_i}, \ldots, e_1$.

Note that one has by definition $\nu_\bullet^{(1)}(M') = \mu$ and $\nu_\bullet^{(j)}(M') = \nu_\bullet^{(j)}(M)$ for $2 \leq j \leq k$. The generating series of k-layered maps M' which are obtained from M as described above can be expressed as follows

$$\sum_{M'} (-t)^{|M'|}\kappa(M') \frac{1}{z_{\nu_\bullet^{(k)}(M')}}$$
$$= \sum_{\mu \in \mathbb{Y}} \left(\prod_{1 \leq i \leq \ell(\mu)} \frac{1}{m_i(\mu)!} \right) \frac{(-t)^{\mu_{\ell(\mu)}} B_{\mu_{\ell(\mu)}}(\mathbf{p}, -\alpha s_1)}{\mu_{\ell(\mu)}} \cdots \frac{(-t)^{\mu_1} B_{\mu_1}(\mathbf{p}, -\alpha s_1)}{\mu_1} (-t)^{|M|}\kappa(M).$$

Since the operators B_{μ_i} commute (see Proposition 4.6 above), and since there are $\ell(\mu)! \prod_{j \geq 1} \frac{1}{m_j(\mu)!}$ reorderings γ of μ, the right-hand side of the last equation can be rewritten as follows

$$\sum_{\gamma} \frac{1}{|\ell(\gamma)|!} \frac{(-t)^{\gamma_{\ell(\gamma)}} B_{\gamma_{\ell(\gamma)}}(\mathbf{p}, -\alpha s_1)}{\gamma_{\ell(\gamma)}} \cdots \frac{(-t)^{\gamma_1} B_{\gamma_1}(\mathbf{p}, -\alpha s_1)}{\gamma_1} (-t)^{|M|}\kappa(M),$$

where the sum is taken over all integer compositions γ. Hence

$$\sum_{M'} (-t)^{|M'|}\kappa(M') \frac{1}{z_{\nu_\bullet^{(k)}(M')}} = \exp\left(\sum_{n \geq 1} (-t)^n \frac{B_n(\mathbf{p}, -\alpha s_1)}{n} \right) \cdot \kappa(M),$$

and we deduce that

$$F^{(k)}(t, \mathbf{p}, s_1, \dots, s_k) = \exp\left(\sum_{n \geq 1} (-t)^n \frac{B_n(\mathbf{p}, -\alpha s_1)}{n} \right) \cdot F^{(k-1)}(t, \mathbf{p}, s_2, \dots, s_k). \quad \square$$

5. The vanishing property

In this section we consider functions in two different alphabets $\mathbf{p} := (p_1, p_2, \dots,)$ and $\mathbf{q} := (q_1, q_2, \dots)$. We will use repeatedly without further mention the fact that operators and coefficient extraction which depend on different alphabets trivially commute.

5.1. The function τ_b.

We consider the function τ_b introduced in [CD22].

$$\tau_b(t, \mathbf{p}, \mathbf{q}, \underline{u}) = \sum_{\xi \in \mathbb{Y}} t^{|\xi|} \frac{J_\xi^{(\alpha)}(\mathbf{p}) J_\xi^{(\alpha)}(\mathbf{q}) J_\xi^{(\alpha)}(\underline{u})}{j_\xi^{(\alpha)}}.$$

The following theorem, due to Chapuy and Dołęga, will play a crucial role in this section.

Theorem 5.1. [CD22, Theorem 5.7] *For any $m \geq 1$, we have*

$$t^m \frac{B_m(\mathbf{p}, u)}{m} \cdot \tau_b(t, \mathbf{p}, \mathbf{q}, \underline{u}) = \frac{\partial}{\partial q_m} \tau_b(t, \mathbf{p}, \mathbf{q}, \underline{u}).$$

We start by giving a construction of the function τ_b using the operators $B_m(\mathbf{p}, u)$. This construction is related to the fact that τ_b is the generating series of bipartite maps, see [CD22, BD22a].

Corollary 5.2. *The function τ_b has the following expression:*

$$\tau_b(t, \mathbf{p}, \mathbf{q}, \underline{u}) = \exp\left(\sum_{m\geq 1} \frac{t^m q_m}{m} B_m(\mathbf{p}, u)\right) \cdot 1.$$

Proof. Fix an integer $n \geq 1$ and a partition $\mu \vdash n$. We prove that

$$[t^n q_\mu]\tau_b(t, \mathbf{p}, \mathbf{q}, \underline{u}) = \frac{1}{\ell(\mu)!} \sum_{\gamma \models \mu} \frac{B_{\gamma_{\ell(\mu)}}(\mathbf{p}, u)}{\gamma_{\ell(\mu)}} \cdots \frac{B_{\gamma_1}(\mathbf{p}, u)}{\gamma_1} \cdot 1, \tag{24}$$

where the sum is taken over all the reorderings γ of μ. We start by noticing that, for any reordering γ of μ,

$$[t^n q_\mu]\tau_b(t, \mathbf{p}, \mathbf{q}, \underline{u}) = [t^n 1_\mathbf{q}] \left(\prod_{j\geq 1} \frac{1}{m_j(\mu)!}\right) \left(\prod_{1\leq i\leq \ell(\mu)} \frac{\partial}{\partial q_{\gamma_i}}\right) \tau_b(t, \mathbf{p}, \mathbf{q}, \underline{u}),$$

where $[1_\mathbf{q}]$ denotes the extraction of the constant term in the variables $\mathbf{q}$. Since there are $\ell(\mu)! \prod_{j\geq 1} \frac{1}{m_j(\mu)!}$ reorderings γ of μ, we can rewrite the last equation as follows

$$[t^n q_\mu]\tau_b(t, \mathbf{p}, \mathbf{q}, \underline{u}) = [t^n 1_\mathbf{q}] \frac{1}{\ell(\mu)!} \sum_{\gamma \models \mu} \frac{\partial}{\partial q_{\gamma_1}} \cdots \frac{\partial}{\partial q_{\gamma_{\ell(\mu)}}} \tau_b(t, \mathbf{p}, \mathbf{q}, \underline{u}).$$

Using Theorem 5.1 and the fact that the operators $\frac{\partial}{\partial q_i}$ commute with the operators $B_j(\mathbf{p}, u)$ we obtain

$$\begin{aligned}
[t^n q_\mu]\tau_b(t, \mathbf{p}, \mathbf{q}, \underline{u}) &= [t^n 1_\mathbf{q}] \frac{1}{\ell(\mu)!} \sum_{\gamma \models \mu} \frac{t^{\gamma_{\ell(\mu)}} B_{\gamma_{\ell(\mu)}}(\mathbf{p}, u)}{\gamma_{\ell(\mu)}} \cdots \frac{t^{\gamma_1} B_{\gamma_1}(\mathbf{p}, u)}{\gamma_1} \tau_b(t, \mathbf{p}, \mathbf{q}, u) \\
&= \frac{1}{\ell(\mu)!} \sum_{\gamma \models \mu} \frac{B_{\gamma_{\ell(\mu)}}(\mathbf{p}, u)}{\gamma_{\ell(\mu)}} \cdots \frac{B_{\gamma_1}(\mathbf{p}, u)}{\gamma_1} [t^0 1_\mathbf{q}] \tau_b(t, \mathbf{p}, \mathbf{q}, \underline{u}) \\
&= \frac{1}{\ell(\mu)!} \sum_{\gamma \models \mu} \frac{B_{\gamma_{\ell(\mu)}}(\mathbf{p}, u)}{\gamma_{\ell(\mu)}} \cdots \frac{B_{\gamma_1}(\mathbf{p}, u)}{\gamma_1} \cdot 1.
\end{aligned}$$

This concludes the proof of Eq. (24) and hence the proof of the proposition. □

A second consequence of Theorem 5.1 is Proposition 4.6. The key idea of the proof is to consider the action of the commutators $B_\ell(\mathbf{p}, u)$ and $B_m(\mathbf{p}, u)$ on the function $\tau_b(\mathbf{p}, \mathbf{q}, \underline{u})$ and then extract some coefficient.

Proof of Proposition 4.6. We start by proving that the two operators $B_\ell(\mathbf{p}, u)$ and $B_m(\mathbf{p}, u)$ commute when acting on the function $\tau_b(t, \mathbf{p}, \mathbf{q}, \underline{u})$. Indeed,

$$t^{\ell+m} B_\ell(\mathbf{p}, u) B_m(\mathbf{p}, u) \cdot \tau_b(t, \mathbf{p}, \mathbf{q}, \underline{u}) = t^\ell B_\ell(\mathbf{p}, u) \cdot \frac{\partial}{\partial q_m} \tau_b(t, \mathbf{p}, \mathbf{q}, \underline{u}).$$

by Theorem 5.1. Since the operators $\frac{\partial}{\partial q_m}$ and $B_\ell(\mathbf{p},\underline{u})$ commute, we obtain

$$\begin{aligned} t^{\ell+m}B_\ell(\mathbf{p},u)B_m(\mathbf{p},u)\cdot\tau_b(t,\mathbf{p},\mathbf{q},\underline{u}) &= t^\ell\frac{\partial}{\partial q_m}B_\ell(\mathbf{p},u)\cdot\tau_b(t,\mathbf{p},\mathbf{q},\underline{u}) \\ &= t^\ell\frac{\partial}{\partial q_m}\frac{\partial}{\partial q_\ell}\tau_b(t,\mathbf{p},\mathbf{q},\underline{u}). \end{aligned}$$

Since the right-hand side of the last equation is symmetric in ℓ and m, this is also the case for the left-hand side, and we deduce that

$$[B_\ell(\mathbf{p},u),B_m(\mathbf{p},u)]\cdot\tau_b(t,\mathbf{p},\mathbf{q},\underline{u})=0. \tag{25}$$

By extracting the coefficient of $J_\xi^{(\alpha)}(\mathbf{q})$ in Eq. (25):

$$[B_\ell(\mathbf{p},u),B_m(\mathbf{p},u)]\cdot\frac{J_\xi^{(\alpha)}(\mathbf{p})J_\xi^{(\alpha)}(\underline{u})}{j_\xi^{(\alpha)}}=0.$$

This concludes the proof, since Jack polynomials form a basis of $\mathcal{P}$. □

5.2. **The space $\mathcal{P}_{\leq s}$.** We define for any integer $s\geq 1$, the space $\mathcal{P}_{\leq s}\subset\mathcal{P}$ by

$$\mathcal{P}_{\leq s}:=\mathrm{Span}_{\mathbb{Q}(b)}\left\{J_\xi^{(\alpha)}(\mathbf{p})\right\}_{\xi\in\mathbb{Y},\xi_1\leq s}.$$

In this section we prove some properties of the operators B_m when applied on $\mathcal{P}_{\leq s}$. As in the proof of Proposition 4.6, we start by considering the action of these operators on the function τ_b, but with some particular specialization. We start by the following lemma.

Lemma 5.3. *Let s be a positive integer and let ξ be a partition. Then $J_\xi^{(\alpha)}(\underline{-\alpha s})=0$ if and only if $\xi_1>s$.*

Proof. From Theorem 2.1, we know that $J_\xi^{(\alpha)}(\underline{-\alpha s})=0$ if and only if ξ contains the box $\square_0=(s+1,1)$, the only box satisfying $c_\alpha(\square_0)=\alpha s$. From the definition of the α-content this condition is satisfied if and only if $\xi_1>s$. □

The space $\mathcal{P}_{\leq s}$ satisfies a stability property and a vanishing property with respect to the operators B_m, which will play a key role in the proof of Theorem 1.7.

Proposition 5.4 (Stability property)**.** *For any $s\geq 1$, the space $\mathcal{P}_{\leq s}$ is stable by the operators $B_m(\mathbf{p},-\alpha s)$ for every $m\geq 1$.*

Proof. It is enough to prove that

$$B_m(\mathbf{p},\underline{-\alpha s})\cdot J_\xi^{(\alpha)}(\mathbf{p})\in\mathcal{P}_{\leq s} \tag{26}$$

for every partition ξ such that $\xi_1\leq s$. Fix such partition ξ.

From Theorem 5.1, we know that

$$t^mB_m(\mathbf{p},-\alpha s)\cdot\tau_b(t,\mathbf{p},\mathbf{q},\underline{-\alpha s})=\frac{m\partial}{\partial q_m}\tau_b(t,\mathbf{p},\mathbf{q},-\alpha s).$$

Fix a partition ξ such that $\xi_1 \leq s$. By extracting the coefficient of $t^{|\xi|+m} J^{(\alpha)}_\xi(\mathbf{q})$ in the last equation, we get

$$B_m(\mathbf{p}, -\alpha s) \cdot \frac{J^{(\alpha)}_\xi(\mathbf{p}) J^{(\alpha)}_\xi(\underline{-\alpha s})}{j^{(\alpha)}_\xi} = \left[t^{|\xi|+m} J_\xi(\mathbf{q})\right] \frac{m\partial}{\partial q_m} \tau_b(t, \mathbf{p}, \mathbf{q}, -\alpha s)$$
$$= \sum_{\pi \vdash |\xi|+m} \frac{J^{(\alpha)}_\pi(\mathbf{p}) J^{(\alpha)}_\pi(\underline{-\alpha s})}{j^{(\alpha)}_\pi} [J_\xi(\mathbf{q})] \frac{m\partial}{\partial q_m} J_\pi(\mathbf{q}).$$

Using Lemma 5.3, we know that $\frac{J^{(\alpha)}_\xi(\underline{-\alpha s})}{j^{(\alpha)}_\xi} \neq 0$, and that the right hand-side of the last equation is in $\mathcal{P}_{\leq s}$. This finishes the proof of Eq. (26) and hence the proof of the proposition. $\square$

Proposition 5.5. *Fix an integer $s \geq 1$. The operator $\exp\left(B_\infty(t, \mathbf{p}, -\alpha s)\right)$, as operator on $\mathcal{P}_{\leq s}$, is a polynomial in t of degree less or equal than s. In other terms, for every ξ, such that $\xi_1 \leq s$ and for every $\ell > s$, we have*

$$\left([t^\ell] \exp\left(B_\infty(t, \mathbf{p}, -\alpha s)\right)\right) \cdot J^{(\alpha)}_\xi(\mathbf{p}) = 0.$$

Proof. We start by considering the action of this operator on the function $\tau_b(z, \mathbf{p}, \mathbf{q}, \underline{-\alpha s})$, where z is a formal parameter. Using Corollary 5.2 we have

$$\left([t^\ell] \exp\left(B_\infty(t, \mathbf{p}, -\alpha s)\right)\right) \cdot \tau_b(z, \mathbf{p}, \mathbf{q}, \underline{-\alpha s})$$
$$= \left([t^\ell] \exp\left(B_\infty(t, \mathbf{p}, -\alpha s)\right)\right) \exp\left(\sum_{m \geq 1} \frac{z^m q_m}{m} B_m(\mathbf{p}, \underline{-\alpha s})\right) \cdot 1.$$

But since the operators $B_m(\mathbf{p}, \underline{-\alpha s})$ for $m \geq 1$ commute (see Proposition 4.6), the last equation can be rewritten as follows:

$$\left([t^\ell] \exp\left(B_\infty(t, \mathbf{p}, -\alpha s)\right)\right) \tau_b(z, \mathbf{p}, \mathbf{q}, \underline{-\alpha s})$$
$$= \exp\left(\sum_{m \geq 1} \frac{z^m q_m}{m} B_m(\mathbf{p}, \underline{-\alpha s})\right) \left([t^\ell] \exp\left(B_\infty(t, \mathbf{p}, -\alpha s)\right)\right) \cdot 1$$
$$= \exp\left(\sum_{m \geq 1} \frac{z^m q_m}{m} B_m(\mathbf{p}, \underline{-\alpha s})\right) \cdot [t^\ell] \tau_b(t, \mathbf{p}, \underline{1}, \underline{-\alpha s}).$$

Let us now prove that

$$[t^\ell] \tau_b(t, \mathbf{p}, \underline{1}, \underline{-\alpha s}) = 0. \tag{27}$$

Recall that

$$[t^\ell] \tau_b(t, \mathbf{p}, \underline{1}, \underline{-\alpha s}) = \sum_{\xi \vdash \ell} \frac{J^{(\alpha)}_\xi(\mathbf{p}) J^{(\alpha)}_\xi(\mathbf{q}) J^{(\alpha)}_\xi(\underline{-\alpha s})}{j^{(\alpha)}_\xi}.$$

Since $\ell > s$, we know that any partition of size ℓ contains at least one of the two boxes $(s+1, 1)$ and $(1, 2)$, of respective α-content αs and -1. Hence, from Theorem 2.1, we know that

$$\frac{J_\xi(\underline{1})J_\xi(\underline{-\alpha s})}{j_\xi^{(\alpha)}}=0$$

for any partition ξ of size ℓ. This proves Eq. (27), and as a consequence,

$$\left([t^\ell]\exp\left(B_\infty(t,\mathbf{p},-\alpha s)\right)\right)\cdot\tau_b(z,\mathbf{p},\mathbf{q},\underline{-\alpha s})=0.$$

Let ξ be a partition with $\xi_1\leq s$. We extract the coefficient of $t^{|\xi|}J_\xi^{(\alpha)}(\mathbf{q})$ in the last equation, and we use the fact that this extraction commutes with the action of $\exp\left(B_\infty(t,\mathbf{p},-\alpha s)\right)$:

$$\left([t^\ell]\exp\left(B_\infty(t,\mathbf{p},-\alpha s)\right)\right)\cdot[t^{|\xi|}J_\xi^{(\alpha)}(\mathbf{q})]\tau_b(z,\mathbf{p},\mathbf{q},\underline{-\alpha s})=0.$$

Hence

$$\left([t^\ell]\exp\left(B_\infty(t,\mathbf{p},-\alpha s)\right)\right)\cdot\frac{J_\xi^{(\alpha)}(\mathbf{p})J_\xi^{(\alpha)}(\underline{-\alpha s})}{j_\xi^{(\alpha)}}=0.$$

But from Lemma 5.3 we know that $J_\xi^{(\alpha)}(\underline{-\alpha s})\neq 0$, which concludes the proof. ☐

5.3. **Proof of Theorem 1.7.** We now prove the main theorem of this section.

Proof of Theorem 1.7. Fix a partition λ and an integer $\ell>|\lambda|$, and let us prove that

$$[t^\ell]F(t,\mathbf{p},\lambda)=0.$$

Let m denote the length of λ. From the definitions of Eqs. (3) and (4), we know that

$$[t^\ell]F(t,\mathbf{p},\lambda)=\sum_{\gamma\models\ell,\,\ell(\gamma)=m}\left([t^{\gamma_1}]\exp\left(B_\infty(-t,\mathbf{p},-\alpha\lambda_1)\right)\right)\cdots\left([t^{\gamma_m}]\exp\left(B_\infty(-t,\mathbf{p},-\alpha\lambda_m)\right)\right)\cdot 1,\tag{28}$$

where the sum is taken over compositions γ of ℓ of length m. Fix such a composition γ, and $1\leq i\leq m$ such that $\gamma_i>\lambda_i$ (such an integer exists because $\ell>|\lambda|$).

Since $1\in\mathcal{P}_{\leq\lambda_m}$, we have from Proposition 5.4, that $[t^{\lambda_m}]\exp(B_\infty(-t,\mathbf{p},-\alpha\lambda_m))\cdot 1\in\mathcal{P}_{\leq\lambda_m}$. Since $\lambda_m\leq\lambda_{m-1}$, one has $\mathcal{P}_m\subset\mathcal{P}_{m-1}$. Using Proposition 5.4 a second time, we obtain $\left([t^{\lambda_{m-1}}]\exp(B_\infty(-t,\mathbf{p},-\alpha\lambda_{m-1}))\right)\left([t^{\lambda_m}]\exp(B_\infty(-t,\mathbf{p},-\alpha\lambda_m))\right)\cdot 1\in\mathcal{P}_{\leq\lambda_{m-1}}$. Iterating this argument, we obtain

$$\left([t^{\gamma_{i+1}}]\exp\left(B_\infty(-t,\mathbf{p},-\alpha\lambda_{i+1})\right)\right)\cdots\left([t^{\gamma_m}]\exp\left(B_\infty(-t,\mathbf{p},-\alpha\lambda_m)\right)\right)\cdot 1\in\mathcal{P}_{\leq\lambda_{i+1}}.$$

We now apply Proposition 5.5 with $s=\lambda_i$ and $\ell=\gamma_i$, we get

$$\left([t^{\gamma_i}]\exp\left(B_\infty(-t,\mathbf{p},-\alpha\lambda_i)\right)\right)\cdots\left([t^{\gamma_m}]\exp\left(B_\infty(-t,\mathbf{p},-\alpha\lambda_m)\right)\right)\cdot 1=0.\tag{29}$$

As a consequence, the contribution of any composition γ in Eq. (28) is zero, which concludes the proof of the theorem. ☐

6. Preliminary commutation relations

6.1. **The operators $Y_{\ell,k}$.** We consider the following catalytic version of the operators $C_{\ell,k}$ defined in Section 4.3. If ℓ and k are two integers, then $Y_{\ell,k}$ is defined by

$$Y_{\ell,k} := \begin{cases} [u^\ell]\left(\Gamma_Y + uY_+\right)^{\ell+k} & \text{if } \ell, k \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

on $\mathcal{P}_Y$. In practice, we think of $Y_{\ell,k}$ as the sum of all successions of operators Y_+ and Γ_Y in which Y_+ appears ℓ times and Γ_Y appears k times. $Y_{\ell,k}$ and $C_{\ell,k}$ are related by

$$C_{\ell,k} = \Theta_Y Y_{\ell,k} \frac{y_0}{1+b}. \tag{30}$$

These operators satisfy the following recursive relations.

Lemma 6.1. *Fix a pair of integers (ℓ, k). One has*

$$Y_{\ell,k} = Y_{\ell,k-1}\Gamma_Y + Y_{\ell-1,k}Y_+ + \delta_{\ell,0}\delta_{k,0}, \tag{31}$$

where δ denotes the Kronecker delta. Moreover, if $1 \leq m \leq \ell$ then

$$Y_{\ell,k} = \sum_{0 \leq i \leq k} Y_{m-1,i} Y_+ Y_{\ell-m,k-i}, \tag{32}$$

and if $1 \leq m \leq \ell + k$, then

$$Y_{\ell,k} = \sum_{0 \leq j \leq m} Y_{j,m-j} Y_{\ell-j,k-m+j}. \tag{33}$$

Proof. If $\ell \leq 0$ or $k \leq 0$ then the equations are immediate from the definition. Let us suppose that $\ell > 0$ and $k > 0$. In order to obtain Eq. (31), we expand $Y_{\ell,k}$ according to the rightmost operator; the sum of terms ending with Γ_Y (reps. Y_+) give $Y_{\ell,k-1}\Gamma_Y$ (resp. $Y_{\ell-1,k}Y_+$).

Similarly, we obtain Eq. (32) by expanding $Y_{\ell,k}$ according to the position of the m-th occurrence of the operator Y_+, and we obtain Eq. (33) by expanding on the number of occurrences of Y_+ in the m left operators. □

6.2. **Catalytic operators in $\widetilde{Y}$ and $\widetilde{Z}$.** We consider three new alphabets

$$Y' := \{y'_0, y'_1 \dots\}, \qquad Z := \{z_0, z_1, \dots\} \quad \text{and} \quad Z' := \{z'_0, z'_1, \dots\}.$$

We also denote

$$\widetilde{Y} := Y \cup Y', \quad \text{and} \quad \widetilde{Z} := Z \cup Z'.$$

Let $\mathcal{P}_{\widetilde{Y},\widetilde{Z}}$ be the space

$$\mathcal{P}_{\widetilde{Y},\widetilde{Z}} = \operatorname{Span}_{\mathbb{Q}(b)} \left\{ y_i z_j p_\lambda, y'_i z'_j p_\lambda \right\}_{i,j \in \mathbb{N}, \lambda \in \mathbb{Y}}.$$

In this subsection, we use several differential operators acting on the space $\mathcal{P}_{\widetilde{Y},\widetilde{Z}}$ which have been introduced in [CD22]. We briefly discuss the combinatorial interpretation of each one of these operators, but such interpretation is not necessary for the understanding of the computation in the next subsections. The following definition is inspired from [CD22].

Definition 6.2. *A map M is a double-rooted map if it has two distinguished oriented black corners c_1 and c_2 (not necessarily distinct), such that if c_1 and c_2 are incident to the same face, then they induce on it the same orientation. We associate to a double-rooted k-layered map (M, c_1, c_2) the marking $\kappa(M, c_1, c_2)$ defined as follows:*

- *If c_1 and c_2 lie on two distinct faces of degrees i and j respectively, then*
$$\kappa(M, c_1, c_2) := y_i z_j \prod_f p_{\deg(f)} \prod_{1\leq i\leq k} (-\alpha s_i)^{|\mathcal{V}_\circ^{(i)}(M)|},$$
where the product runs over the faces of M different from the two root-faces.
- *If c_1 and c_2 are incident to the same face, and are separated by i black corners (resp. j black corners), when travelling in the orientation induced by the roots (resp. the opposite orientation), then*
$$\kappa(M, c_1, c_2) := y'_i z'_j \prod_f p_{\deg(f)} \prod_{1\leq i\leq k} (-\alpha s_i)^{|\mathcal{V}_\circ^{(i)}(M)|},$$
where the product runs over the faces of M different from the root-face.

We define the operators Y'_+ and $\Lambda_{Y'}$ by replacing y_i by y'_i in Eq. (22) and Eq. (23), respectively. Similarly, we define Z_+, Z'_+, Λ_Z and $\Lambda_{Z'}$. We also consider the catalytic operators in the two variables $\widetilde{Y}$ and $\widetilde{Z}$.

$$\Gamma^{Y,Y'}_{Z,Z'} = (1+b)\cdot \sum_{i,j,k\geq 1} \frac{y'_{i+j-1} z'_k \partial^2}{\partial y_{i+k-1}\partial z_{j-1}} + \sum_{i,j,k\geq 1} \frac{y_{i+j-1} z_k \partial^2}{\partial y'_{i+k-1}\partial z'_{j-1}} + b\cdot \sum_{i,j,k\geq 1} \frac{y'_{i+j-1} z'_k \partial^2}{\partial y'_{i+k-1}\partial z'_{j-1}},$$

$$\Gamma_{\widetilde{Z}} = \Gamma_Z + \Gamma_{Z'} + \Gamma^{Y,Y'}_{Z,Z'}, \qquad \text{and} \qquad \widetilde{Z}_+ = Z_+ + Z'_+.$$

Let (M, c_1, c_2) be a double-rooted map, then

$$\widetilde{Z}_+ \cdot \kappa(M, c_1, c_2) = \sum_{(M', c'_1, c'_2)} \kappa(M', c'_1, c'_2),$$

where the sum runs over all double-rooted maps obtained by adding one leaf-edge to (M, c_1, c_2) on the root corner c_2. Similarly, the operator $\widetilde{Z}_+$ allows to add a non leaf-edge on the $\widetilde{Z}$-root corner of double-rooted maps. We also consider the following operator from $\mathcal{P}_{\widetilde{Y},\widetilde{Z}}$ to $\mathcal{P}_Y$

$$\Theta_{\widetilde{Z}} = \sum_{i\geq 0} p_i \frac{\partial}{\partial z_i} + \sum_{i,j\geq 0} y_{i+j} \frac{\partial^2}{\partial y'_i \partial z'_j}.$$

$\Theta_{\widetilde{Z}}$ acts by forgetting the root c_2:

$$\Theta_{\widetilde{Z}} \cdot \kappa(M, c_1, c_2) = \kappa(M, c_1).$$

Similarly, the operators $\Lambda^{Z,Z'}_{Y,Y'}$, $\Lambda_{\widetilde{Y}}$, $\widetilde{Y}_+$, $\Theta_{\widetilde{Y}}$ are defined by exchanging $z_i \leftrightarrow y_i$ and $z'_i \leftrightarrow y'_i$ in the previous definitions. Finally, let Δ be the operator

$$\Delta := (1+b)\cdot \sum_{i,j\geq 0} \frac{y'_j z'_i \partial^2}{\partial y_i \partial z_j} + \sum_{i,j\geq 0} \frac{y_j z_i \partial^2}{\partial y'_i \partial z'_j} + b\cdot \sum_{i,j\geq 0} \frac{y'_j z'_i \partial^2}{\partial y'_i \partial z'_j}.$$

Δ acts by adding an edge from the Y-root to the black corner succeeding the $\widetilde{Z}$-root (with respect to the canonical orientation). Hence, it "exchanges" the variables $\widetilde{Y}$ and $\widetilde{Z}$ (see Lemma 6.3).

6.3. **Preliminary commutation relations.** In this subsection, we prove some commutation relations satisfied by the catalytic variables in the variables $\widetilde{Y}$ and $\widetilde{Z}$.

Lemma 6.3. *We have the following equalities between operators on $\mathcal{P}_{\widetilde{Y},\widetilde{Z}}$.*

$$\Delta\Gamma_{\widetilde{Y}} = \Gamma_{\widetilde{Z}}\Delta, \qquad \text{and} \qquad \Delta\widetilde{Y}_+ = \widetilde{Z}_+\Delta.$$

As a consequence, for every $\ell, k \geq 0$, we have

$$\Delta\widetilde{Y}_{\ell,k} = \widetilde{Z}_{\ell,k}\Delta. \tag{34}$$

Proof. The first equation corresponds to [CD22, Eq. (28a)], and the second one is direct from the definitions. Eq. (34) follows immediately. □

We have the following commutation relations between $\widetilde{Y}$ and $\widetilde{Z}$ operators.

Lemma 6.4. [CD22] *We have the following commutation relations on $\mathcal{P}_{\widetilde{Y},\widetilde{Z}}$,*

$$\left[\widetilde{Z}_+, \widetilde{Y}_+\right] = 0, \qquad \left[\Gamma_{\widetilde{Z}}, \Gamma_{\widetilde{Y}}\right] = 0, \tag{35}$$

$$\left[\Gamma_{\widetilde{Z}}, \widetilde{Y}_+\right] = -\left[\widetilde{Z}_+, \Gamma_{\widetilde{Y}}\right] = \widetilde{Y}_+\Delta\widetilde{Y}_+. \tag{36}$$

Proof. These equations are a consequence of [CD22, Eq. (30)] (for $m = n = 0$, $m = n = 1$ and ($m = 1$,$n = 0$) resp.) and Lemma 6.3. □

We now consider a two catalytic variables version of $Y_{\ell,k}$, defined as the operators on $\mathcal{P}_{\widetilde{Y},\widetilde{Z}}$ given by

$$\widetilde{Z}_{\ell,k} := \begin{cases} [u^\ell]\left(\Gamma_{\widetilde{Z}} + uZ_+\right)^{\ell+k} & \text{for } \ell, k \geq 0 \\ 0 & \text{if } \ell < 0 \text{ or } k < 0. \end{cases}$$

In practice, we think of $\widetilde{Z}_{\ell,k}$ as the sum of all succession of operators $\widetilde{Z}_+$ and $\Gamma_{\widetilde{Z}}$ in which $\widetilde{Z}_+$ appears ℓ times and $\Gamma_{\widetilde{Z}}$ appears k times. Finally, let the operator on $\mathcal{P}_Y$:

$$C^Y_{\ell,k} := \Theta_{\widetilde{Z}}\widetilde{Z}_{\ell,k}\frac{z_0}{1+b}.$$

Combinatorially, $C^Y_{\ell,k}$ acts by adding a black vertex of degree $\ell + k$ with ℓ new white vertices, on an a rooted map. $C^Y_{\ell,k}$ and $C_{\ell,k}$ are related by Lemma 6.6.

Lemma 6.5. *The following equalities hold,*

$$\Theta_{\widetilde{Z}}\widetilde{Y}_{i,j} = Y_{i,j}\Theta_{\widetilde{Z}} \text{ for } i, j \geq 0, \text{ as operators from } \mathcal{P}_{\widetilde{Y},\widetilde{Z}} \text{ to } \mathcal{P}_Y, \tag{37a}$$

$$\Theta_Y\Theta_{\widetilde{Z}} = \Theta_Z\Theta_{\widetilde{Y}}, \text{ as operators from } \mathcal{P}_{\widetilde{Y},\widetilde{Z}} \text{ to } \mathcal{P}, \tag{37b}$$

$$\Theta_{\widetilde{Y}}z_i = z_i\Theta_Y, \text{ for } i \geq 0, \text{ as operators from } \mathcal{P}_Y \text{ to } \mathcal{P}_Z, \tag{37c}$$

$$\Theta_{\widetilde{Z}}\Delta\frac{z_0}{1+b} = 1 \text{ as operators on } \mathcal{P}_Y. \tag{37d}$$

(37e) $$\widetilde{Y}_{i,j}\frac{z_0}{1+b} = \frac{z_0}{1+b}Y_{i,j} \text{ for } i,j \geq 0, \text{ as operators from } \mathcal{P}_Y \text{ to } \mathcal{P}_{\widetilde{Y},\widetilde{Z}}.$$

(37f) $$C^Y_{i,j}\frac{y_0}{1+b} = \frac{y_0}{1+b}C_{i,j} \text{ for } i,j \geq 0, \text{ as operators from } \mathcal{P} \text{ to } \mathcal{P}_Y.$$

Proof. Eq. (37a) is a consequence of [CD22, Eqs. (28e) and (28f)]. Eqs. (37b) to (37e) are direct from the definitions. By exchangin $\widetilde{Y}$ and $\widetilde{Z}$ in Eq. (37e), we get

$$\widetilde{Z}_{i,j}\frac{y_0}{1+b} = \frac{y_0}{1+b}Z_{i,j}, \text{as operators from } \mathcal{P}_Z \text{ to } \mathcal{P}_{\widetilde{Y},\widetilde{Z}}.$$

Applying $\Theta_{\widetilde{Z}}$ on the left and $\frac{z_0}{1+b}$ on the right we get

$$C^Y_{i,j}\frac{y_0}{1+b} = \Theta_{\widetilde{Z}}\frac{y_0}{1+b}Z_{i,j}\frac{z_0}{1+b}.$$

We deduce Eq. (37f) by applying Eq. (37c). □

We conclude this subsection with the following lemma.

Lemma 6.6. *Let $\ell, k \geq 0$. Then,*

$$\Theta_Y C^Y_{\ell,k} = C_{\ell,k}\Theta_Y, \text{ as operators from } \mathcal{P}_Y \text{ to } \mathcal{P}.$$

Proof. Applying Eqs. (37b), (37a) and (37c) successively, we get that

$$\begin{aligned}
\Theta_Y C^Y_{\ell,k} &= \Theta_Y \Theta_{\widetilde{Z}}\widetilde{Z}_{\ell,k}\frac{z_0}{1+b}\\
&= \Theta_Z \Theta_{\widetilde{Y}}\widetilde{Z}_{\ell,k}\frac{z_0}{1+b}\\
&= \Theta_Z Z_{\ell,k}\Theta_{\widetilde{Y}}\frac{z_0}{1+b}\\
&= \Theta_Z Z_{\ell,k}\frac{z_0}{1+b}\Theta_Y\\
&= C_{\ell,k}\Theta_Y.
\end{aligned}$$

□

7. Proof of Theorem 1.9

In this section, we prove Theorem 1.9, which give the commutation relations for the operators C_ℓ. The idea of the proof is to start from Lemma 6.4 which gives the commutators of a $\widetilde{Y}$ and a $\widetilde{Z}$ operators of length 1, and use inductions to obtain the commutators of such operators of arbitrary lengths. The proof is organized as follows: we start from Lemma 6.4 which gives an expression for the commutator $[\widetilde{Z}_{\ell,k}, \widetilde{Y}_+]$ and $[\widetilde{Z}_{\ell,k}, \Gamma_{\widetilde{Z}}]$ when $\ell + k = 1$. By induction we obtain in Lemma 7.1 an expression for these commutators for any ℓ and k. By "forgetting" the first catalytic operator $\widetilde{Z}$, we deduce in Corollary 7.2 the commutators $[C^Y_{\ell,k}, \widetilde{Y}_+]$. We then use induction to obtain an expression for $\sum_{0\leq i\leq k}\left[\frac{1}{\ell+i}C^Y_{\ell,i}, Y_{m,k-i}\right]$. Finally, we deduce Theorem 1.9 by forgetting the catalytic variable $\widetilde{Y}$.

We start by following lemma which is a generalization of Lemma 6.4.

Lemma 7.1. *For any integers $\ell, k \geq -1$, we have the following equalities between operators on $\mathcal{P}_{\widetilde{Y},\widetilde{Z}}$.*

$$\left[\widetilde{Z}_{\ell,k}, \widetilde{Y}_+\right] = \sum_{i=0}^{\ell}\sum_{j=0}^{k-1} \widetilde{Y}_+\widetilde{Y}_{i,j}\Delta\widetilde{Y}_{\ell-i,k-1-j}\widetilde{Y}_+, \tag{38}$$

and

$$\left[\widetilde{Z}_{\ell,k}, \Gamma_{\widetilde{Y}}\right] = -\sum_{i=0}^{\ell-1}\sum_{j=0}^{k} \widetilde{Y}_+\widetilde{Y}_{i,j}\Delta\widetilde{Y}_{\ell-i,k-1-j}\widetilde{Y}_+. \tag{39}$$

In particular, by exchanging the variables $\widetilde{Y}$ and $\widetilde{Z}$, we get that

$$\left[\widetilde{Y}_{\ell-1,k}, \widetilde{Z}_+\right] + \left[\widetilde{Y}_{\ell,k-1}, \Gamma_{\widetilde{Z}}\right] = 0, \tag{40}$$

Proof. We prove simultaneously the three equations by induction on $\ell + k$. If $\ell = -1$ or $k = -1$ or $\ell = k = 0$ the result is immediate from the definitions. For $(\ell = 0, k = 1)$ and $(\ell = 1, k = 0)$ it corresponds to Lemma 6.4.

We now fix $\ell, k \geq 0$ such that $(\ell, k) \neq (0,0)$ and we suppose that Eqs. (38) and (39) hold for all (i,j) such that $i + j < \ell + k$. First, since $(\ell, k) \neq (0,0)$, Eq. (31) becomes

$$\widetilde{Z}_{\ell,k} = \widetilde{Z}_{\ell,k-1}\Gamma_{\widetilde{Z}} + \widetilde{Z}_{\ell-1,k}\widetilde{Z}_+ \tag{41}$$

Applying Eq. (38) with the pairs $(\ell, k-1)$ and $(\ell-1, k)$, and using Lemma 6.4 we get that

$$\begin{aligned}
\left[\widetilde{Z}_{\ell,k}, \widetilde{Y}_+\right] &= \left[\widetilde{Z}_{\ell,k-1}\Gamma_{\widetilde{Z}}, \widetilde{Y}_+\right] + \left[\widetilde{Z}_{\ell-1,k}\widetilde{Z}_+, \widetilde{Y}_+\right] \\
&= \sum_{i=0}^{\ell}\sum_{j=0}^{k-2} \widetilde{Y}_+\widetilde{Y}_{i,j}\Delta\widetilde{Y}_{\ell-i,k-2-j}\widetilde{Y}_+\Gamma_{\widetilde{Z}} + \widetilde{Z}_{\ell,k-1}\widetilde{Y}_+\Delta\widetilde{Y}_+ \\
&\quad + \sum_{i=0}^{\ell-1}\sum_{j=0}^{k-1} \widetilde{Y}_+\widetilde{Y}_{i,j}\Delta\widetilde{Y}_{\ell-1-i,k-1-j}\widetilde{Y}_+\widetilde{Z}_+ \\
&= \sum_{i=0}^{\ell}\sum_{j=0}^{k-2} \widetilde{Y}_+\widetilde{Y}_{i,j}\Delta\widetilde{Y}_{\ell-i,k-2-j}\Gamma_{\widetilde{Z}}\widetilde{Y}_+ - \sum_{i=0}^{\ell}\sum_{j=0}^{k-2} \widetilde{Y}_+\widetilde{Y}_{i,j}\Delta\widetilde{Y}_{\ell-i,k-2-j}\widetilde{Y}_+\Delta\widetilde{Y}_+ \\
&\quad + \widetilde{Z}_{\ell,k-1}\widetilde{Y}_+\Delta\widetilde{Y}_+ + \sum_{i=0}^{\ell-1}\sum_{j=0}^{k-1} \widetilde{Y}_+\widetilde{Y}_{i,j}\Delta\widetilde{Y}_{\ell-1-i,k-1-j}\widetilde{Z}_+\widetilde{Y}_+.
\end{aligned}$$

In order to deal with the operators containing the variable $\widetilde{Z}$, we make them pass through the operator Δ which transforms them into $\widetilde{Y}$ operators (switch $\widetilde{Y}$ and $\widetilde{Z}$ in Lemma 6.3). To this purpose, we apply Eq. (40) with $(\ell - i, k-1-j)$ for $0 \leq i \leq \ell$ and $0 \leq j \leq k-1$, we get that the sum of the first and the fourth item of the last equation is equal to

$$\sum_{i=0}^{\ell}\sum_{j=0}^{k-2} \widetilde{Y}_+\widetilde{Y}_{i,j}\Gamma_{\widetilde{Y}}\Delta\widetilde{Y}_{\ell-i,k-2-j}\widetilde{Y}_+ + \sum_{i=0}^{\ell-1}\sum_{j=0}^{k-1} \widetilde{Y}_+\widetilde{Y}_{i,j}\widetilde{Y}_+\Delta\widetilde{Y}_{\ell-1-i,k-1-j}\widetilde{Y}_+.$$

On the other hand, applying Eq. (38) with the pair $(\ell, k-1)$, we obtain that

$$\widetilde{Z}_{\ell,k-1}\widetilde{Y}_+\Delta\widetilde{Y}_+ = \widetilde{Y}_+\Delta\widetilde{Y}_{\ell,k-1}\widetilde{Y}_+ + \sum_{i=0}^{\ell}\sum_{j=0}^{k-2}\widetilde{Y}_+\widetilde{Y}_{i,j}\Delta\widetilde{Y}_{\ell-i,k-2-j}\widetilde{Y}_+\Delta\widetilde{Y}_+.$$

Hence

$$\begin{aligned}\left[\widetilde{Z}_{\ell,k},\widetilde{Y}_+\right] &= \sum_{i=0}^{\ell}\sum_{j=0}^{k-2}\widetilde{Y}_+\widetilde{Y}_{i,j}\Gamma_{\widetilde{Y}}\Delta\widetilde{Y}_{\ell-i,k-2-j}\widetilde{Y}_+ - \sum_{i=0}^{\ell}\sum_{j=0}^{k-2}\widetilde{Y}_+\widetilde{Y}_{i,j}\Delta\widetilde{Y}_{\ell-i,k-2-j}\widetilde{Y}_+\Delta\widetilde{Y}_+ + \widetilde{Y}_+\Delta\widetilde{Y}_{\ell,k-1}\widetilde{Y}_+ \\ &+ \sum_{i=0}^{\ell}\sum_{j=0}^{k-2}\widetilde{Y}_+\widetilde{Y}_{i,j}\Delta\widetilde{Y}_{\ell-i,k-2-j}\widetilde{Y}_+\Delta\widetilde{Y}_+ + \sum_{i=0}^{\ell-1}\sum_{j=0}^{k-1}\widetilde{Y}_+\widetilde{Y}_{i,j}\widetilde{Y}_+\Delta\widetilde{Y}_{\ell-1-i,k-1-j}\widetilde{Y}_+.\end{aligned}$$

Observe that the second and the fourth terms of the last equation cancel each other. We then have

$$\begin{aligned}\left[\widetilde{Z}_{\ell,k},\widetilde{Y}_+\right] &= \sum_{i=0}^{\ell}\sum_{j=1}^{k-1}\widetilde{Y}_+\widetilde{Y}_{i,j-1}\Gamma_{\widetilde{Y}}\Delta\widetilde{Y}_{\ell-i,k-1-j}\widetilde{Y}_+ + \widetilde{Y}_+\Delta\widetilde{Y}_{\ell,k-1}\widetilde{Y}_+ \\ &\quad + \sum_{i=1}^{\ell}\sum_{j=0}^{k-1}\widetilde{Y}_+\widetilde{Y}_{i-1,j}\widetilde{Y}_+\Delta\widetilde{Y}_{\ell-i,k-1-j}\widetilde{Y}_+, \\ &= \sum_{i=0}^{\ell}\sum_{j=0}^{k-1}\widetilde{Y}_+\widetilde{Y}_{i,j-1}\Gamma_{\widetilde{Y}}\Delta\widetilde{Y}_{\ell-i,k-1-j}\widetilde{Y}_+ + \widetilde{Y}_+\Delta\widetilde{Y}_{\ell,k-1}\widetilde{Y}_+ \\ &\quad + \sum_{i=0}^{\ell}\sum_{j=0}^{k-1}\widetilde{Y}_+\widetilde{Y}_{i-1,j}\widetilde{Y}_+\Delta\widetilde{Y}_{\ell-i,k-1-j}\widetilde{Y}_+.\end{aligned}$$

We use here the fact that $Y_{i,j} = 0$ if $i < 0$ or $j < 0$. For each couple of indices $(i,j) \neq (0,0)$, we regroup the terms in the two sums of the last equation by applying Eq. (41). On the other hand, note that the second term in the last equation can be written $\widetilde{Y}_+\widetilde{Y}_{0,0}\Delta\widetilde{Y}_{\ell,k-1}\widetilde{Y}_+$. We deduce that

$$\left[\widetilde{Z}_{\ell,k},\widetilde{Y}_+\right] = \sum_{i=0}^{\ell}\sum_{j=0}^{k-1}\widetilde{Y}_+\widetilde{Y}_{i,j}\Delta\widetilde{Y}_{\ell-i,k-1-j}\widetilde{Y}_+.$$

We prove Eq. (39) in a similar way. Using Eq. (41) and the induction hypothesis, we have

$$\begin{aligned}\left[\widetilde{Z}_{\ell,k},\Gamma_{\widetilde{Y}}\right] &= \left[\widetilde{Z}_{\ell,k-1}\Gamma_{\widetilde{Z}},\Gamma_{\widetilde{Y}}\right] + \left[\widetilde{Z}_{\ell-1,k}\widetilde{Z}_+,\Gamma_{\widetilde{Y}}\right] \\ &= -\sum_{i=0}^{\ell-1}\sum_{j=0}^{k-1}\widetilde{Y}_+\widetilde{Y}_{i,j}\Delta\widetilde{Y}_{\ell-1-i,k-1-j}\widetilde{Y}_+\Gamma_{\widetilde{Z}} - \sum_{i=0}^{\ell-2}\sum_{j=0}^{k}\widetilde{Y}_+\widetilde{Y}_{i,j}\Delta\widetilde{Y}_{\ell-2-i,k-j}\widetilde{Y}_+\widetilde{Z}_+ \\ &\quad - \widetilde{Z}_{\ell-1,k}\widetilde{Y}_+\Delta\widetilde{Y}_+.\end{aligned}$$

From Lemma 6.4, we have

$$\begin{aligned}\left[\widetilde{Z}_{\ell,k},\Gamma_{\widetilde{Y}}\right] &= -\sum_{i=0}^{\ell-1}\sum_{j=0}^{k-1}\widetilde{Y}_+\widetilde{Y}_{i,j}\Delta\widetilde{Y}_{\ell-1-i,k-1-j}\Gamma_{\widetilde{Z}}\widetilde{Y}_+ + \sum_{i=0}^{\ell-1}\sum_{j=0}^{k-1}\widetilde{Y}_+\widetilde{Y}_{i,j}\Delta\widetilde{Y}_{\ell-1-i,k-1-j}\widetilde{Y}_+\Delta\widetilde{Y}_+\\ &- \sum_{i=0}^{\ell-2}\sum_{j=0}^{k}\widetilde{Y}_+\widetilde{Y}_{i,j}\Delta\widetilde{Y}_{\ell-2-i,k-j}\widetilde{Z}_+\widetilde{Y}_+ - \widetilde{Y}_+\Delta\widetilde{Y}_{\ell-1,k}\widetilde{Y}_+\\ &- \sum_{i=0}^{\ell-1}\sum_{j=0}^{k-1}\widetilde{Y}_+\widetilde{Y}_{i,j}\Delta\widetilde{Y}_{\ell-1-i,k-1-j}\widetilde{Y}_+\Delta\widetilde{Y}_+.\end{aligned}$$

Applying Eq. (40) with $(\ell-i,k-1-j)$ for $0\le i\le\ell$ and $0\le j\le k-1$, we get that

$$\begin{aligned}\left[\widetilde{Z}_{\ell,k},\Gamma_{\widetilde{Y}}\right] &= -\sum_{i=0}^{\ell-1}\sum_{j=0}^{k-1}\widetilde{Y}_+\widetilde{Y}_{i,j}\Gamma_{\widetilde{Y}}\Delta\widetilde{Y}_{\ell-1-i,k-1-j}\widetilde{Y}_+ - \sum_{i=0}^{\ell-2}\sum_{j=0}^{k}\widetilde{Y}_+\widetilde{Y}_{i,j}\widetilde{Y}_+\Delta\widetilde{Y}_{\ell-2-i,k-j}\widetilde{Y}_+\\ &- \widetilde{Y}_+\Delta\widetilde{Y}_{\ell-1,k}\widetilde{Y}_+\\ &= -\sum_{i=0}^{\ell-1}\sum_{j=0}^{k}\widetilde{Y}_+\widetilde{Y}_{i,j}\Delta\widetilde{Y}_{\ell-1-i,k-j}\widetilde{Y}_+\end{aligned}$$ □

We deduce the following corollary.

Corollary 7.2. *Let $\ell,k\ge 0$. As operators on $\mathcal{P}_Y$,*

$$\left[C^Y_{\ell,k},Y_+\right]=(\ell+k)Y_+Y_{\ell,k-1}Y_+,\qquad \left[C^Y_{\ell,k},\Gamma_Y\right]=-(\ell+k)Y_+Y_{\ell-1,k}Y_+.$$

Proof. We start by multiplying Eq. (38) and Eq. (39) by $\Theta_{\widetilde{Z}}$ on the left and $\frac{z_0}{1+b}$ on the right, and we use Equations (37a), (37f) and (37d) to obtain:

$$\begin{aligned}\left[C^Y_{\ell,k},Y_+\right] &= \Theta_{\widetilde{Z}}\sum_{i=0}^{\ell}\sum_{j=0}^{k-1}\widetilde{Y}_+\widetilde{Y}_{i,j}\Delta\widetilde{Y}_{\ell-i,k-1-j}\widetilde{Y}_+\frac{z_0}{1+b}\\ &= \sum_{i=0}^{\ell}\sum_{j=0}^{k-1}Y_+Y_{i,j}\Theta_{\widetilde{Z}}\Delta\frac{z_0}{1+b}Y_{\ell-i,k-1-j}Y_+\\ &= \sum_{i=0}^{\ell}\sum_{j=0}^{k-1}Y_+Y_{i,j}Y_{\ell-i,k-1-j}Y_+.\\ &= \sum_{m=0}^{\ell+k-1}\sum_{i=0}^{\ell}Y_+Y_{i,m-i}Y_{\ell-i,k-1-m+i}Y_+.\end{aligned}$$

From Eq. (33), we know that in the last line, for each m the second sum is equal to $Y_+Y_{\ell,k-1}Y_+$, which concludes the proof of the first equation. Similarly, we obtain the second equation of the corollary from Eq. (39). □

We deduce the following proposition.

Proposition 7.3. *Fix $m \geq 0$ and $k \geq -1$. If $\ell > 0$ then*

$$\sum_{0 \leq i \leq k} \left[\frac{1}{\ell+i} C^Y_{\ell,i}, Y_{m,k-i}\right] = -Y_{\ell+m+1,k-1}. \tag{42}$$

Moreover, if $\ell = 0$ then

$$\sum_{1 \leq i \leq k} \left[\frac{1}{i} C^Y_{0,i}, Y_{m,k-i}\right] = mY_{m+1,k-1}. \tag{43}$$

Proof. We proceed by induction on $k+m$. For $k = -1$ the two equations are immediate from the definitions.

Let us start by proving Eq. (42). Fix (k,m) with $k \geq 0$, such that Eq. (42) is satisfied for every (j,s) such that $s+j < k+m$. Fix $0 \leq i \leq k$. We rewrite Eq. (31) as follows.

$$Y_{m,k-i} = \mathbb{1}_{m>0} Y_{m-1,k-i} Y_+ + Y_{m,k-i-1} \Gamma_{\widetilde{Y}} + \delta_{m,0}\delta_{k,i}.$$

Hence, using Corollary 7.2, we get for each $0 \leq i \leq k$

$$\left[\frac{1}{\ell+i} C^Y_{\ell,i}, Y_{m,k-i}\right] = \mathbb{1}_{m>0} \left[\frac{1}{\ell+i} C^Y_{\ell,i}, Y_{m-1,k-i}\right] Y_+ + \mathbb{1}_{m>0} Y_{m-1,k-i} Y_+ Y_{\ell,i-1} Y_+ \\ + \left[\frac{1}{\ell+i} C^Y_{\ell,i}, Y_{m,k-i-1}\right] \Gamma_{\widetilde{Y}} - Y_{m,k-1-i} Y_+ Y_{\ell-1,i} Y_+.$$

Applying the induction hypothesis on the pairs $(m-1,k)$ and $(m,k-1)$, we get that

$$\sum_{0 \leq i \leq k} \left[\frac{1}{\ell+i} C^Y_{\ell,i}, Y_{m,k-i}\right] = -\mathbb{1}_{m>0} Y_{\ell+m,k-1} Y_+ + \mathbb{1}_{m>0} \sum_{0 \leq i \leq k} Y_{m-1,k-i} Y_+ Y_{\ell,i-1} Y_+ \\ - Y_{\ell+m+1,k-2} \Gamma_{\widetilde{Y}} - \sum_{0 \leq i \leq k} Y_{m,k-1-i} Y_+ Y_{\ell-1,i} Y_+.$$

Using Eq. (32), we know that the two sums in the right-hand side of the last equality are both equal to $Y_{\ell+m,k-1} Y_+$. On the other hand, from Eq. (31), we know that

$$Y_{\ell+m+1,k-2} \Gamma_Y + Y_{\ell+m,k-1} Y_+ = Y_{\ell+m+1,k-1},$$

which concludes the proof of Eq. (42).

We now prove Eq. (43) in a similar way. Let (k,m) be two non-negative integers such that Eq. (43) is satisfied for every (j,s) such that $s+j < k+m$. For each $1 \leq i \leq k$, one has

$$\left[\frac{1}{i} C^Y_{0,i}, Y_{m,k-i}\right] = \mathbb{1}_{m>0} \left[\frac{1}{i} C^Y_{0,i}, Y_{m-1,k-i}\right] Y_+ + \mathbb{1}_{m>0} Y_{m-1,k-i} Y_+ Y_{0,i-1} Y_+ \\ + \left[\frac{1}{i} C^Y_{0,i}, Y_{m,k-1-i}\right] \Gamma_Y.$$

Applying the induction hypothesis on the pairs $(m-1,k)$ and $(m,k-1)$, we get that

$$\sum_{1\leq i\leq k}\left[\frac{1}{i}C^Y_{0,i},Y_{m,k-i}\right]=(m-1)\mathbb{1}_{m>0}Y_{m,k-1}Y_+ + \mathbb{1}_{m>0}\sum_{0\leq i\leq k}Y_{m-1,k-i}Y_+Y_{0,i-1}Y_+ + mY_{m+1,k-2}\Gamma_Y.$$

But from Eq. (32), we know that the sum in the right hand is equal to $Y_{m,k-1}Y_+$. Hence

$$\begin{aligned}\sum_{1\leq i\leq k}\left[\frac{1}{i}C^Y_{0,i},Y_{m,k-i}\right]&=m\mathbb{1}_{m>0}Y_{m,k-1}Y_+ + mY_{m+1,k-2}\Gamma_Y\\&=mY_{m,k-1}Y_+ + mY_{m+1,k-2}\Gamma_Y\\&=mY_{m+1,k-1},\end{aligned}$$

where we use Eq. (31) to obtain the last equality. □

We deduce the following corollary by forgetting the catalytic variable Y.

Corollary 7.4. *Fix $m,k\geq 0$. As operators on $\mathcal{P}$, we have*

$$\sum_{0\leq i\leq k}\left[\frac{1}{\ell+i}C_{\ell,i},C_{m,k-i}\right]=-C_{\ell+m+1,k-1},\ \textit{if } \ell>0, \tag{44}$$

and

$$\sum_{0\leq i\leq k}\left[C_{\ell,i},\frac{1}{m+k-i}C_{m,k-i}\right]=C_{\ell+m+1,k-1},\ \textit{if } m>0,. \tag{45}$$

Moreover, if $\ell=0$ then

$$\sum_{1\leq i\leq k}\left[\frac{1}{i}C_{0,i},C_{m,k-i}\right]=mC_{m+1,k-1}. \tag{46}$$

Proof. Let us prove Eq. (44). Starting from Eq. (42), and applying Θ_Y on the left and $\frac{y_0}{1+b}$ on the right, we get

$$\sum_{0\leq i\leq k}\frac{1}{\ell+i}\left(\Theta_Y C^Y_{\ell,i}Y_{m,k-i}\frac{y_0}{1+b}-\Theta_Y Y_{m,k-i}C^Y_{\ell,i}\frac{y_0}{1+b}\right)=-\Theta_Y Y_{\ell+m+1,k-1}\frac{y_0}{1+b}.$$

Using Lemma 6.6 and Eq. (37f) we obtain

$$\sum_{0\leq i\leq k}\frac{1}{\ell+i}\left(C_{\ell,i}\Theta_Y Y_{m,k-i}\frac{y_0}{1+b}-\Theta_Y Y_{m,k-i}\frac{y_0}{1+b}C_{\ell,i}\right)=-\Theta_Y Y_{\ell+m+1,k-1}\frac{y_0}{1+b}.$$

We deduce Eq. (44) using Eq. (30). We obtain in a similar way Eq. (46) from Eq. (43). Finally, we deduce Eq. (45) from Eq. (44) by exchanging m and ℓ. □

We now prove the main result of this section.

Proof of Theorem 1.9. Let $\ell, m > 0$. Then

$$[C_\ell, C_m] = \sum_{k\geq 0} t^{k+\ell+m} \sum_{0\leq i\leq k} \frac{[C_{\ell,i}, C_{m,k-i}]}{(i+\ell)(k-i+m)}$$
$$= \sum_{k\geq 0} \frac{t^{k+\ell+m}}{k+\ell+m} \left(\sum_{0\leq i\leq k} \left[\frac{C_{\ell,i}}{i+\ell}, C_{m,k-i}\right] + \sum_{0\leq i\leq k} \left[C_{\ell,i}, \frac{C_{m,k-i}}{k-i+m}\right] \right).$$

Applying Eq. (44), we deduce that the last quantity is equal to 0. Fix now $m > 0$. We have

$$[C_0, C_m] = \sum_{k\geq 0} t^{k+m} \sum_{1\leq i\leq k} \frac{[C_{0,i}, C_{m,k-i}]}{i(k-i+m)}$$
$$= \sum_{k\geq 0} \frac{t^{k+m}}{k+m} \left(\sum_{1\leq i\leq k} \left[\frac{C_{0,i}}{i}, C_{m,k-i}\right] + \sum_{1\leq i\leq k} \left[C_{0,i}, \frac{C_{m,k-i}}{k-i+m}\right] \right).$$

Since $C_{0,0} = 1$ and by consequence commutes trivially with any operator, we can write

$$[C_0, C_m] = \sum_{k\geq 0} \frac{t^{k+m}}{k+m} \left(\sum_{1\leq i\leq k} \left[\frac{C_{0,i}}{i}, C_{m,k-i}\right] + \sum_{0\leq i\leq k} \left[C_{0,i}, \frac{C_{m,k-i}}{k-i+m}\right] \right)$$
$$= \sum_{k\geq 0} \frac{t^{k+m}}{k+m} \left(mC_{m+1,k-1} + C_{m+1,k-1}\right)$$
$$= (m+1)C_{m+1}.$$

□

8. Shifted symmetry property

We start by proving some general commutation relations which will be useful in the proof of Theorem 1.8.

8.1. **Preliminaries.** Let C be an operator on $\mathcal{P}$. We consider the adjoint action of C, denoted by ad_C, as the linear map defined by

$$\mathrm{ad}_C(A) := [C, A]$$

for every operator A. By a direct induction we obtain the following lemma.

Lemma 8.1. *Let A_1, A_2 and C be three operators such that*

$$[A_2, \mathrm{ad}^m_C(A_1)] = 0, \text{ for every } m \geq 0.$$

Then

$$\mathrm{ad}^m_{A_2+C}(A_1) = \mathrm{ad}^m_C(A_1), \text{ for } m \geq 0.$$

The two following standard results of mathematical physics will be useful in this section, see e.g [Wil67, Müg19].

$$e^C A e^{-C} = e^{\mathrm{ad}_C} A, \tag{47}$$

$$\frac{d}{dt} e^{tA+C} = e^{tA+C} \frac{e^{\mathrm{ad}_{-C-tA}} - 1}{\mathrm{ad}_{-C-tA}}(A), \tag{48}$$

where $\frac{e^{\mathrm{ad}_{-C-tA}-1}}{\mathrm{ad}_{-C-tA}}$ denotes the formal power series

$$\frac{e^{\mathrm{ad}_{-C-tA}}-1}{\mathrm{ad}_{-C-tA}} := \sum_{k\geq 0} \frac{\mathrm{ad}^k_{-C-tA}}{(k+1)!}$$

The purpose of the following lemma is to establish a simple expression of for logarithm of the product of the exponential of non commutative operators, satisfying some particular conditions (this can be considered as a particular case of the Baker-Cambell-Hausdorff formula).

Lemma 8.2. *Let A and C be two operators such that $[A, \mathrm{ad}^m_C(A)] = 0$ for every $m \geq 0$. Then*

$$e^{A+C}e^{-C} = \exp\left(\frac{e^{\mathrm{ad}_C}-1}{\mathrm{ad}_C}(A)\right).$$

Proof. We consider the following function

$$\Phi(t) := e^C e^{-tA-C} \exp\left(\frac{e^{\mathrm{ad}_C}-1}{\mathrm{ad}_C}(tA)\right).$$

We want to prove that $\Phi(1) = 1$. Since $\Phi(0) = 1$, it is enough then to prove that $\frac{d}{dt}\Phi(t) = 0$. But

$$\frac{d}{dt}\Phi(t) = e^C\left(\frac{d}{dt}e^{-tA-C}\right)\exp\left(\frac{e^{\mathrm{ad}_C}-1}{\mathrm{ad}_C}(tA)\right) + e^C e^{-tA-C}\frac{d}{dt}\exp\left(\frac{e^{\mathrm{ad}_C}-1}{\mathrm{ad}_C}(tA)\right).$$

On one hand, using Eq. (48) and Lemma 8.1 we have that

$$\frac{d}{dt}e^{-tA-C} = -e^{-tA-C}\cdot\frac{e^{\mathrm{ad}_C}-1}{\mathrm{ad}_C}(A).$$

On the other hand,

$$\frac{d}{dt}\exp\left(\frac{e^{\mathrm{ad}_C}-1}{\mathrm{ad}_C}(tA)\right) = \frac{e^{\mathrm{ad}_C}-1}{\mathrm{ad}_C}(A)\exp\left(\frac{e^{\mathrm{ad}_C}-1}{\mathrm{ad}_C}(tA)\right).$$

This finishes the proof of the lemma. □

We deduce the main result of this subsection.

Proposition 8.3. *Let A_1, A_2 and C be three operators such that*

$$\left[\mathrm{ad}^\ell_C A_i, \mathrm{ad}^m_C A_j\right] = 0, \text{ for } 1 \leq i,j \leq 2 \text{ and } m,\ell \geq 0.$$

Then

$$e^{A_1+C}e^{-C}e^{A_2+C} = e^{A_1+A_2+C}.$$

Proof. We prove that

$$e^{A_1+C}e^{-C} = \exp\left(\frac{e^{\mathrm{ad}_C}-1}{\mathrm{ad}_C}(A_1)\right) = e^{A_1+A_2+C}e^{-A_2-C}.$$

The left equality in the last line is guaranteed by Lemma 8.2. Let us prove the right equality. Since the operators A_1, A_2 and C satisfy the conditions of Lemma 8.1, then

$$\mathrm{ad}^m_{A_2+C}(A_1) = \mathrm{ad}^m_C(A_1) \text{ for } m \geq 0. \tag{49}$$

Hence,

$$[A_1, \mathrm{ad}^m_{A_2+C}(A_1)] = [A_1, \mathrm{ad}^m_C(A_1)] = 0.$$

By consequence, the operators A_1 and $A_2 + C$ fulfill then the conditions of Lemma 8.2, and we obtain that

$$e^{A_1+A_2+C}e^{-A_2-C} = \exp\left(\frac{e^{\mathrm{ad}_{A_2+C}} - 1}{\mathrm{ad}_{A_2+C}} A_1\right).$$

We conclude using Eq. (49). □

8.2. **Proof of Theorem 1.8.** It will be convenient in this section to separate the constant part of the operator B_∞ in the variable u. Namely, we consider

$$B^>_\infty(t, \mathbf{p}, u) := \sum_{\ell \geq 1} u^\ell C_\ell(t, \mathbf{p}).$$

In the following, when there is no ambiguity we will simply denote

$$B_\infty(u) \equiv B_\infty(t, \mathbf{p}, u) \text{ and } B^>_\infty(u) \equiv B^>_\infty(t, \mathbf{p}, u).$$

As a consequence of Theorem 1.9, we have

$$[B^>_\infty(u), B^>_\infty(v)] = 0. \tag{50}$$

From Proposition 1.6 we know that

$$F^{(k)}(-t, \mathbf{p}, s_1 - (k-1)/\alpha, \dots, s_k) = \exp\left(B_\infty(-\alpha s_1 + k - 1)\right) \cdots \exp\left(B_\infty(-\alpha s_k)\right) \cdot 1.$$

The purpose of this section is to prove that this function is symmetric in the variables $s_1 \dots s_k$ and to give a symmetric expression of it.

Theorem 8.4. *For every $k \geq 1$, we have*

$$\begin{aligned}(51)\quad & F^{(k)}\left(-t, \mathbf{p}, s_1, s_2 \dots, s_k\right) \\ &= \exp\left(B_\infty(k-1)\right) \cdots \exp\left(B_\infty(1)\right) \exp\left(C_0 + B^>_\infty(-\alpha s'_1 - k) + \cdots + B^>_\infty(-\alpha s'_k - k)\right) \cdot 1,\end{aligned}$$

where $s'_i := s_i - i/\alpha$.

The following lemma establishes a relation between the operators $B^>_\infty(u+1)$ and $B^>_\infty(u)$.

Proposition 8.5. *For any variable u, we have*

$$B_\infty(u+1) = B_\infty(1) + e^{C_0} B^>_\infty(u) e^{-C_0}.$$

Proof. From the definition, we have

$$\begin{aligned} B_\infty(u+1) &= \sum_{\ell \geq 0} (u+1)^\ell C_\ell \\ &= \sum_{\ell \geq 0} \sum_{0 \leq k \leq \ell} \binom{\ell}{k} u^k C_\ell \\ &= B_\infty(1) + \sum_{k \geq 1} \sum_{\ell \geq k} \binom{\ell}{k} u^k C_\ell. \end{aligned}$$

Applying Theorem 1.9 inductively, we get that

$$\frac{\ell!}{k!}C_\ell = \mathrm{ad}_{C_0}^{\ell-k} C_k, \text{ for } 1 \leq k \leq \ell.$$

Hence,

$$\begin{aligned} B_\infty(u+1) &= B_\infty(1) + \sum_{k\geq 1} u^k \left(\sum_{i\geq 0} \frac{(\mathrm{ad}_{C_0})^i}{i!}\right) C_k \\ &= B_\infty(1) + \sum_{k\geq 1} u^k e^{C_0} C_k e^{-C_0} \\ &= B_\infty(1) + e^{C_0} B_\infty^{>}(u) e^{-C_0}, \end{aligned}$$

where we use Eq. (47) for the second equality. □

Remark 4. Note that the operators we consider here $B_\infty^{>}(u)$, $B_\infty(u)$ and C_0 are formal power series in t in which the term in t^0 are zero. Hence, the exponential of such operators is well defined as a formal power series in t.

We now prove that the operators $B_\infty^{>}(u)$, $B_\infty^{>}(v)$ and C_0 satisfy the conditions of Proposition 8.3.

Lemma 8.6. *Let u and v be two variables, and let m and ℓ be two non negative integers. Then*

$$\left[\mathrm{ad}_{C_0}^{\ell} B_\infty^{>}(u), \mathrm{ad}_{C_0}^{m} B_\infty^{>}(v)\right] = 0.$$

Proof. This is a consequence of Theorem 1.9. □

Lemma 8.7. *For any variable u, we have*

$$\exp(B_\infty(u+1)) = \exp\big(B_\infty(1)\big) \cdot \exp\big(B_\infty(u)\big) \cdot \exp(-C_0).$$

Proof. We know from Proposition 8.5 that

$$\begin{aligned} \exp(B_\infty(u+1)) &= \exp\left(B_\infty(1) + e^{C_0} B_\infty^{>}(u) e^{-C_0}\right) \\ &= \exp\left(C_0 + B_\infty^{>}(1) + e^{C_0} B_\infty^{>}(u) e^{-C_0}\right). \end{aligned}$$

But from Lemma 8.6 we know that the triplet of operators $B_\infty^{>}(1)$, $B_\infty^{>}(v)$ and C_0 satisfy the conditions of Proposition 8.3. Moreover, this is also the case for the triplet $B_\infty^{>}(1)$, $e^{C_0} B_\infty^{>}(u) e^{-C_0}$ and C_0, since $e^{C_0} B_\infty^{>}(u) e^{-C_0} = e^{\mathrm{ad}_{C_0}} B_\infty^{>}(u)$. Hence,

$$\begin{aligned} \exp(B_\infty(u+1)) &= \exp\left(C_0 + B_\infty^{>}(1)\right) \cdot \exp(-C_0) \cdot \exp(C_0 + e^{C_0} B_\infty^{>}(u) e^{-C_0}) \\ &= \exp\left(C_0 + B_\infty^{>}(1)\right) \cdot \exp(-C_0) \cdot \exp\left(e^{C_0}\left(C_0 + B_\infty^{>}(u)\right) e^{-C_0}\right) \\ &= \exp\big(B_\infty(1)\big) \cdot \exp\big(C_0 + B_\infty^{>}(u)\big) \cdot \exp(-C_0). \end{aligned}$$ □

Remark 5. Fix now two variables u and v. By applying Lemma 8.7 and Proposition 8.3 with $B_\infty^{>}(1)$, $B_\infty^{>}(v)$ and C_0, we get that

$$\begin{aligned} &\exp(B_\infty(u+1)) \exp(B_\infty(v)) \\ &\quad = \exp\big(B_\infty(1)\big) \cdot \exp\big(C_0 + B_\infty^{>}(u)\big) \cdot \exp(-C_0) \cdot \exp(B_\infty(v)) \\ &\quad = \exp\big(B_\infty(1)\big) \cdot \exp\big(C_0 + B_\infty^{>}(u) + B_\infty^{>}(v)\big). \end{aligned} \tag{52}$$

In particular, we deduce that

$$\exp(B_\infty(u+1))\exp(B_\infty(v)) = \exp(B_\infty(v+1))\exp(B_\infty(u)). \tag{53}$$

One can see that this implies that the series $F^{(k)}(-t, \mathbf{p}, s_1 - (k-1)/\alpha, \dots, s_k)$ is symmetric in the variables $s_1 \dots s_k$.

We now prove the main theorem of this section.

Proof of Theorem 1.8 . We prove the theorem by induction on k. For $k = 1$ the result is straightforward from the definition and for $k = 2$ it corresponds to Remark 5. Fix $k \geq 0$ and suppose that the theorem holds for $F^{(k)}$. We have then

$$\begin{aligned}F^{(k+1)}&(-t, \mathbf{p}, s_1, s_2, \dots, s_{k+1})\\ &= \exp\Big(B_\infty\big(-\alpha s_1' - 1\big)\Big) \cdot \exp\left(B_\infty(k-1)\right) \cdots \exp\left(B_\infty(1)\right) \cdot\\ &\qquad \exp\left(C_0 + B_\infty^{>}(-\alpha s_2' - k - 1) + \cdots + B_\infty^{>}(-\alpha s_{k+1}' - k - 1)\right) \cdot 1.\end{aligned}$$

Using $k-1$ times Eq. (53), we obtain

$$\begin{aligned}F^{(k+1)}&(-t, \mathbf{p}, s_1, s_2, \dots, s_{k+1})\\ &= \exp\left(B_\infty(k)\right) \cdots \exp\left(B_\infty(2)\right) \cdot \exp\left(B_\infty(-\alpha s_1' - k)\right) \cdot\\ &\qquad \exp\left(C_0 + B_\infty^{>}(-\alpha s_2 - k - 1) + \cdots + B_\infty^{>}(-\alpha s_{k+1}' - k - 1)\right) \cdot 1.\end{aligned}$$

Using Lemma 8.7, this can be rewritten as follows

$$\begin{aligned}F^{(k+1)}&(-t, \mathbf{p}, s_1, s_2, \dots, s_{k+1})\\ &= \exp\left(B_\infty(k)\right) \cdots \exp\left(B_\infty(1)\right) \exp(B_\infty(-\alpha s_1' - k - 1)) \exp(-C_0)\\ &\qquad \exp\left(C_0 + B_\infty^{>}(-\alpha s_2' - k - 1) + \cdots + B_\infty^{>}(-\alpha s_{k+1}' - k - 1)\right) \cdot 1.\end{aligned}$$

Finally, we apply Proposition 8.3 with the operators $B_\infty^{>}(-\alpha s_1' - k - 1)$, $B_\infty^{>}(-\alpha s_2' - k - 1) + \cdots + B_\infty^{>}(-\alpha s_{k+1}' - k - 1)$ and C_0 in order to reassemble the last three exponentials, which concludes the proof of the theorem. □

9. Proof of the first main result and positivity in Lassalle's conjecture

9.1. **End of proof of Theorem 1.3.**

Lemma 9.1. *The functions $F^{(k)}$ satisfy the conditions Eq. (7):*

$$F^{(k+1)}(t, \mathbf{p}, s_1, \dots, s_k, 0) = F^{(k)}(t, \mathbf{p}, s_1, \dots, s_k)$$

As a consequence, the projective limit $F^{(\infty)} := \varprojlim F^{(k)}$ is well defined in $\mathcal{S}_\alpha^*[\![t, p_1, p_2, \dots]\!]$, and

$$F^{(\infty)}(t, \mathbf{p}, s_1, s_2, \dots) = \sum_{M \in \mathcal{M}^{(\infty)}} (-t)^{|M|} p_{\nu_\diamond(M)} \frac{b^{\vartheta_\rho(M)}}{2^{|\mathcal{V}_\bullet(M)| - cc(M)} \alpha^{cc(M)}} \prod_{i \geq 1} \frac{(-\alpha s_i)^{|\mathcal{V}_\circ^{(i)}(M)|}}{z_{\nu_\bullet^{(i)}(M)}}.$$

Proof. We know from the recursive expression of $F^{(k)}$ given in Proposition 1.6 that

$$F^{(k+1)}\left(t,\mathbf{p},s_1,\dots,s_k,0\right)=\exp\left(B_\infty(-t,\mathbf{p},-\alpha s_1)\right)\cdots\exp\left(B_\infty(-t,\mathbf{p},-\alpha s_k)\right)F^{(1)}\left(t,\mathbf{p},0\right).$$

Using the combinatorial expression of the function $F^{(1)}$ (see Eq. (2)), one can see that $F^{(1)}(t,\mathbf{p},0)=1$, since the only bipartite map without any white vertex is the empty map. Reapplying Proposition 1.6, the last equation is equal to $F^{(k)}(t,\mathbf{p},s_1,\dots,s_k)$. The second part of the lemma is a consequence of Theorem 1.8. □

We now give the end of proof of the first main result.

End Proof of Theorem 1.3. Fix a partition μ. We prove that the coefficient of $t^{|\mu|}p_\mu$ in $F^{(\infty)}(t,\mathbf{p},\lambda)$ satisfies the conditions of Theorem 1.5. The fact that this coefficient vanishes on partitions λ of size $|\lambda|<|\mu|$ is given by Theorem 1.7, and we know that that it is α-shifted symmetric from Lemma 9.1.

Let us now prove that the top homogeneous part in $[t^{|\mu|}p_\mu]F^{(k)}(t,\mathbf{p},\lambda)$ is equal to $\alpha^{|\mu|-\ell(\mu)/z_\mu}p_\mu(\lambda_1,\lambda_2,\dots,\lambda_k)$ for any $k\geq 1$. This part corresponds to labelled k-layered maps of face-type μ and with maximal number of white vertices, *i.e* maps with $|\mu|$ white vertices which are all of degree 1. Note that adding a black vertex connected to n white vertices in a layer i of a map M corresponds to multiplying its marking $\kappa(M)$ by $p_n(-\alpha\lambda_i)^n/\alpha$. Thus, in order to obtain the top homogeneous part in $F^{(k)}$ we replace $B_n(\mathbf{p},\lambda_i)$ by $p_n(-\alpha\lambda_i)^n$ in Eq. (4). As a consequence, the top homogeneous part in $[t^{|\mu|}p_\mu]F^{(k)}$ is given by

$$\begin{aligned}[t^{|\mu|}p_\mu]\exp\left(\sum_{n\geq 1}\frac{(-t)^n\cdot(-\alpha\lambda_1)^n p_n}{\alpha n}\right)&\cdots\exp\left(\sum_{n\geq 1}\frac{(-t)^n\cdot(-\alpha\lambda_k)^n p_n}{\alpha n}\right)\\ &=[t^{|\mu|}p_\mu]\exp\left(\sum_{n\geq 1}\frac{t^n\alpha^{n-1}\cdot p_n(\lambda_1,\dots,\lambda_k)p_n}{n}\right)\\ &=\frac{\alpha^{|\mu|-\ell(\mu)}}{z_\mu}p_\mu(\lambda_1,\dots,\lambda_k).\end{aligned}$$ □

9.2. **Proof of Theorem 1.12.** In this section, we give a direct way to construct Jack polynomials by adding the rows in increasing order of their size. This formula is more efficient to generate Jack polynomials than the one given Proposition 1.6.

Theorem 9.2. *Fix a partition λ. Let μ be the partition obtained from λ by removing the largest part; $\mu=\lambda\backslash\lambda_1$. Then*

$$\begin{aligned}J^{(\alpha)}_\lambda&=[t^{\lambda_1}]\exp\left(B_\infty(-t,\mathbf{p},-\alpha\lambda_1)\right)\cdot J^{(\alpha)}_\mu\\ &=\sum_{\mu\vdash\lambda_1}t^{\lambda_1}\prod_{1\leq i\leq\ell(\mu)}\left(\frac{1}{m_i(\mu)!}\right)\frac{B_{\mu_1}(\mathbf{p},-\alpha\lambda_1)}{\mu_1}\cdots\frac{B_{\mu_{\ell(\mu)}}(\mathbf{p},-\alpha\lambda_1)}{\mu_{\ell(\mu)}}\cdot J^{(\alpha)}_\mu.\end{aligned}$$

Proof. From the definition of the Jack characters and Theorem 1.3 and Proposition 1.6, we have

$$\begin{aligned} J_\lambda^{(\alpha)} &= [t^{|\lambda|}] \exp\left(B_\infty(-t,\mathbf{p},-\alpha\lambda_1)\right) \cdots \exp\left(B_\infty(-t,\mathbf{p},-\alpha\lambda_{\ell(\lambda)})\right) \cdot 1 \\ &= \sum_\gamma \left([t^{\gamma_1}] \exp\left(B_\infty(-t,\mathbf{p},-\alpha\lambda_1)\right)\right) \cdots \left([t^{\gamma_{\ell(\lambda)}}] \exp\left(B_\infty(-t,\mathbf{p},-\alpha\lambda_{\ell(\lambda)})\right)\right) \cdot 1, \end{aligned}$$

where the sum runs over compositions γ of $|\lambda|$ of length $\ell(\lambda)$. But from the proof of Theorem 1.7, only the composition $\gamma = \lambda$ contributes to this sum. Indeed, if $\gamma \neq \lambda$, there exists i such that $\gamma_i > \lambda_i$, and from Eq. (29), we have

$$\left([t^{\gamma_i}] \exp\left(B_\infty(-t,\mathbf{p},-\alpha\lambda_i)\right)\right) \cdots \left([t^{\gamma_{\ell(\lambda)}}] \exp\left(B_\infty(-t,\mathbf{p},-\alpha\lambda_{\gamma_{\ell(\lambda)}})\right)\right) \cdot 1 = 0.$$

As a consequence, we obtain

$$J_\lambda^{(\alpha)} = \left([t^{\lambda_1}] \exp\left(B_\infty(-t,\mathbf{p},-\alpha\lambda_1)\right)\right) \cdots \left([t^{\lambda_{\ell(\lambda)}}] \exp\left(B_\infty(-t,\mathbf{p},-\alpha\lambda_{\ell(\lambda)})\right)\right) \cdot 1. \tag{54}$$

Comparing this expression for the partitions λ and μ we obtain the first part of the equation of the theorem. To obtain the second part of the equation we develop the exponential and we use the fact the operators $B_n(\mathbf{p}, -\alpha\lambda_1)$ commute for $n \geq 1$, see Proposition 4.6. ☐

9.3. **Positivity in Lassalle's conjecture.** We consider two sequences of variables $s_1, \dots, s_k$ and $r_1, \dots, r_k$.

We introduce the generating series $\widetilde{F}^{(k)}\left(t,\mathbf{p}, \begin{matrix} s_1 & \dots & s_k \\ r_1 & \dots & r_k \end{matrix}\right)$ inductively by $F^{(0)} = 1$ and for every $k \geq 0$

$$\widetilde{F}^{(k+1)}\left(t,\mathbf{p}, \begin{matrix} s_1 & \dots & s_{k+1} \\ r_1 & \dots & r_{k+1} \end{matrix}\right) = \exp\left(r_{k+1} B_\infty(-t,\mathbf{p},-\alpha s_{k+1})\right) \cdot \widetilde{F}^{(k)}\left(t,\mathbf{p}, \begin{matrix} s_1 & \dots & s_k \\ r_1 & \dots & r_k \end{matrix}\right).$$

Hence the functions $F^{(k)}$ defined in Eq. (4) are obtained as a specialization of $\widetilde{F}^{(k)}$:

$$F^{(k)}(t,\mathbf{p},s_1,\dots,s_k) = \widetilde{F}^{(k)}\left(t,\mathbf{p}, \begin{matrix} s_1 & \dots & s_k \\ 1 & \dots & 1 \end{matrix}\right).$$

Using the same argument used in Section 4.2 with the function $F^{(k)}$, we have

$$\widetilde{F}^{(k)}\left(t,\mathbf{p}, \begin{matrix} s_1 & \dots & s_k \\ r_1 & \dots & r_k \end{matrix}\right) = \sum_{M \in \mathcal{M}^{(k)}} \frac{(-t)^{|M|} p_{\nu_\diamond(M)} b^{\vartheta_\rho(M)}}{2^{|\nu_\bullet(M)| - cc(M)} \alpha^{cc(M)}} \prod_{1 \leq i \leq k} \frac{r_i^{|\nu_\bullet^{(i)}(M)|} (-\alpha s_i)^{|\nu_\circ^{(i)}(M)|}}{z_{\nu_\bullet^{(i)}(M)}}, \tag{55}$$

The two following properties follow from the definition of the functions $\widetilde{F}^{(k)}$:

(i) For every $1 \leq i \leq k-1$,

$$\widetilde{F}^{(k)}\left(t,\mathbf{p}, \begin{matrix} s_1 & \dots & s_k \\ r_1 & \dots & r_k \end{matrix}\right)_{|s_i = s_{i+1}} = \widetilde{F}^{(k-1)}\left(t,\mathbf{p}, \begin{matrix} s_1 & \dots & s_i & s_{i+2} & \dots & s_k \\ r_1 & \dots & r_i + r_{i+1} & r_{i+2} & \dots & r_k \end{matrix}\right).$$

(ii) For every $1 \leq i \leq k$,

$$\widetilde{F}^{(k)}\left(t,\mathbf{p}, \begin{matrix} s_1 & \dots & s_k \\ r_1 & \dots & r_k \end{matrix}\right)_{|r_i = 0} = \widetilde{F}^{(k-1)}\left(t,\mathbf{p}, \begin{matrix} s_1 & \dots & s_{i-1} & s_{i+1} & \dots & s_k \\ r_1 & \dots & r_{i-1} & r_{i+1} & \dots & r_k \end{matrix}\right)$$

We deduce the following proposition.

Proposition 9.3. *Let λ be a partition, and let* $\begin{pmatrix} s_1 & \dots & s_k \\ r_1 & \dots & r_k \end{pmatrix}$ *be multirectangular coordinates of λ. Then*

$$\widetilde{F}^{(k)}\left(t, \mathbf{p}, \begin{matrix} s_1 & \dots & s_k \\ r_1 & \dots & r_k \end{matrix}\right) = F^{(\ell(\lambda))}(t, \mathbf{p}, \lambda_1, \dots, \lambda_{\ell(\lambda)}).$$

In particular, the quantity $\widetilde{F}^{(k)}\left(t, \mathbf{p}, \begin{matrix} s_1 & \dots & s_k \\ r_1 & \dots & r_k \end{matrix}\right)$ *does not depend on the multirectangular coordinates of λ chosen.*

Proof. We start by removing all the pairs (s_i, r_i) for which $r_i = 0$, using property (ii) above. And then, we use property (i) in order to decrease the remaining coordinates r_i until they are all equal 1. More precisely, we apply attractively the following equation

$$\widetilde{F}^{(k)}\left(t, \mathbf{p}, \begin{matrix} s_1 & \dots & s_k \\ r_1 & \dots & r_k \end{matrix}\right) = \widetilde{F}^{(k+1)}\left(t, \mathbf{p}, \begin{matrix} s_1 & \dots & s_i & s_i & \dots & s_k \\ r_1 & \dots & r_i - 1 & 1 & \dots & r_k \end{matrix}\right).$$

Note that in each one of these operations the new coordinates obtained are also multirectangular coordinates for λ. Hence when $r_i = 1$ for every i, we have $k = \ell(\lambda)$ and $s_i = \lambda_i$ for each $1 \leq i \leq \ell(\lambda)$. ☐

As a consequence of Theorem 1.3 and Proposition 9.3, we obtain that for any partition μ

$$\tilde{\theta}^{(\alpha)}_\mu(\mathbf{s}, \mathbf{r}) = [t^{|\mu|} p_\mu] \widetilde{F}^{(k)}\left(t, \mathbf{p}, \begin{matrix} s_1 & \dots & s_k \\ r_1 & \dots & r_k \end{matrix}\right), \tag{56}$$

where $\mathbf{s} = (s_1 \dots, s_k, 0, \dots)$ and $\mathbf{r} = (r_1, \dots, r_k, 0, \dots)$. We now prove the positivity in Lassalle's conjecture.

Proof of positivity in Theorem 1.11. Comparing Eq. (55) and Eq. (56) and taking the limit on k we get

$$\tilde{\theta}^{(\alpha)}_\mu(\mathbf{s}, \mathbf{r}) = \sum_M \frac{b^{\vartheta_\rho(M)}}{2^{|\mathcal{V}_\bullet(M)| - cc(M)} \alpha^{cc(M)}} \prod_{i \geq 1} \frac{r_i^{|\mathcal{V}^{(i)}_\bullet(M)|} (-\alpha s_i)^{|\mathcal{V}^{(i)}_\circ(M)|}}{z_{\nu^{(i)}_\bullet(M)}}, \tag{57}$$

where the sum is taken over layered maps M of face type μ. Since every connected component of a bipartite map component contains at least one white vertex, the α-term which appears in the denominator is compensated. This concludes the proof of the theorem. ☐

9.4. **A connected version of Lassalle's conjecture.** We introduce a "connected" version $\hat{\theta}^{(\alpha)}_\mu$ of the characters $\theta^{(\alpha)}_\mu$ defined by the following expansion

$$(1 + b) \log \left(\sum_{\mu \in \mathbb{Y}} t^{|\mu|} \theta^{(\alpha)}_\mu(\mathbf{s}, \mathbf{r}) p_\mu \right) = \sum_{\mu \in \mathbb{Y}} t^{|\mu|} \hat{\theta}^{(\alpha)}_\mu(\mathbf{s}, \mathbf{r}) p_\mu.$$

Then we have the following theorem.

Theorem 9.4. *For any partition μ, the signed connected character $\hat{\theta}^{(\alpha)}_\mu$ is a polynomial in the variables $b, -\alpha s_1, -\alpha s_2, \dots, r_1, r_2 \dots$ with non-negative integers.*

Proof. Fix $k \geq 1$. Then

(58)
$$(1+b)\log\left(\widetilde{F}^{(k)}\left(t,\mathbf{p},\begin{array}{ccc} s_1 & \dots & s_k \\ r_1 & \dots & r_k\end{array}\right)\right) = \sum_M \frac{b^{\vartheta_\rho(M)}}{2^{|\mathcal{V}_\bullet(M)|-1}} \prod_{1\leq i\leq k} \frac{r_i^{|\mathcal{V}_\bullet^{(i)}(M)|}(-\alpha s_i)^{|\mathcal{V}_\circ^{(i)}(M)|}}{z_{\nu_\bullet^{(i)}(M)}},$$

where the sum runs over connected labelled k-layered maps. Indeed, this is a variant of the exponential formula for combinatorial labelled classes see *e.g* [FS09, Chapter II]. However, we use here a labelling for each pair (i,d) where $1 \leq i \leq k$ is layer and d is a vertex degree (see also [BD22a, Lemma 4.6]). We deduce then that for $\mathbf{r} = (r_1 \dots r_k)$ and $\mathbf{s} = (s_1, \dots, s_k)$ and for any partition μ, we have

$$\hat{\theta}_\mu^{(\alpha)}(\mathbf{r},\mathbf{s}) = \sum_M \frac{b^{\vartheta_\rho(M)}}{2^{|\mathcal{V}_\bullet(M)|-1}} \prod_{1\leq i\leq k} \frac{r_i^{|\mathcal{V}_\bullet^{(i)}(M)|}(-\alpha s_i)^{|\mathcal{V}_\circ^{(i)}(M)|}}{z_{\nu_\bullet^{(i)}(M)}},$$

where the sum is taken over connected labelled k-layered maps of face-type μ. We conclude by taking the limit in k. ☐

Remark 6. Note that the advantage of taking the logarithm is that, unlike Theorem 1.11, we keep here the parameter α in front of the variables s_i which reflects better the combinatorial formula of the Jack characters obtained in Eq. (57).

Supported by the this positivity result and by numerical data, we formulate the following conjecture.

Conjecture 1. *The normalized Jack connected characters* $(-1)^{|\mu|}z_\mu\hat{\theta}_\mu^{(\alpha)}$ *are polynomials in the variables* $b, -\alpha s_1, \dots, -\alpha s_k$, $r_1, \dots, r_k$, *with non-negative integer coefficients.*

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