Jack polynomials as generating series of non-oriented bipartite maps

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Enumerative Combinatorics Oberwolfach

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Jack polynomials

Jack polynomials $J_\lambda^{(\alpha)}$ is a family of symmetric functions:

- λ is an integer partition.
- α is a formal parameter.

For $\alpha = 1$ we obtain Schur functions.

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Main result

A combinatorial interpretation of the power-sum expansion of Jack polynomials in terms of bipartite maps (oriented or not) .

This generalizes a known interpretation of Schur functions in terms of oriented bipartite maps and gives an answer in some sense to a conjecture of Hanlon 1988.
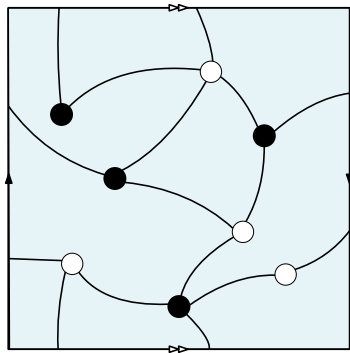
Maps

- A map is a graph embedded into a surface, oriented or not.
  A map is oriented if it is the case of the underlying surface.

- A map is bipartite if its vertices are colored in white and black,
  and each white vertex has only black neighbors.
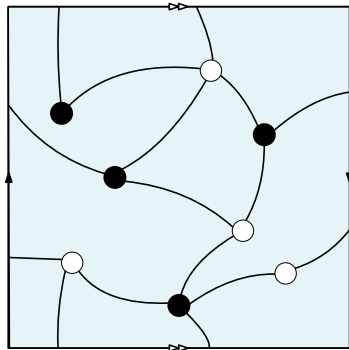
A non-oriented bipartite map on the Klein bottle.

Maps

- The face-type of a bipartite map M, denoted by $\diamond(M)$, is the is the partition given by the face degrees, divided by 2.
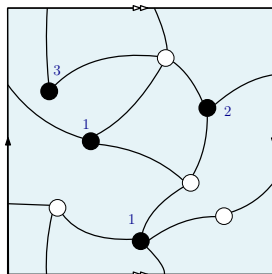

A non-oriented map of face-type $[4, 4, 2, 2]$.

Layered maps

Let k be a positive integer. A map M is k-layered if

- each black vertex has a label in $1, 2, \ldots, k$.

A 3-layered map on the Klein bottle

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- each white vertex is labelled by the maximal label among the labels of its white neighbors.

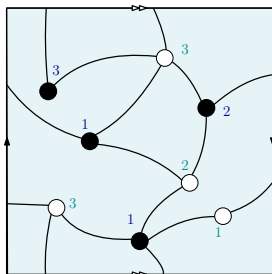

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Definition (Goulden–Jackson '96)

A statistic of non-orientability on k-layered maps is a statistic which associates to each k-layered map M a non-negative integer such that $\vartheta(M) = 0$ if and only if M is oriented.

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Maps will be counted with a weight $b^{\vartheta(M)}$, where $b := \alpha - 1$ is the shifted Jack parameter.

Jack polynomials in the power-sum basis

Theorem (B.D–Dołęga '22)

There exists an explicit statistic of non-orientability ϑ, such that

$$J_\lambda^{(\alpha)} = (-1)^{|\lambda|} \sum_{\substack{\ell(\lambda)-layered \\ maps\ M}} \frac{p_{\diamond(M)} b^{\vartheta(M)}}{2^{|\mathcal{V}_\bullet(M)|-cc(M)} \alpha^{cc(M)}} \prod_{1 \leq i \leq \ell(\lambda)} \frac{(-\alpha\lambda_i)^{|\mathcal{V}_\circ^{(i)}(M)|}}{z_{\nu_\bullet^{(i)}(M)}},$$

where

- $p_{\diamond(M)}$ *is the power-sum function associated to the partition $\diamond(M)$.*
- $|\mathcal{V}_\bullet(M)|$ *is the number of black vertices of M.*
- $|\mathcal{V}_\circ^{(i)}(M)|$ *is the number of white vertices of M labelled by i.*
- $cc(M)$ *is the number of connected components of M.*

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Well known for $\alpha = 1$ (Young symmetrizers) and for $\alpha = 2$ (Féray–Śniady's 2010).

Jack characters (a dual approach)

$$\theta_\mu^{(\alpha)}(\lambda) := \left\{ \begin{array}{ll} 0, & \text{if } |\lambda| < |\mu|. \\ \binom{|\lambda|-|\mu|+m_1(\mu)}{m_1(\mu)} [p_{\mu,1^{|\lambda|-|\mu|}}] J_\lambda^{(\alpha)}, & \text{if } |\lambda| \geq |\mu|. \end{array} \right.$$

where $m_1(\mu)$ is the number of parts of size 1 in μ.

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Theorem (B.D–Dołęga '22)

There exists a statistic of non-orientability ϑ on layered maps, such that

$$\theta_\mu^{(\alpha)}(\lambda) = (-1)^{|\mu|} \sum_{\substack{layered\ maps\ M \\ of\ face\text{-}type\ \mu}} \frac{b^{\vartheta(M)}}{2^{|\mathcal{V}_\bullet(M)|-cc(M)}\alpha^{cc(M)}} \prod_{i\geq 1} \frac{(-\alpha\lambda_i)^{|\mathcal{V}_\circ^{(i)}(M)|}}{z_{\nu_\bullet^{(i)}(M)}}, \quad (1)$$

- For $\alpha = 1$: Stanley-Féray formula 2010.
- For $\alpha = 2$: Féray–Śniady formula for zonal characters 2010.

Idea of the proof

$\theta_\mu^{(\alpha)}(\lambda_1, \lambda_2, \dots)$ is symmetric in the variables $\lambda_1 - 1/\alpha, \lambda_2 - 2/\alpha, \dots$
(such a function is called an α-shifted symmetric function).

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Theorem (Féray '19)

Fix a partition μ. The Jack character $\theta_\mu^{(\alpha)}$ is the unique α-shifted symmetric function of degree $|\mu|$ with top homogeneous part $\alpha^{|\mu|-\ell(\mu)}/z_\mu \cdot p_\mu$, such that $\theta_\mu^{(\alpha)}(\lambda) = 0$ for any partition $|\lambda| < |\mu|$.

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- We introduce the generating series of k-layered maps

$$F^{(k)}(t, \mathbf{p}, s_1, \dots, s_k) :=$$
$$\sum_{k-\text{layered maps } M} (-t)^{|M|} p_{\diamond(M)} \frac{b^{\vartheta(M)}}{2^{|\mathcal{V}_\bullet(M)|-cc(M)}\alpha^{cc(M)}} \prod_{1 \le i \le k} \frac{(-\alpha s_i)^{|\mathcal{V}_\circ^{(i)}(M)|}}{z_{\nu_\bullet^{(i)}(M)}}.$$

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We prove that this generating series satisfies the three conditions of the chracterization theorem.

Idea of the proof

- For a well-chosen statistic of non-orientability ϑ, this generating series can be constructed inductively using differential operators (Tutte decomposition):

$$F^{(k)}(t, \mathbf{p}, s_1, \ldots, s_k)$$
$$= \exp\left(\sum_{n \geq 1} \frac{(-t)^n}{n} B_n(\mathbf{p}, -\alpha s_1)\right) \cdot F^{(k-1)}(t, \mathbf{p}, s_2, \ldots, s_k),$$

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where

$$B_n(\mathbf{p}, -\alpha s_1) := \Theta_Y \left(\Gamma_Y - \alpha s_1 Y_+\right)^n \frac{y_0}{1+b}$$

is an operator which adds a black vertex of degree n with label 1.
$Y := (y_0, y_1, y_2, \ldots,)$ is a catalytic variable, and

$$\left.\begin{array}{l} \Theta_Y := \sum_{i \geq 1} p_i \frac{\partial}{\partial y_i}, \qquad Y_+ = \sum_{i \geq 0} y_{i+1} \frac{\partial}{\partial y_i}, \\ \Gamma_Y = (1+b) \cdot \sum_{i,j \geq 1} y_{i+j} \frac{i\partial^2}{\partial p_i \partial y_{j-1}} \\ \qquad + \sum_{i,j \geq 1} y_i p_j \frac{\partial}{\partial y_{i+j-1}} + b \cdot \sum_{i \geq 1} y_{i+1} \frac{i\partial}{\partial y_i}. \end{array}\right\} \text{Chapuy–Dołęga operators.}$$

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A key step of the proof: Two commutation relations

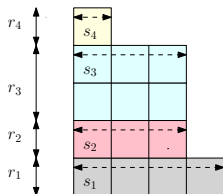$$[B_n(\mathbf{p}, u), B_m(\mathbf{p}, u)] = 0, \text{ for } n, m \geq 1,$$

$$\left[\sum_{n \geq 1} \frac{t^n}{n} B_n^>(\mathbf{p}, u), \sum_{n \geq 1} \frac{t^n}{n} B_n^>(\mathbf{p}, v)\right] = 0,$$

where

$$B_n^>(\mathbf{p}, u) := B_n(\mathbf{p}, u) - B_n(\mathbf{p}, 0).$$

Application: Lassalle's conjecture 2008
Stanley's coordinates of a Young diagram

The Young diagram of the partition $[4, 3, 3, 3, 1]$, with $\mathbf{s} = (4, 3, 3, 1)$ and $\mathbf{r} = (1, 1, 2, 1)$ as Stanley coordinates.

Theorem (B.D–Dołęga '22: Lassalle's conjecture on Jack characters)

The normalized Jack character $(-1)^{|\mu|} z_\mu \theta_\mu^{(\alpha)}$ is a polynomial in Stanley's coordinates $b, r_1, r_2 \ldots, -s_1, -s_2 \ldots$, with non-negative integer coefficients.

Proof:

- Non-negativity: the combinatorial interpretation in terms of maps.
- Integrality: (a different approach) the integrable system of Nazarov–Sklyanin.