Goulden and Jackson's b-conjecture and Matching-Jack conjecture

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Maps

Maps

- A map is a graph embedded into a surface, oriented or not. A map is oriented if it is the case of the underlying surface.
- A map is bipartite if its vertices are colored in white and black, and each white vertex has only black neighbors.

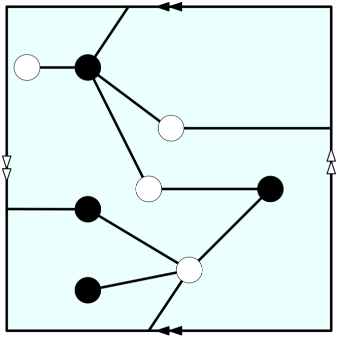

Figure 1: A non-oriented bipartite map on the Klein bottle.

Maps

- A bipartite map is rooted by distinguishing an oriented white corner.
- **Example:**

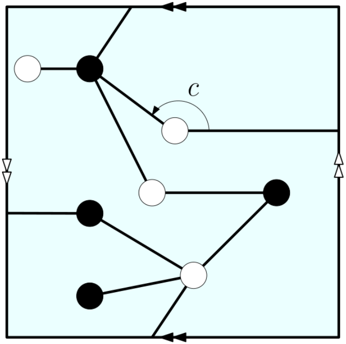

Figure 1: A rooted non-oriented bipartite map on the Klein bottle.

Maps

- (λ, μ, ν) is the profile of the bipartite map *M* if λ is the partition given by the face degrees divided by 2, and μ (resp. ν) is the partition given by the degrees of the white (resp. black) vertices.

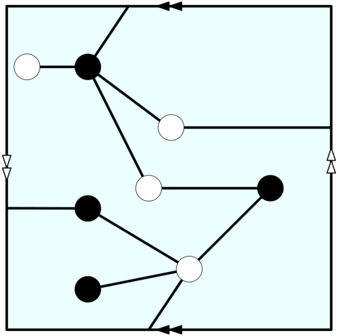

Figure 1: A non-oriented bipartite map on the Klein bottle with profile $([9], [4, 2, 2, 1], [4, 2, 2, 1])$.

Generating series of oriented bipartite maps

Oriented bipartite maps

1. For every triplet (λ, μ, ν), we have the bijection

Oriented (edge-) labelled bipartite maps of profile (λ, μ, ν)	$\longleftrightarrow$	couples of permutations (σ_1, σ_2) such that the cyclic type of σ_1, σ_2 and $\sigma_1\sigma_2$ are respectively λ, μ and ν

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2. [Representation theory of the symmetric group]

$$\sum_{\theta} t^{|\theta|} \frac{|\theta|!}{\dim(\theta)} s_\theta(\mathbf{p}) s_\theta(\mathbf{q}) s_\theta(\mathbf{r}) = \sum_{n \geq 0} \frac{t^n}{n!} \sum_{\sigma_1, \sigma_2 \in \mathfrak{S}_n} p_{type(\sigma_1)} q_{type(\sigma_2)} r_{type(\sigma_1\sigma_2)},$$

s_θ : the Schur function associated to the partition θ, expressed in the power-sum basis.

$\mathbf{p} := (p_i)_{i\geq 1}$; $\mathbf{q} := (q_i)_{i \geq 1}$; ; $\mathbf{r} := (r_i)_{i \geq 1}$.

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- [Classical]

$$\frac{t\partial}{\partial t} \log\left(\sum_{\theta} t^{|\theta|} \frac{|\theta|!}{\dim(\theta)} s_\theta(\mathbf{p}) s_\theta(\mathbf{q}) s_\theta(\mathbf{r})\right) = \sum_{\substack{M \text{ connected rooted} \\ \text{oriented bipartite maps}}} t^{|M|} p_{\Lambda^\circ(M)} q_{\Lambda^\bullet(M)} r_{\Lambda^\diamond(M)}.$$

Generating series of non-oriented maps

Labelled Maps

- A map is labelled if it is equipped with a bijection between its edge-sides and the set $\mathcal{A}_n := \{1, \hat{1}, ..., n, \hat{n}\}$.
- **Example:**

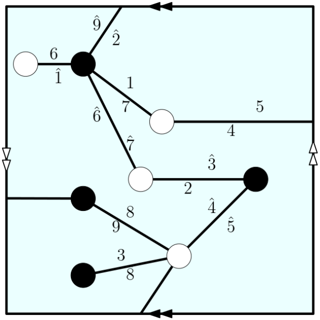


Figure 2: A labelled non-oriented bipartite map on the Klein bottle

Matchings

- A matching δ on $\mathcal{A}_n = \{1, \hat{1}, ..., n, \hat{n}\}$ is a 1-regular graph.

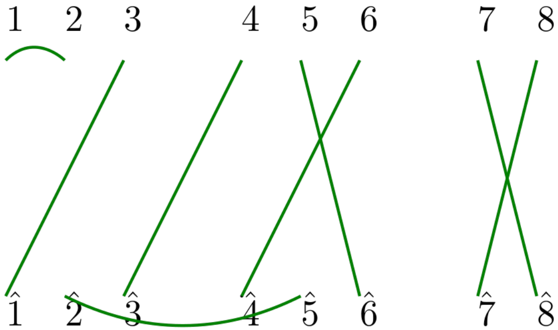


Figure 3: An example of a matching on $\mathcal{A}_8$.

Matchings

- A matching is bipartite if each one of its edges is of the form $(i, \hat{j})$.

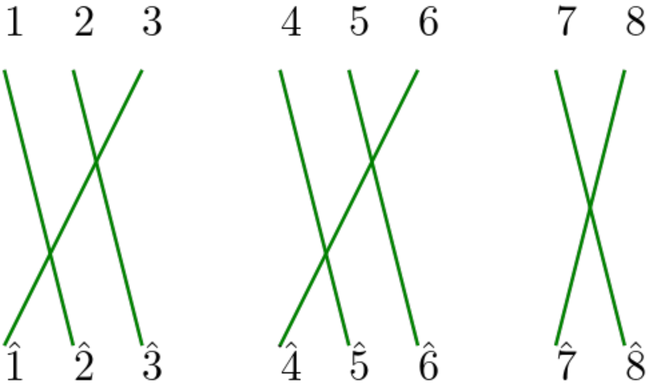


Figure 3: An example of a bipartite matching on $\mathcal{A}_8$.

Matchings

- For every $n \geq 1$, we denote by ε the bipartite matching on $\mathcal{A}_n$ formed by the pairs of the form $(i, \hat{i})$.

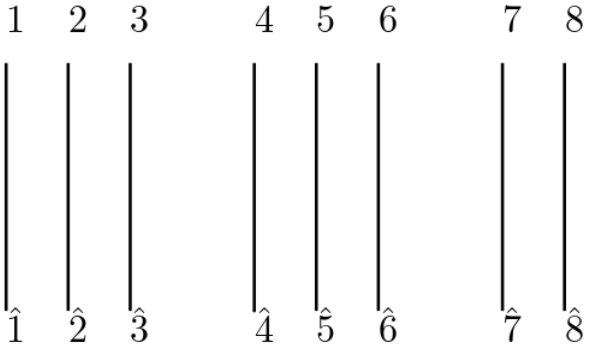


Figure 3: The matching ε on $\mathcal{A}_8$.

Matchings

- For two matchings δ and δ' on $\mathcal{A}_n$, we define $\Lambda(\delta, \delta')$ as the partition given by half-sizes of the connected components of the graph $\delta \cup \delta'$.
- Once and for all, we fix for every partition λ a bipartite matching δ_λ such that $\Lambda(\varepsilon, \delta_\lambda) = \lambda$.

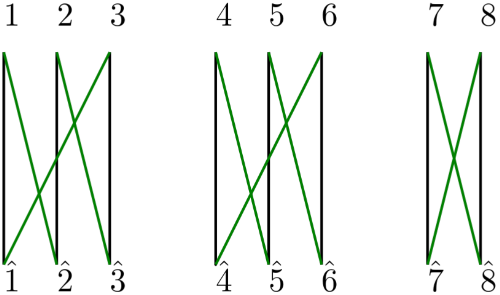
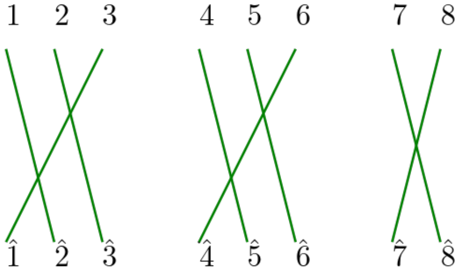


Figure 3: An example of the graph of $\varepsilon \cup \delta_\lambda$ for $\lambda = [3, 3, 2]$

Matchings

- We have a bijection between $\mathfrak{S}_n$ and bipartite matchings on $\mathcal{A}_n$:
  $\sigma \longmapsto$ the matching formed by $(i, \sigma(\hat{j}))$.
- **Example:**

$(1,2,3)(4,5,6)(7,8) \longmapsto$



Remark

A permutation of cycle type λ is associated to a matching δ such that $\Lambda(\varepsilon, \delta) = \lambda$.

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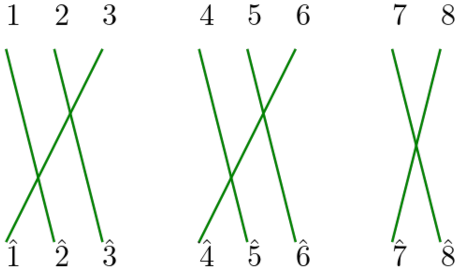


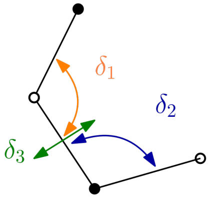
Remark

A permutation of cycle type λ is associated to a matching δ such that $\Lambda(\varepsilon, \delta) = \lambda$.

- The profile of $(\delta_1, \delta_2, \delta_3)$ is the triplet of partitions $(\Lambda(\delta_1, \delta_2), \Lambda(\delta_1, \delta_3), \Lambda(\delta_2, \Lambda_3))$.

Correspondence between bipartite maps and matchings

For a labelled bipartite map M we define three matchings;



δ_1 relating the labels of edge-sides forming a white corner.
δ_2 relating the labels of edge-sides forming a black corner.
δ_3 relating the labels of the two sides of a same edge.

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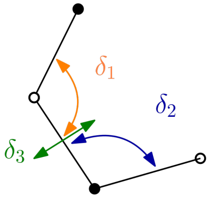

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δ_2 relating the labels of edge-sides forming a black corner.
δ_3 relating the labels of the two sides of a same edge.

$\Lambda(\delta_1, \delta_2)$ gives the face degrees.
$\Lambda(\delta_1, \delta_3)$ gives the white vertices degrees.
$\Lambda(\delta_2, \delta_3)$ gives the black vertices degrees.

Generating series of non-oriented maps [Goulden and Jackson '96]

1. We obtain the following bijection :

Labelled bipartite maps of profile (λ, μ, ν)	$\longleftrightarrow$	$(\delta_1, \delta_2, \delta_3)$ of profile (λ, μ, ν)

2. [Representation Theory of the Gelfand pair $(\mathfrak{S}_{2n}, \mathfrak{B}_n)$]

$$\sum_{\theta} t^{|\theta|} \frac{\dim(2\theta)}{|2\theta|!} Z_\theta(\mathbf{p}) Z_\theta(\mathbf{q}) Z_\theta(\mathbf{r}) = \sum_{n \geq 0} \frac{t^n}{(2n)!} \sum_{\substack{\delta_0, \delta_1, \delta_2 \\ \text{matchings on } \mathcal{A}_n}} p_{\Lambda(\delta_0, \delta_1)} q_{\Lambda(\delta_1, \delta_2)} r_{\Lambda(\delta_1, \delta_2)}$$

Z_θ : the zonal polynomial associated to the partition θ, expressed in the power-sum basis.

$\mathbf{p} := (p_i)_{i \geq 1}$; $\mathbf{q} := (q_i)_{i \geq 1}$; $\mathbf{r} := (r_i)_{i \geq 1}$.

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3.

$$2 \frac{t\partial}{\partial t} \log \left(\sum_{\theta} t^{|\theta|} \frac{\dim(2\theta)}{|2\theta|!} Z_\theta(\mathbf{p}) Z_\theta(\mathbf{q}) Z_\theta(\mathbf{r}) \right) = \sum_{\substack{M \text{ connected rooted} \\ \text{bipartite maps}}} t^{|M|} p_{\Lambda^\circ(M)} q_{\Lambda^\bullet(M)} r_{\Lambda^\diamond(M)}$$

Jack polynomials and a one parameter deformation of the generating series of bipartite maps

Jack polynomials

We consider the following deformation of the Hall scalar product $\langle ., . \rangle_b$ defined on symmetric functions by

$$\langle p_\lambda, p_\mu \rangle_b = \delta_{\lambda\mu} z_\lambda (1+b)^{\ell(\lambda)}.$$

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Definition

Jack polynomials of parameter $1+b$, denoted $J_\lambda^{(b)}$ are defined as follows :

1. Triangularity and normalisation: if $\lambda \vdash n$, then

$$J_\lambda^{(b)} = \sum_{\mu \vdash n, \mu \leq \lambda} u_{\lambda\mu} m_\mu,$$

   such that $u_{\lambda[1^n]} = n!$.
   (predominance order $\mu \leq \lambda : \mu_1 + \mu_2 + ... + \mu_i \leq \lambda_1 + \lambda_2 ... + \lambda_i \; \forall i$)

2. Orthogonality: if $\lambda \neq \mu$ then $\langle J_\lambda^{(b)}, J_\mu^{(b)} \rangle_b = 0$.

Jack polynomials

- For $b = 0 \longrightarrow$ Schur functions $J_\lambda^{(0)} = \frac{|\lambda|!}{\dim(\lambda)} s_\lambda$.
- For $b = 1 \longrightarrow$ Zonal polynomials $J_\lambda^{(1)} = Z_\lambda$.

We define

$$\tau_b(t, \mathbf{p}, \mathbf{q}, \mathbf{r}) := \sum_\theta \frac{t^{|\theta|}}{j_\theta^{(b)}} J_\theta^{(b)}(\mathbf{p}) J_\theta^{(b)}(\mathbf{q}) J_\theta^{(b)}(\mathbf{r}),$$

where $j_\theta^{(b)} = \langle J_\theta^{(b)}, J_\theta^{(b)} \rangle_b$.

b=0

$$\tau_0(t,\mathbf{p},\mathbf{q},\mathbf{r}) = \sum_{n\geq 0}\sum_{\lambda\vdash n}\frac{t^n}{z_\lambda}\sum_{\substack{\delta \text{ bipartite} \\ \text{matching on } \mathcal{A}_n}} p_\lambda q_{\Lambda(\varepsilon,\delta)} r_{\Lambda(\delta_\lambda,\delta)}.$$

$$\frac{t\partial}{\partial t}\log\left(\tau_0(t,\mathbf{p},\mathbf{q},\mathbf{r})\right) = \sum_{\substack{M \text{ oriented rooted} \\ \text{connected bipartite map}}} t^{|M|} p_{\Lambda^\circ(M)} q_{\Lambda^\bullet(M)} r_{\Lambda^\diamond(M)}.$$

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b=1

$$\tau_1(t,\mathbf{p},\mathbf{q},\mathbf{r}) = \sum_{n\geq 0}\sum_{\lambda\vdash n}\frac{t^n}{z_\lambda 2^{\ell(\lambda)}}\sum_{\delta \text{ matching on } \mathcal{A}_n} p_\lambda q_{\Lambda(\varepsilon,\delta)} r_{\Lambda(\delta_\lambda,\delta)}.$$

$$2\frac{t\partial}{\partial t}\log\left(\tau_1(t,\mathbf{p},\mathbf{q},\mathbf{r})\right) = \sum_{\substack{M \text{ rooted} \\ \text{connected bipartite map}}} t^{|M|} p_{\Lambda^\circ(M)} q_{\Lambda^\bullet(M)} r_{\Lambda^\diamond(M)}.$$

Goulden and Jackson's conjectures '96

Matching-Jack conjecture

$$\tau_b(t,\mathbf{p},\mathbf{q},\mathbf{r}) = \sum_{n\geq 0}\sum_{\lambda\vdash n}\frac{t^n}{z_\lambda(1+b)^{\ell(\lambda)}}\sum_{\delta \text{ matching on } \mathcal{A}_n} b^{\vartheta_\lambda(\delta)} p_\lambda q_{\Lambda(\varepsilon,\delta)} r_{\Lambda(\delta_\lambda,\delta)},$$

where for every partition $\lambda\vdash n$, ϑ_λ a function on the matchings of $\mathcal{A}_n$ with non-negative integer values, such that $\vartheta_\lambda(\delta)=0$ iff δ is a bipartite matching.

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b-conjecture (Hypermap-Jack conjecture)

$$(1+b)\frac{t\partial}{\partial t}\log\left(\tau_b(t,\mathbf{p},\mathbf{q},\mathbf{r})\right) = \sum_{\substack{M \text{ rooted connected}\\ \text{bipartite map}}} t^{|M|} b^{\vartheta(M)} p_{\Lambda^\circ(M)} q_{\Lambda^\bullet(M)} r_{\Lambda^\diamond(M)}$$

where ϑ is a function on connected rooted maps with non-negative integer value, such that $\vartheta(M)=0$ iff M is oriented.

Some partial results

Theorem (Dołęga-Féray '15)

The coefficient of $p_\lambda q_\mu r_\nu$ *in the function* $\tau_b(t, \mathbf{p}, \mathbf{q}, \mathbf{r})$ *multiplied by* $z_\lambda(1+b)^{\ell(\lambda)}$ *is a polynomial in b with rational coefficients.*

Theorem (Dołęga-Féray '17)

The coefficient of $p_\lambda q_\mu r_\nu$ *in the function* $(1+b)\frac{t\partial}{\partial t}\log\left(\tau_b(t, \mathbf{p}, \mathbf{q}, \mathbf{r})\right)$ *is a polynomial in b with rational coefficients.*

Some partial results

Theorem (Chapuy-Dołęga '20)

$$(1+b)\frac{t\partial}{\partial t}\log\left(\tau_b(t,\mathbf{p},\mathbf{q},\underline{u})\right) = \sum_{\substack{M \text{ rooted connected} \\ \text{bipartite map}}} t^{|M|} b^{\vartheta(M)} p_{\Lambda^\diamond} q_{\Lambda^\circ(M)} u^{\ell(\Lambda^\bullet(M))}$$

where ϑ is a function on connected rooted maps with non-negative integer value, such that $\vartheta(M)=0$ iff M is oriented.

$\mathbf{p} := (p_1, p_2, p_3, \ldots)$,
$\mathbf{q} := (q_1, q_2, q_3, \ldots)$,
$\underline{u} := (u, u, u\ldots)$.

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Theorem (B.D. '21, arXiv:2106.15414)

$$\tau_b(t,\mathbf{p},\mathbf{q},\underline{u}) = \sum_{n\geq 0}\sum_{\lambda\vdash n}\frac{t^n}{z_\lambda(1+b)^{\ell(\lambda)}}\sum_{\delta \text{ matching on } \mathcal{A}_n} b^{\vartheta_\lambda(\delta)}p_\lambda q_{\Lambda(\varepsilon,\delta)}u^{\ell(\Lambda(\delta_\lambda,\delta))},$$

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Remark

All the precedent results can be generalized to the case of k-constellations.

Recall:

Theorem (Dołęga-Féray '15)

The coefficient of $p_\lambda q_\mu r_\nu$ *in the function* $\tau_b(t, \mathbf{p}, \mathbf{q}, \mathbf{r})$ *multiplied by* $z_\lambda(1+b)^{\ell(\lambda)}$ *is a polynomial in b with rational coefficients.*

Recall:

Theorem (Dołęga-Féray '15)

The coefficient of $p_\lambda q_\mu r_\nu$ in the function $\tau_b(t, \mathbf{p}, \mathbf{q}, \mathbf{r})$ multiplied by $z_\lambda(1+b)^{\ell(\lambda)}$ is a polynomial in b with rational coefficients.

Upcoming result (joint work with Chapuy and Dołęga):

Theorem

The coefficient of $p_\lambda q_\mu r_\nu$ in the function $\tau_b(t, \mathbf{p}, \mathbf{q}, \mathbf{r})$ multiplied by $z_\lambda(1+b)^{\ell(\lambda)}$ is a polynomial in b with integer coefficients.

Proof: The integrality of the coefficients $\tau_b(t, \mathbf{p}, \mathbf{q}, \underline{u})$ + Farahat-Higman Algebra.