Goulden and Jackson’s b-conjecture and Matching-Jack conjecture

Houcine Ben Dali

Université de Paris, CNRS, IRIF, Paris
Université de Lorraine, CNRS, IECL, Nancy

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Maps
Maps

- A map is a graph embedded into a surface, oriented or not. A map is oriented if it is the case of the underlying surface.
- A map is bipartite if its vertices are colored in white and black, and each white vertex has only black neighbors.

Figure 1: A non-oriented bipartite map on the Klein bottle.
Maps

- A bipartite map is rooted by distinguishing an oriented white corner.
- Example:

![Diagram of a rooted non-oriented bipartite map on the Klein bottle.]

Figure 1: A rooted non-oriented bipartite map on the Klein bottle.
Maps

$(\lambda, \mu, \nu)$ is the profile of the bipartite map $M$ if $\lambda$ is the partition given by the face degrees divided by 2, and $\mu$ (resp. $\nu$) is the partition given by the degrees of the white (resp. black) vertices.

Figure 1: A non-oriented bipartite map on the Klein bottle with profile $([9], [4, 2, 2, 1], [4, 2, 2, 1])$. 

Houcine Ben Dali

Goulden and Jackson’s conjectures

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Generating series of oriented bipartite maps
Oriented bipartite maps

1. For every triplet \((\lambda, \mu, \nu)\), we have the bijection
   Oriented (edge-) labelled bipartite maps of profile \((\lambda, \mu, \nu)\) \(\leftrightarrow\) couples of permutations \((\sigma_1, \sigma_2)\) such that the cyclic type of \(\sigma_1, \sigma_2\) and \(\sigma_1\sigma_2\) are respectively \(\lambda, \mu\) and \(\nu\)
Oriented bipartite maps

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\[\longleftrightarrow\]

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type of \(\sigma_1, \sigma_2\) and \(\sigma_1 \sigma_2\) are
respectively \(\lambda, \mu\) and \(\nu\).

2 [Representation theory of the symmetric group]

\[
\sum_{\theta} t^{\mid \theta \mid} \frac{\mid \theta \mid !}{\dim(\theta)} \mathbb{C}_{\theta}(p) \mathbb{C}_{\theta}(q) \mathbb{C}_{\theta}(r) = \sum_{n \geq 0} \frac{t^n}{n!} \sum_{\sigma_1, \sigma_2 \in S_n} \rho_{\text{type}(\sigma_1)} q_{\text{type}(\sigma_2)} r_{\text{type}(\sigma_1 \sigma_2)},
\]

\(s_{\theta} : \) the Schur function associated to the partition \(\theta\), expressed in the
power-sum basis.

\(p := (p_i)_{i \geq 1}; \ q := (q_i)_{i \geq 1}; \ r := (r_i)_{i \geq 1}.\)
Oriented bipartite maps

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Oriented (edge-) labelled bipartite maps of profile \((\lambda, \mu, \nu)\) ↔ couples of permutations \((\sigma_1, \sigma_2)\) such that the cyclic type of \(\sigma_1, \sigma_2\) and \(\sigma_1 \sigma_2\) are respectively \(\lambda, \mu\) and \(\nu\).

2 [Representation theory of the symmetric group]

\[
\sum_{\theta} t^{\mid \theta \mid} \frac{\mid \theta \mid!}{\dim(\theta)} s_{\theta}(p)s_{\theta}(q)s_{\theta}(r) = \sum_{n \geq 0} \frac{t^n}{n!} \sum_{\sigma_1, \sigma_2 \in S_n} p_{\text{type}(\sigma_1)} q_{\text{type}(\sigma_2)} r_{\text{type}(\sigma_1 \sigma_2)},
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\(p := (p_i)_{i \geq 1}; q := (q_i)_{i \geq 1}; r := (r_i)_{i \geq 1}\).

[Classical]

\[
\frac{t \partial}{\partial t} \log \left( \sum_{\theta} t^{\mid \theta \mid} \frac{\mid \theta \mid!}{\dim(\theta)} s_{\theta}(p)s_{\theta}(q)s_{\theta}(r) \right) = \sum_{M \text{ connected rooted oriented bipartite maps}} t^{\mid M \mid} p_{\Lambda^{\circ}(M)} q_{\Lambda^{\bullet}(M)} r_{\Lambda^{\circ}(M)}. \]
Generating series of non-oriented maps
Labelled Maps

- A map is labelled if it is equipped with a bijection between its edge-sides and the set $\mathcal{A}_n := \{1, \hat{1}, \ldots, n, \hat{n}\}$.

- Example:

![Figure 2: A labelled non-oriented bipartite map on the Klein bottle](image)

**Figure 2:** A labelled non-oriented bipartite map on the Klein bottle
A matching $\delta$ on $\mathcal{A}_n = \{1, \hat{1}, \ldots, n, \hat{n}\}$ is a 1-regular graph.

Figure 3: An example of a matching on $\mathcal{A}_8$. 
Matchings

- A matching is bipartite if each one of its edges is of the form \((i, \hat{j})\).

Figure 3: An example of a bipartite matching on \(A_8\).
Matchings

- For every $n \geq 1$, we denote by $\varepsilon$ the bipartite matching on $A_n$ formed by the pairs of the form $(i, \hat{i})$.

Figure 3: The matching $\varepsilon$ on $A_8$. 
Matchings

- For two matchings $\delta$ and $\delta'$ on $A_n$, we define $\Lambda(\delta, \delta')$ as the partition given by half-sizes of the connected components of the graph $\delta \cup \delta'$.
- Once and for all, we fix for every partition $\lambda$ a bipartite matching $\delta_\lambda$ such that $\Lambda(\epsilon, \delta_\lambda) = \lambda$.

Figure 3: An example of the graph of $\epsilon \cup \delta_\lambda$ for $\lambda = [3, 3, 2]$. 
Matchings

- We have a bijection between $\mathfrak{S}_n$ and bipartite matchings on $A_n$:
  $$\sigma \mapsto \text{the matching formed by } (i, \sigma(\hat{j})).$$

- **Example:**
  $$(1, 2, 3)(4, 5, 6)(7, 8) \mapsto$$

![Diagram of matching]

**Remark**

A permutation of cycle type $\lambda$ is associated to a matching $\delta$ such that $\Lambda(\varepsilon, \delta) = \lambda$. 
Matchings

- We have a bijection between $S_n$ and bipartite matchings on $A_n$: 
  $\sigma \mapsto$ the matching formed by $(i, \hat{\sigma}(j))$.

- **Example:**

  \[
  \begin{array}{cccccccc}
    1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
    \hat{1} & \hat{2} & \hat{3} & \hat{4} & \hat{5} & \hat{6} & \hat{7} & \hat{8} \\
  \end{array}
  \]

  \[
  (1, 2, 3)(4, 5, 6)(7, 8) \mapsto
  \]

**Remark**

A permutation of cycle type $\lambda$ is associated to a matching $\delta$ such that $\Lambda(\varepsilon, \delta) = \lambda$.

- The profile of $(\delta_1, \delta_2, \delta_3)$ is the triplet of partitions $(\Lambda(\delta_1, \delta_2), \Lambda(\delta_1, \delta_3), \Lambda(\delta_2, \Lambda_3))$. 
Correspondence between bipartite maps and matchings

For a labelled bipartite map $M$ we define three matchings;

- $\delta_1$ relating the labels of edge-sides forming a white corner.
- $\delta_2$ relating the labels of edge-sides forming a black corner.
- $\delta_3$ relating the labels of the two sides of a same edge.
Correspondence between bipartite maps and matchings

For a labelled bipartite map $M$ we define three matchings;

- $\delta_1$ relating the labels of edge-sides forming a white corner.
- $\delta_2$ relating the labels of edge-sides forming a black corner.
- $\delta_3$ relating the labels of the two sides of a same edge.

$\Lambda(\delta_1, \delta_2)$ gives the face degrees.
$\Lambda(\delta_1, \delta_3)$ gives the white vertices degrees.
$\Lambda(\delta_2, \delta_3)$ gives the black vertices degrees.
Generating series of non-oriented maps

[Goulden and Jackson '96]

1 We obtain the following bijection:
Labelled bipartite maps of profile \((\lambda, \mu, \nu)\)
\(\longleftrightarrow\) \((\delta_1, \delta_2, \delta_3)\) of profile \((\lambda, \mu, \nu)\)

2 [Representation Theory of the Gelfand pair \((\mathfrak{S}_{2n}, \mathfrak{B}_n)\)]

\[
\sum_{\theta} t^{\theta} \frac{\dim(2\theta)}{|2\theta|!} Z_\theta(p)Z_\theta(q)Z_\theta(r) = \sum_{n \geq 0} \frac{t^n}{(2n)!} \sum_{\delta_0, \delta_1, \delta_2 \text{ matchings on } \mathcal{A}_n} p_\Lambda(\delta_0, \delta_1) q_\Lambda(\delta_1, \delta_2) r_\Lambda(\delta_1, \delta_2)
\]

\(Z_\theta\) : the zonal polynomial associated to the partition \(\theta\), expressed in the power-sum basis.
\(p := (p_i)_{i \geq 1}; q := (q_i)_{i \geq 1}; r := (r_i)_{i \geq 1}.\)
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\sum_{\theta} t^{|\theta|} \frac{\dim(2\theta)}{|2\theta|!} Z_{\theta}(p)Z_{\theta}(q)Z_{\theta}(r) = \sum_{n\geq 0} \frac{t^n}{(2n)!} \sum_{\delta_0,\delta_1,\delta_2 \text{ matchings on } \mathfrak{A}_n} p_{\Lambda(\delta_0,\delta_1)} q_{\Lambda(\delta_1,\delta_2)} r_{\Lambda(\delta_1,\delta_2)}
\]

\(Z_{\theta}\): the zonal polynomial associated to the partition \(\theta\), expressed in the power-sum basis.
\(p := (p_i)_{i \geq 1}; \; q := (q_i)_{i \geq 1}; \; r := (r_i)_{i \geq 1}\).

3

\[
2 \frac{t \partial}{\partial t} \log \left( \sum_{\theta} t^{|\theta|} \frac{\dim(2\theta)}{|2\theta|!} Z_{\theta}(p)Z_{\theta}(q)Z_{\theta}(r) \right) = \sum_{M \text{ connected rooted bipartite maps}} t^{|M|} p_{\Lambda^\circ(M)} q_{\Lambda^\bullet(M)} r_{\Lambda^\circ(M)}
\]
Jack polynomials and a one parameter deformation of the generating series of bipartite maps
Jack polynomials
We consider the following deformation of the Hall scalar product $\langle \cdot, \cdot \rangle_b$ defined on symmetric functions by

$$\langle p_\lambda, p_\mu \rangle_b = \delta_{\lambda\mu} z_\lambda (1 + b)^{\ell(\lambda)}.$$
Jack polynomials
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$$\langle p_\lambda, p_\mu \rangle_b = \delta_{\lambda\mu} z_\lambda (1 + b)^{\ell(\lambda)}.$$ 

**Definition**

Jack polynomials of parameter $1 + b$, denoted $J^{(b)}_\lambda$ are defined as follows:

1. **Triangularity and normalisation**: if $\lambda \vdash n$, then

   $$J^{(b)}_\lambda = \sum_{\mu \vdash n, \mu \leq \lambda} u_{\lambda\mu} m_\mu,$$

   such that $u_{\lambda[1^n]} = n!$.

   (predominance order $\mu \leq \lambda : \mu_1 + \mu_2 + \ldots + \mu_i \leq \lambda_1 + \lambda_2 + \ldots + \lambda_i \ \forall i$)

2. **Orthogonality**: if $\lambda \neq \mu$ then $\langle J^{(b)}_\lambda, J^{(b)}_\mu \rangle_b = 0$. 
Jack polynomials

- For $b = 0 \rightarrow$ Schur functions $J^{(0)}_{\lambda} = \frac{|\lambda|!}{\text{dim}(\lambda)} s_{\lambda}$.
- For $b = 1 \rightarrow$ Zonal polynomials $J^{(1)}_{\lambda} = Z_{\lambda}$.

We define

$$\tau_b(t, p, q, r) := \sum_{\theta} \frac{t^{|\theta|}}{j^{(b)}_{\theta}} J^{(b)}_{\theta}(p) J^{(b)}_{\theta}(q) J^{(b)}_{\theta}(r),$$

where $j^{(b)}_{\theta} = \langle J^{(b)}_{\theta}, J^{(b)}_{\theta} \rangle_b$. 
\[ \tau_0(t, p, q, r) = \sum_{n \geq 0} \sum_{\lambda \vdash n} \frac{t^n}{z^\lambda} \sum_{\delta \text{ bipartite matching on } A_n} p_\lambda q_{\Lambda(\varepsilon, \delta)} r_{\Lambda(\delta_\lambda, \delta)} . \]

\[ \frac{t \partial}{\partial t} \log (\tau_0(t, p, q, r)) = \sum_{M \text{ oriented rooted connected bipartite map}} t^{|M|} p_{\Lambda\circ(M)} q_{\Lambda\bullet(M)} r_{\Lambda\diamond(M)}. \]
For $b=0$,

\[
\tau_0(t, p, q, r) = \sum_{n \geq 0} \sum_{\lambda \vdash n} \frac{t^n}{z_\lambda} \sum_{\delta \text{ bipartite matching on } A_n} p_\lambda q_{\Lambda(\varepsilon, \delta)} r_{\Lambda(\delta_\lambda, \delta)}.
\]

\[
\frac{t \partial}{\partial t} \log (\tau_0(t, p, q, r)) = \sum_{M \text{ oriented rooted connected bipartite map}} t^{|M|} p_{\Lambda^\circ (M)} q_{\Lambda^\bullet (M)} r_{\Lambda^\diamond (M)}.
\]

For $b=1$,

\[
\tau_1(t, p, q, r) = \sum_{n \geq 0} \sum_{\lambda \vdash n} \frac{t^n}{z_\lambda 2^{\ell(\lambda)}} \sum_{\delta \text{ matching on } A_n} p_\lambda q_{\Lambda(\varepsilon, \delta)} r_{\Lambda(\delta_\lambda, \delta)}.
\]

\[
2 \frac{t \partial}{\partial t} \log (\tau_1(t, p, q, r)) = \sum_{M \text{ rooted connected bipartite map}} t^{|M|} p_{\Lambda^\circ (M)} q_{\Lambda^\bullet (M)} r_{\Lambda^\diamond (M)}.
\]
Goulden and Jackson’s conjectures ’96

Matching-Jack conjecture

\[
\tau_b(t, p, q, r) = \sum_{n\geq 0} \sum_{\lambda \vdash n} \frac{t^n}{z_\lambda (1 + b)^{\ell(\lambda)}} \sum_{\delta \text{ matching on } A_n} b^{\vartheta_\lambda(\delta)} p_\lambda q_{\Lambda(\epsilon, \delta)}^\Lambda r_{\Lambda(\delta, \lambda, \delta)},
\]

where for every partition \( \lambda \vdash n \), \( \vartheta_\lambda \) a function on the matchings of \( A_n \) with non-negative integer values, such that \( \vartheta_\lambda(\delta) = 0 \) iff \( \delta \) is a bipartite matching.
Goulden and Jackson’s conjectures ’96

Matching-Jack conjecture

\[
\tau_b(t, p, q, r) = \sum_{n \geq 0} \sum_{\lambda \vdash n} \frac{t^n}{z_{\lambda}(1 + b)^{\ell(\lambda)}} \sum_{\delta \text{ matching on } A_n} b^{\vartheta_{\lambda}(\delta)} p_{\lambda} q_{\Lambda(\varepsilon, \delta)} r_{\Lambda(\delta, \lambda)}
\]

where for every partition \( \lambda \vdash n \), \( \vartheta_{\lambda} \) a function on the matchings of \( A_n \) with non-negative integer values, such that \( \vartheta_{\lambda}(\delta) = 0 \) iff \( \delta \) is a bipartite matching.

\( b \)-conjecture (Hypermap-Jack conjecture)

\[
(1 + b) \frac{t \partial}{\partial t} \log (\tau_b(t, p, q, r)) = \sum_{M \text{ rooted connected bipartite map}} t^{|M|} b^{\vartheta(M)} p_{\Lambda^\circ(M)} q_{\Lambda^\bullet(M)} r_{\Lambda^\circ(M)}
\]

where \( \vartheta \) is a function on connected rooted maps with non-negative integer value, such that \( \vartheta(M) = 0 \) iff \( M \) is oriented.
Some partial results

Theorem (Dołęga-Féray ’15)

The coefficient of $p^\lambda q^\mu r^\nu$ in the function $\tau_b(t, p, q, r)$ multiplied by $z_\lambda(1 + b)^{\ell(\lambda)}$ is a polynomial in $b$ with rational coefficients.

Theorem (Dołęga-Féray ’17)

The coefficient of $p^\lambda q^\mu r^\nu$ in the function $(1 + b) \frac{d}{dt} \log (\tau_b(t, p, q, r))$ is a polynomial in $b$ with rational coefficients.
Some partial results

**Theorem (Chapuy-Dołęga ’20)**

\[
(1 + b) \frac{t \partial}{\partial t} \log (\tau_b(t, p, q, u)) = \sum_{M \text{ rooted connected bipartite map}} t^{|M|} b^{\vartheta(M)} p^{\bullet} q^{\circ}(M) u^{\ell}(\Lambda^\bullet(M))
\]

where \( \vartheta \) is a function on connected rooted maps with non-negative integer value, such that \( \vartheta(M) = 0 \) iff \( M \) is oriented.

\[
p := (p_1, p_2, p_3, \ldots),
\]
\[
q := (q_1, q_2, q_3, \ldots),
\]
\[
u := (u, u, u, \ldots).
\]
Some partial results

Theorem (B.D. ’21, arXiv:2106.15414)

\[
\tau_b(t, p, q, u) = \sum_{n \geq 0} \sum_{\lambda \vdash n} \frac{t^n}{z_\lambda (1 + b) \ell(\lambda)} \sum_{\delta \text{ matching on } A_n} b^{\partial_\lambda(\delta)} p_\lambda q_{\Lambda(\epsilon, \delta)} u^{\ell(\Lambda(\delta, \delta))},
\]

where for every partition \( \lambda \vdash n \), \( \partial_\lambda \) a function on the matchings of \( A_n \) with non-negative integer values, such that \( \partial_\lambda(\delta) = 0 \) iff \( \delta \) is a bipartite matching.

\( p := (p_1, p_2, p_3, ...) \),
\( q := (q_1, q_2, q_3, ...) \),
\( u := (u, u, u...) \).
Some partial results

Theorem (B.D. ’21, arXiv:2106.15414)

\[ \tau_b(t, p, q, u) = \sum_{n \geq 0} \sum_{\lambda \vdash n} \frac{t^n}{z_\lambda (1 + b)^\ell(\lambda)} \sum_{\delta \text{ matching on } A_n} b^{\vartheta_\lambda(\delta)} p^{\lambda q_{\Lambda(\varepsilon, \delta)}} u^{\ell(\Lambda(\delta_\lambda, \delta))}, \]

where for every partition \( \lambda \vdash n \), \( \vartheta_\lambda \) a function on the matchings of \( A_n \) with non-negative integer values, such that \( \vartheta_\lambda(\delta) = 0 \) iff \( \delta \) is a bipartite matching.

\( p := (p_1, p_2, p_3, \ldots) \),
\( q := (q_1, q_2, q_3, \ldots) \),
\( u := (u, u, u\ldots) \).

Remark

All the precedent results can be generalized to the case of \( k \)-constellations.
Recall:

**Theorem (Dołęga-Féray ’15)**

The coefficient of $p_\lambda q_\mu r_\nu$ in the function $\tau_b(t, p, q, r)$ multiplied by $z_\lambda(1 + b)^{\ell(\lambda)}$ is a polynomial in $b$ with rational coefficients.
Recall:

Theorem (Dołęga-Féray ’15)

The coefficient of \( p\lambda q_\mu r_\nu \) in the function \( \tau_b(t, p, q, r) \) multiplied by \( z_\lambda(1 + b)^{\ell(\lambda)} \) is a polynomial in \( b \) with rational coefficients.

Upcoming result (joint work with Chapuy and Dołęga):

Theorem

The coefficient of \( p\lambda q_\mu r_\nu \) in the function \( \tau_b(t, p, q, r) \) multiplied by \( z_\lambda(1 + b)^{\ell(\lambda)} \) is a polynomial in \( b \) with integer coefficients.

Proof: The integrality of the coefficients \( \tau_b(t, p, q, u) \) + Farahat-Higman Algebra.