INFORMATIQUE
Sciences
Université Paris Cité

Dijkstra's Token Ring

MPRI

Lélia Blin

lelia.blin@irif.fr

2023

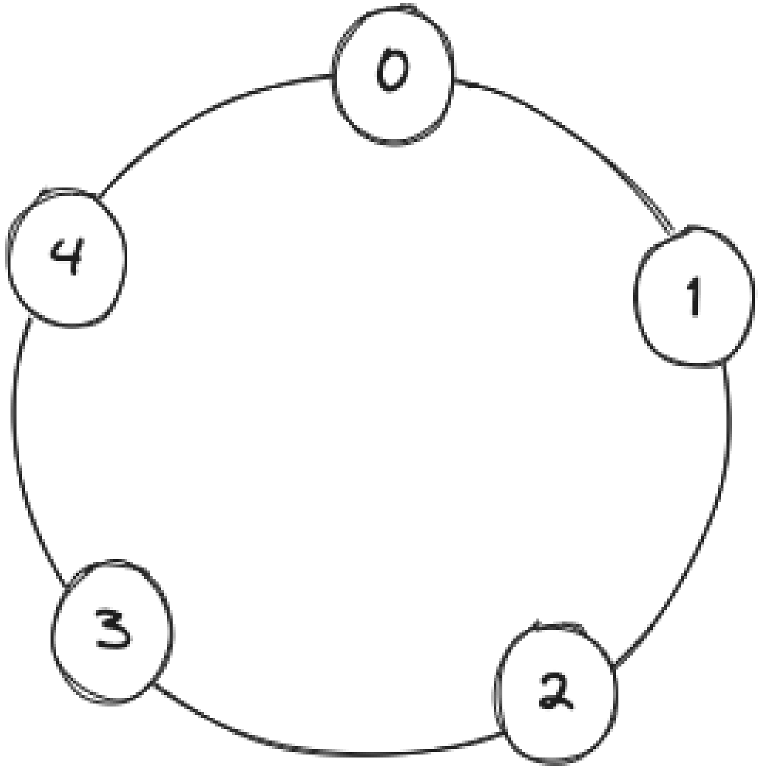
Problem:

There exists n nodes (processes) in the network denoted by $p_o, \ldots, p_{n-1}$

Each p_i can hold a token or not

Goal:

Unique token and the token visits every p_i infinitely often



Assumptions

The nodes are organized in a ring

State model

Each p_i can

- **read** the states of its neighbors
- **updates** its own state

Each p_i is activated infinity often

State of each process

p_i state : $x_i \in \{0, \ldots, n\}$

Remark: the state can take $n + 1$ different values

Token presence:

p_0 holds the token if $\forall p_i : x_i = x_0$

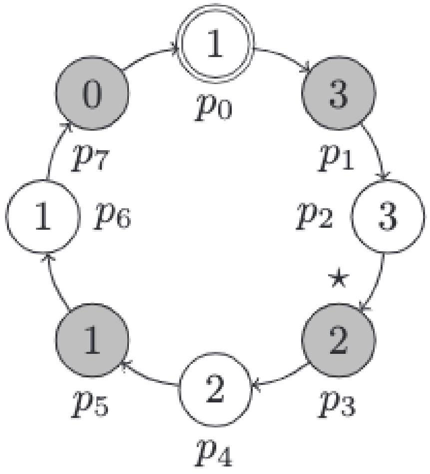
p_i holds the token if

- $\forall j$ with $j > i : p_j$ has the same state of p_j
- $\forall j$ with $j < i : p_j$ has the same state of p_0

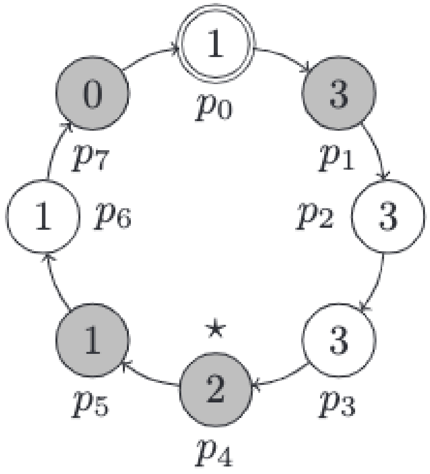
Algorithm

- $R_0 :: (x_{n-1} = x_0) \longrightarrow x_0 := x_0 + 1 \mod n + 1$
- $R_1 :: (x_i \neq x_{i-1}) \wedge (i \neq 0) \longrightarrow x_i := x_{i-1}$

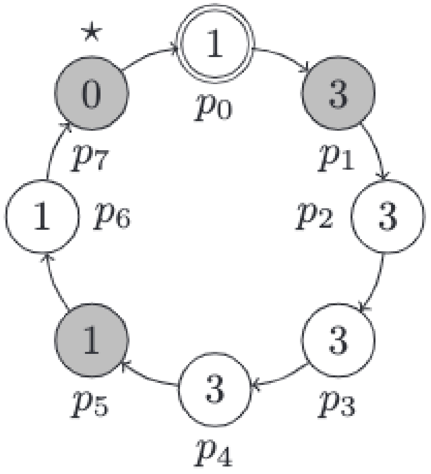
Example



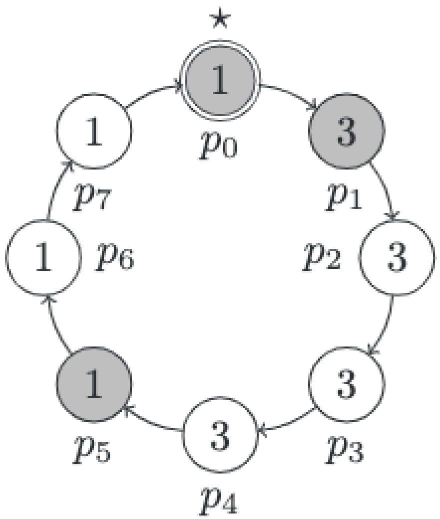
Configuration (i)



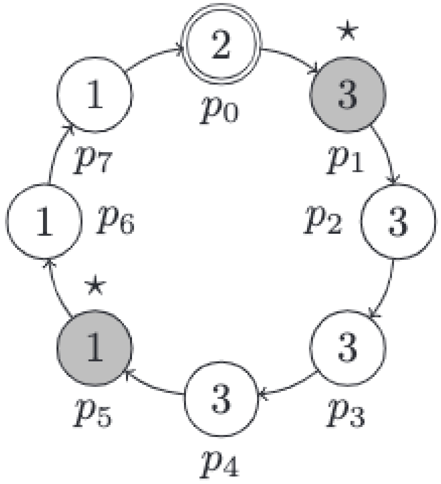
Configuration (ii)



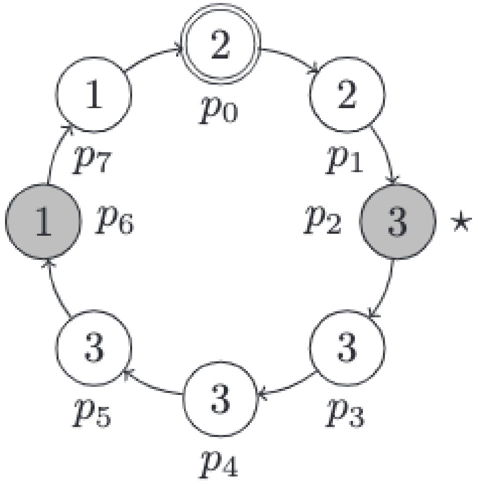
Configuration (iii)



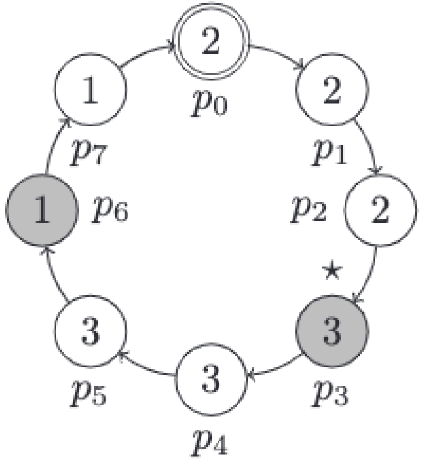
Configuration (iv)



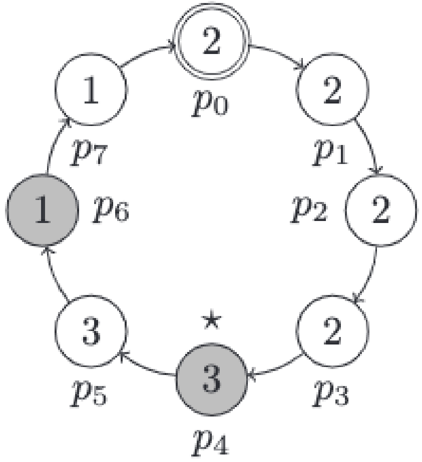
Configuration (v)



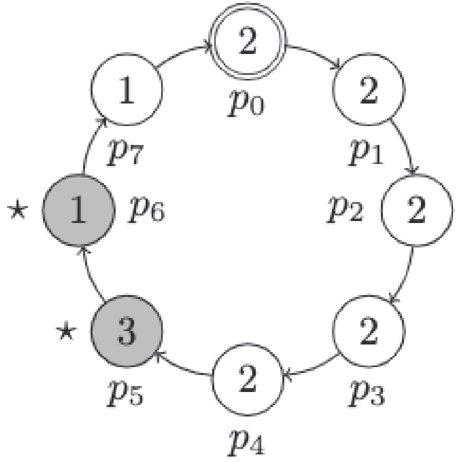
Configuration (vi)



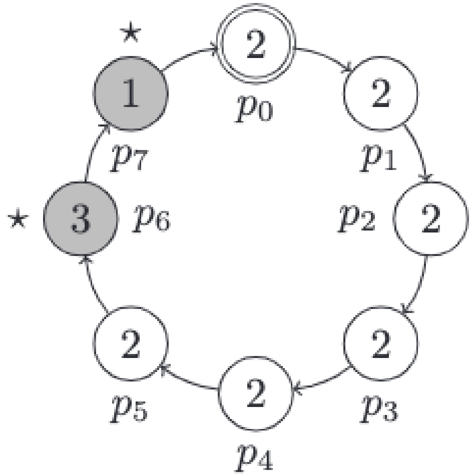
Configuration (vii)



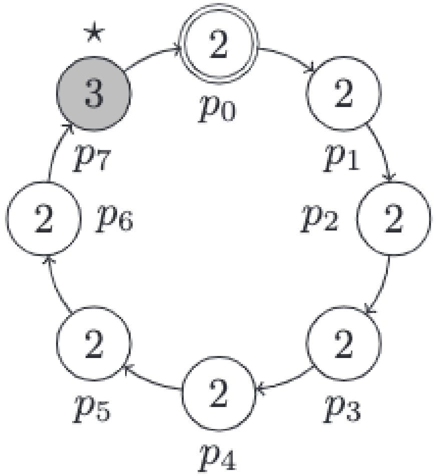
Configuration $(viii)$



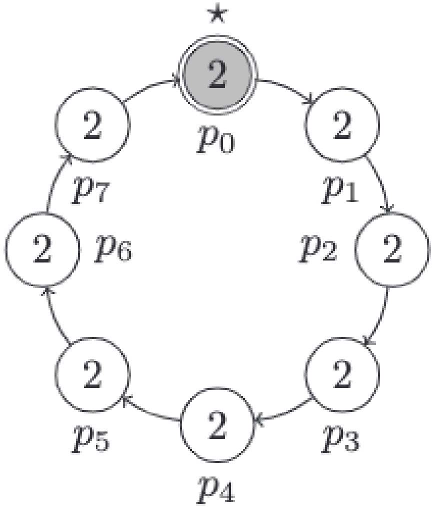
Configuration (ix)



Configuration (x)



Configuration (xi)



Configuration (xii)

Correctness

Closure

> Starting from a legal (legitimate) configuration the system remains in a legal configuration.

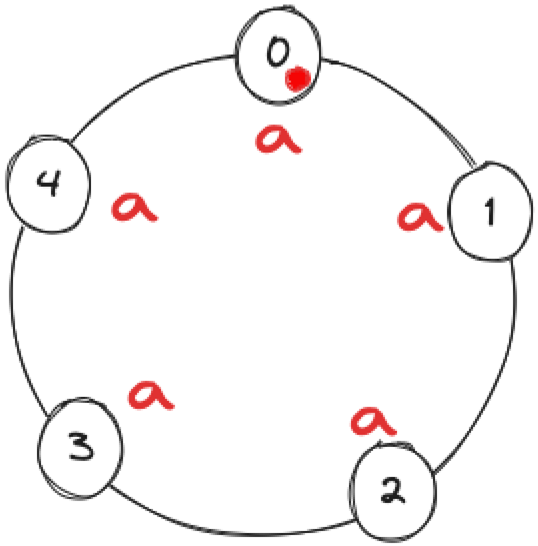
Convergence

> Starting from a illegal configuration the system converges to a legal configuration.

Legal configuration (L_0)

p_0 holds the token in configuration γ, if all the nodes have the same state

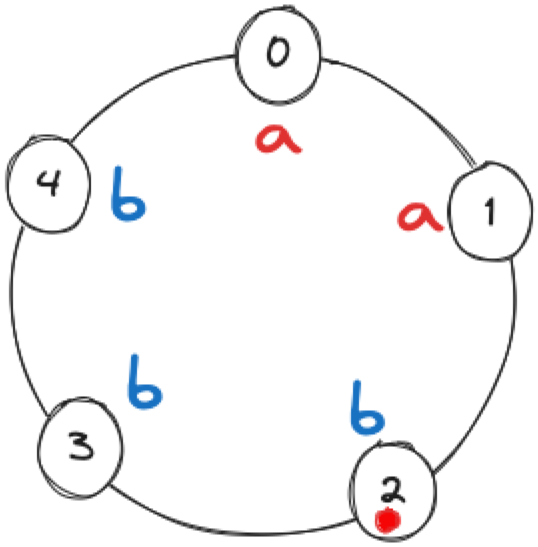
$$L_0(\gamma) \equiv (\forall i \neq 0 : x_i = x_0)$$



Legal configuration (L_i)

p_i with $i \neq 0$ is a unique node with the token in configuration γ, if

$$L_i(\gamma) \equiv \forall j \in \{1, n-1\} : (j < i : x_j = x_0) \wedge (j > i : x_j = x_i)$$



Closure

- We denote by
  - Γ the set of all configurations
  - S the set of legal configurations ($S \in \Gamma$)

Theorem 1 : $\forall \gamma \in S, \gamma(p_i) \to \gamma'$ with $\gamma' \in S$

Closure : proof

1. If p_i does not hold the token then p_i is not activatable and does not change is state $\rightarrow \gamma' = \gamma$ and $\gamma' \in S$
2. If p_i holds the token then p_i is activatable :
   i. If $i = 0$, p_0 increases is state, and p_1 is the only node to hold the token, $\gamma' \in S$
   ii. $i \neq 0$, p_i takes the state of p_{i-1} , and p_i is the only node to hold the token, $\gamma' \in S$

Convergence

Theorem 2: Starting from a configuration $\gamma_0 \notin S$ the system converges to a configuration $\gamma' \in S$

Notations

- $\mathcal{A}(\gamma)$ denoted the set of activatable nodes in configuration $\gamma \in \Gamma$
- $\mathcal{A}^*(\gamma) \subset \mathcal{A}(\gamma)$ denoted the set of nodes activate by the scheduler in configuration γ
- $x_0(\gamma)$ the value of x_0 in γ
- the next configuration : $\gamma \xrightarrow{\mathcal{A}^*(\gamma)} \gamma'$

Deadlock

lemma 1 : $\forall \gamma \in \Gamma : |\mathcal{A}(\gamma)| \neq 0.$

Proof by contradiction:

Suppose that there exists a configuration $\gamma \in \Gamma$ such that $|\mathcal{A}(\gamma)| = 0$, so no node can execute a rule.

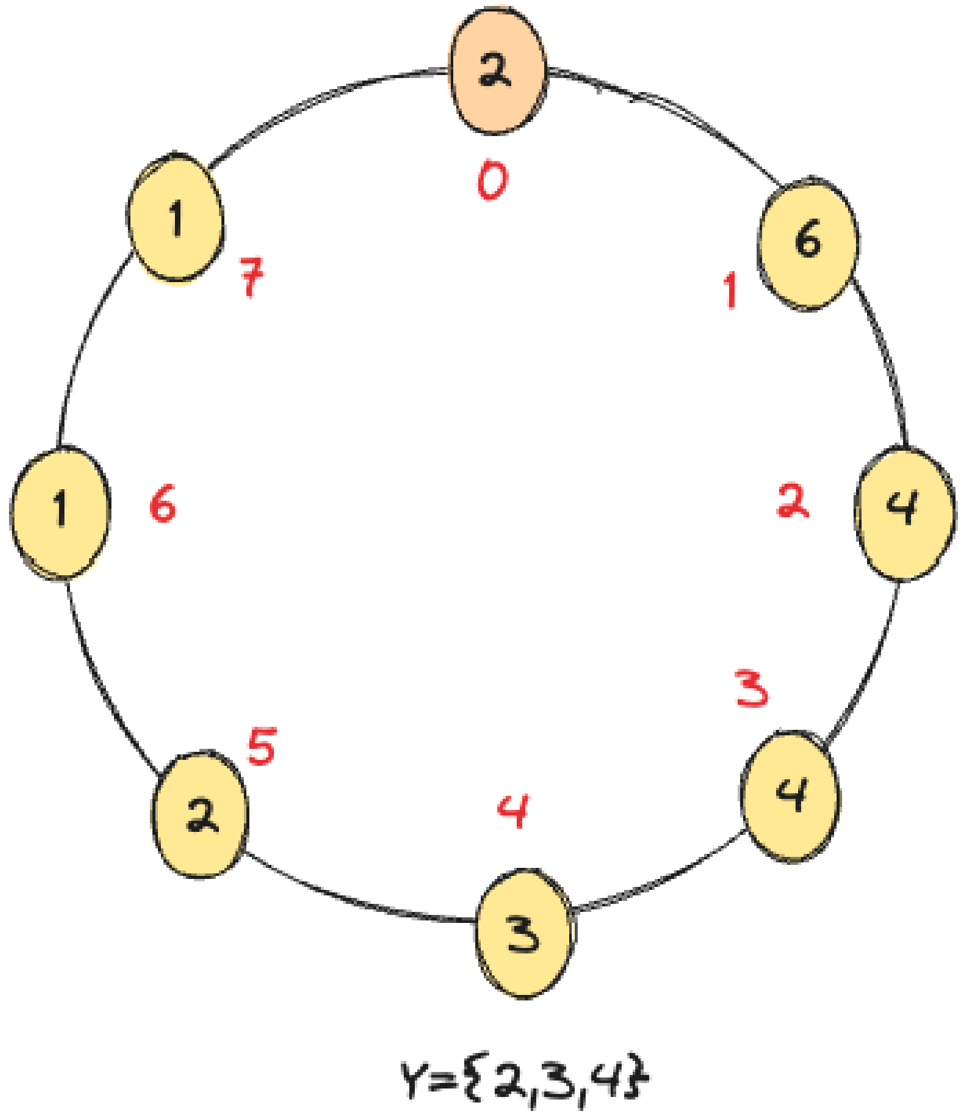
- If $i \neq 0$, then all the nodes have the same value, including the node p_{n-1} and p_0; otherwise, they can execute rule R_1. Contradiction, p_0 can execute the rule R_0.
- If $i = 0$, p_0 mustn't have the same value as p_{n-1}; otherwise, p_0 can execute the rule R_0, and $\forall i : 0 < i < n-1, x_i = x_{n-1}$. Contradiction, p_1 can execute the rule R_1.

Switch elements

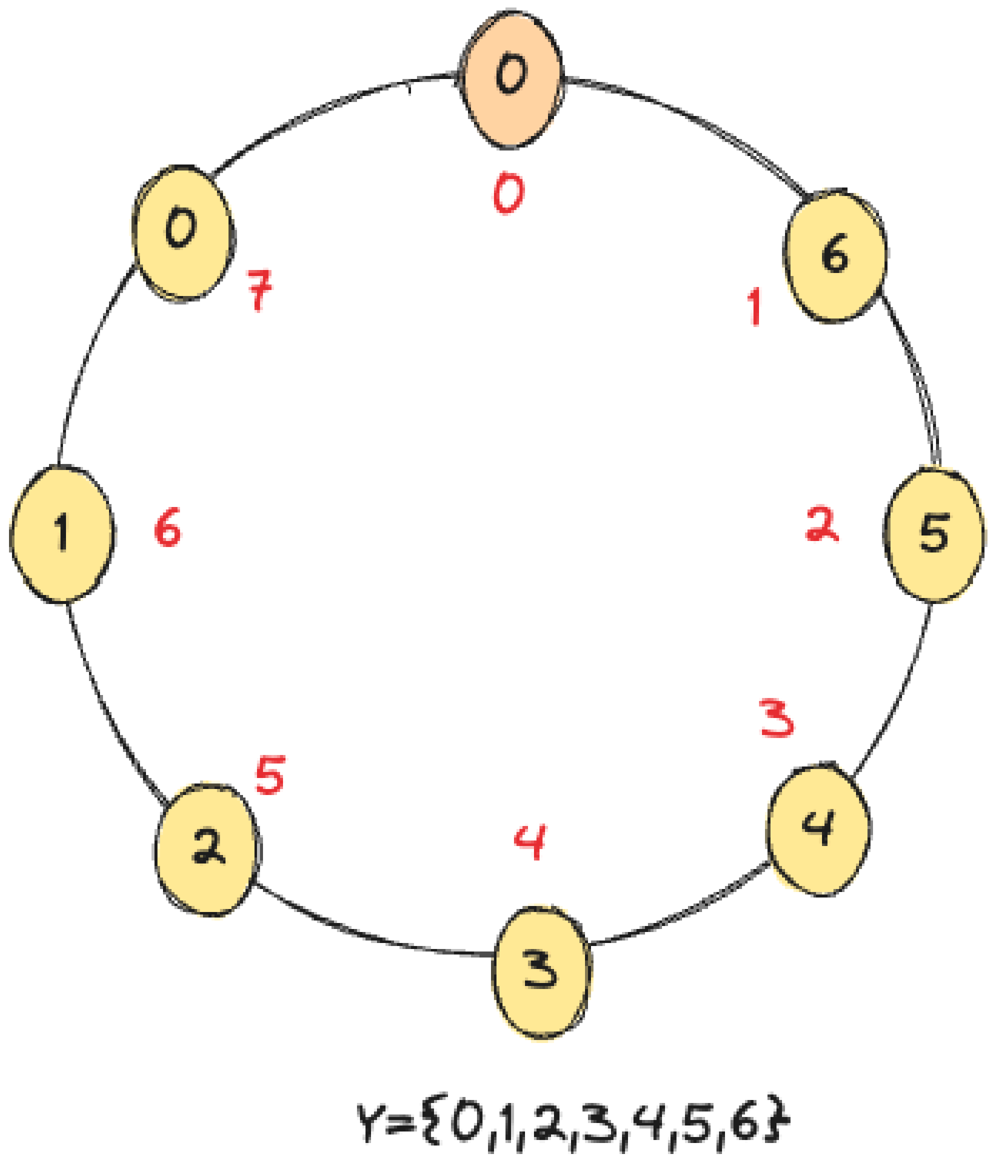
Maximum number of activation of p_0

> $x \in \{0, \ldots, n\}$ is valid if
>
> - $\exists i > 0 : x_i = x$ and
> - $(x = x_0)$ or $\exists j > i | x_j = (x - 1 \mod n + 1)$ and x_j is valid

Let us denote by $Y(\gamma)$ the set of valid elements in configuration γ



- 2 is valid because $\exists i = 5 > 0 | x_5 = 2$ and $x_0 = 2$
- 3 (i=4) is valid because 2(j=5) is valid and $j > i$
- 4 (i=3) is valid because 3(j=4) is valid and $j > i$


0
0
6
1
0
7
1
6
2
5
3
4
5
2
4
3
Y={0,1,2,3,4,5,6}

lemma 2: If p_0 in $\gamma \notin S$ executes R_0 then $|Y(\gamma)| > |Y(\gamma')|$ with $\gamma' > \gamma$

Proof :

- If x_0 executes R_0 we obtain $x_0(\gamma') = x_0(\gamma) + 1 \mod n + 1$.
- By definition of the valid element $x_0(\gamma) \notin Y(\gamma')$
- Consider $x_j(\gamma) \in Y(\gamma)$:
  i. either $x_j(\gamma)$ disappears by application of rule R_1 by p_j.
  ii. or $x_j(\gamma)$ remains and stay valid.
  So $|Y(\gamma)| > |Y(\gamma')|$

The weight of a token

A node p_i with a token is a node that can execute rule R_0 or R_1, so this node has $\delta_i(\gamma) > 0$ in an illegal configuration. In fact, if p_i executes a rule, it changes its value x_i and sends the token to its neighbor p_{i+1}. To simplify the purpose, we call $\delta_i(\gamma)$ the weight of the token of value x_i denoted:

$$T_x(\gamma) = \delta_i(\gamma)$$

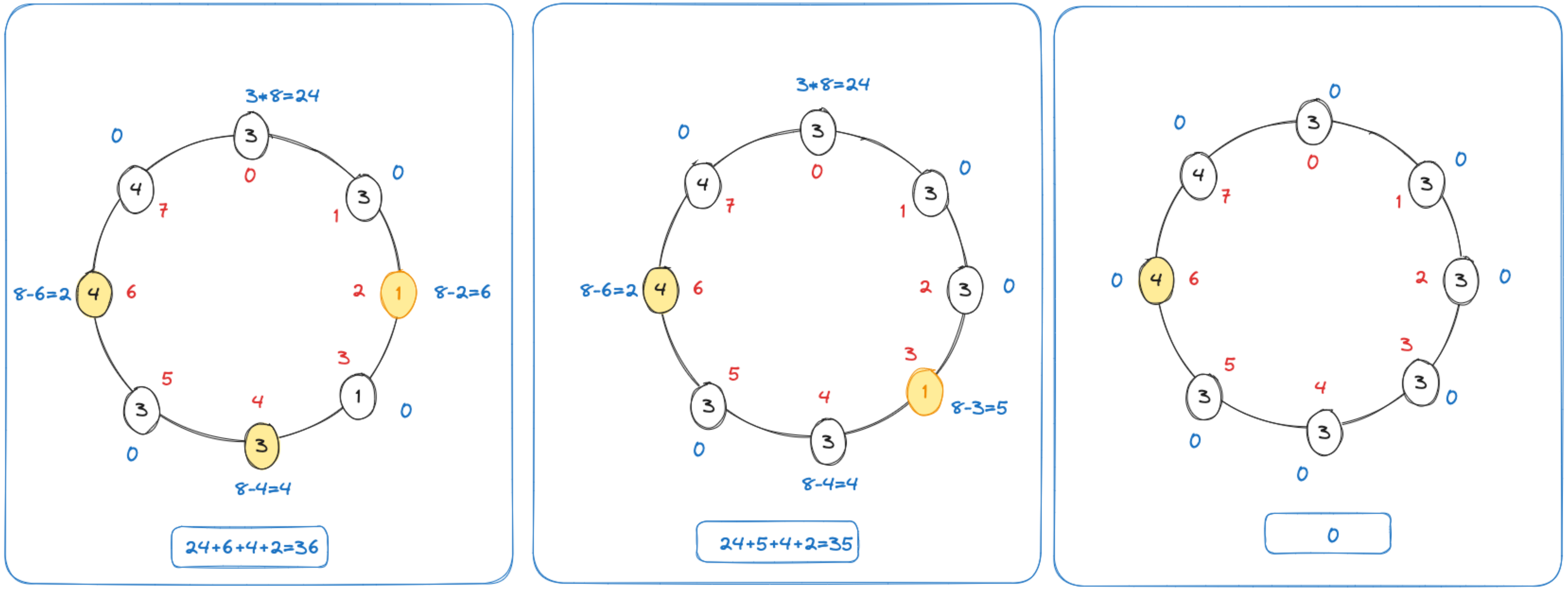
Potential function :

$\delta_i(\gamma) =$

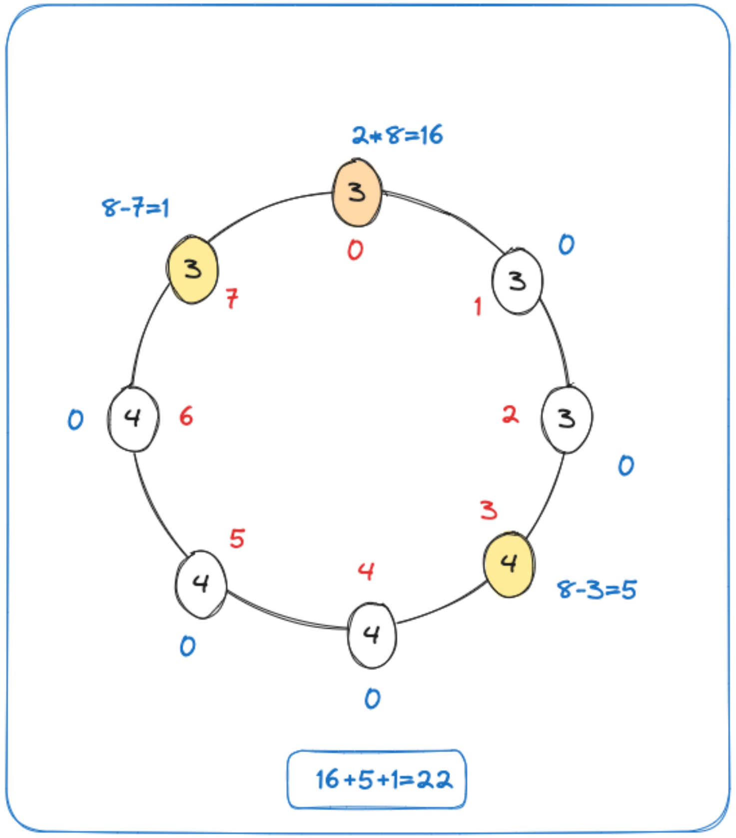
- $J * n$ if $i = 0 \wedge \neg L_0(\gamma) \wedge \neg L_i(\gamma)$ (Where J is the number of token)
- $n - i$ if $i \neq 0 \wedge \neg L_0(\gamma) \wedge \neg L_i(\gamma) \wedge (x_i \neq x_{i-1})$
- 0 otherwise

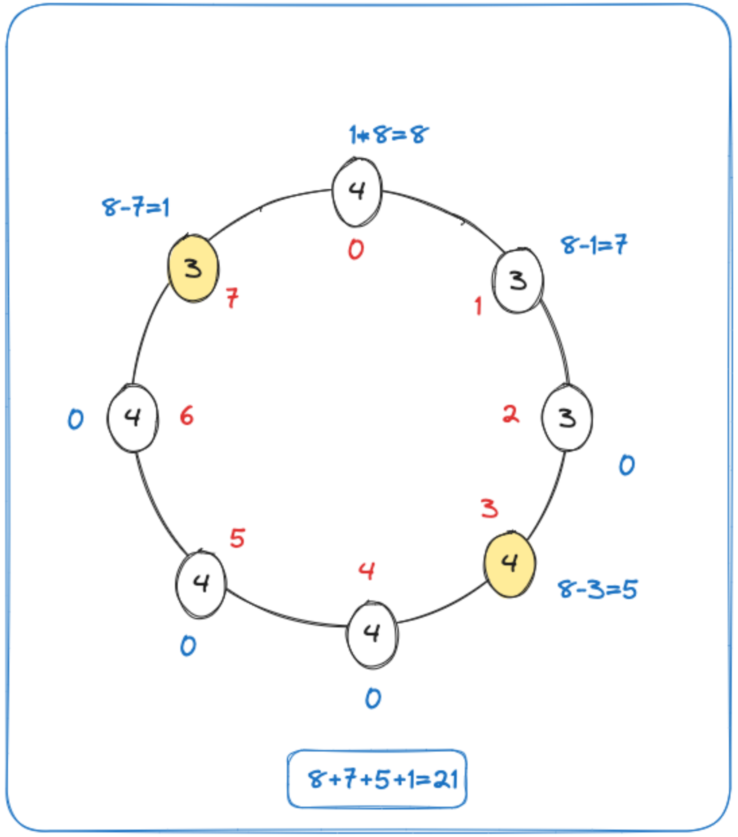
$$\Delta(\gamma) = \sum_{i \in \{0,...,n-1\}} \delta_i(\gamma)$$

Remark: By construction if $\gamma \in S$ we have $\Delta(\gamma) = 0$


3*8=24
8-6=2
8-2=6
8-4=4
24+6+4+2=36
3*8=24
8-6=2
8-3=5
8-4=4
24+5+4+2=35
0

Focus $J * n$


2*8=16
8-7=1
3
0
3
0
7
1
3
0
4
6
2
3
0
5
3
4
4
4
4
8-3=5
0
0
16+5+1=22


1*8=8
8-7=1
4
0
8-1=7
3
7
1
3
0
4
6
2
3
0
5
3
4
4
4
4
8-3=5
0
0
8+7+5+1=21

Convergence

we must prove that the potential function Δ decreases between two illegitimate configurations until Δ reaches zero

The weight of the tokens decrease: p_i

> **Lemma 3**: When $p_i : i \neq 0$ releases the token in $\gamma \notin S$, the weight of the token decreases.

Proof:

Let consider the node p_i with the token in configuration γ so we obtain:

$$T_x(\gamma) = \delta_i(\gamma) = n - i \geq T_x(\gamma') = \delta_{i-1}(\gamma') = n - i - 1$$

The weight of the tokens decrease: p_0

Lemma 4: When p_0 releases the token, the weight of the token decreases.

In other words: If $p_0 \in \mathcal{A}^*(\gamma)$ and $\gamma, \gamma' \notin S$ then $\delta_0(\gamma) + \delta_1(\gamma) > \delta_0(\gamma') + \delta_1(\gamma')$

Proof: Let denote by k an integer inferior at n thanks to the lemma 2 we obtain:

$$\delta_i(\gamma) = k * n > \delta_i(\gamma') \geq (k-1) * n$$

$$\delta_i(\gamma) + \delta_{i+1}(\gamma) = k * n + 0 > \delta_i(\gamma') + \delta_i(\gamma') \geq (k-1) * n + n - 1$$

$$kn > kn - n + n - 1 = kn - 1$$

The number of token does not increased

__Lemma 6_5: $|\mathcal{A}(\gamma)| \geq |\mathcal{A}(\gamma')|$

Proof : If a node p_i with a token in configuration γ executes a rule, then it changes its value x_i. This action impacts only its neighbor p_{i+1} according to rules R_0 and R_1. If, after that, p_i has the token in γ', that means p_{i-1} had the token in γ.

If a node p_i with a token in configuration γ does not execute a rule, then it does not change its value x_i. So if its neighbor p_{i-1} has the token in γ' and executes a rule, either the two tokens merge, or the two tokens disappear.

Remember:

lemma 1 : $\forall \gamma \in \Gamma : |\mathcal{A}(\gamma)| \neq 0.$

lemma 2: If p_0 in $\gamma \notin S$ executes R_0 then $\Psi(\gamma) > \Psi(\gamma')$

Lemma 3: When $p_i : i \neq 0$ releases the token in $\gamma \notin S$, the weight of the token decreases.

Lemma 4: When p_0 releases the token, the weight of the token decreases.

Lemma 5: $|\mathcal{A}(\gamma)| \geq |\mathcal{A}(\gamma')|$

Proof of the theorem 2

Proof: Thanks to the lemma 1 we known that there exist at least one token in a system, and thanks to lemma 5, we saw that the number of tokens does not increase. Moreover, the weight of a token decrease (see lemma 3 and 4), so like $\Delta(\gamma)$ is the sum of the weights of the tokens in the configuration γ, we obtain $\Delta(\gamma') < \Delta(\gamma)$.

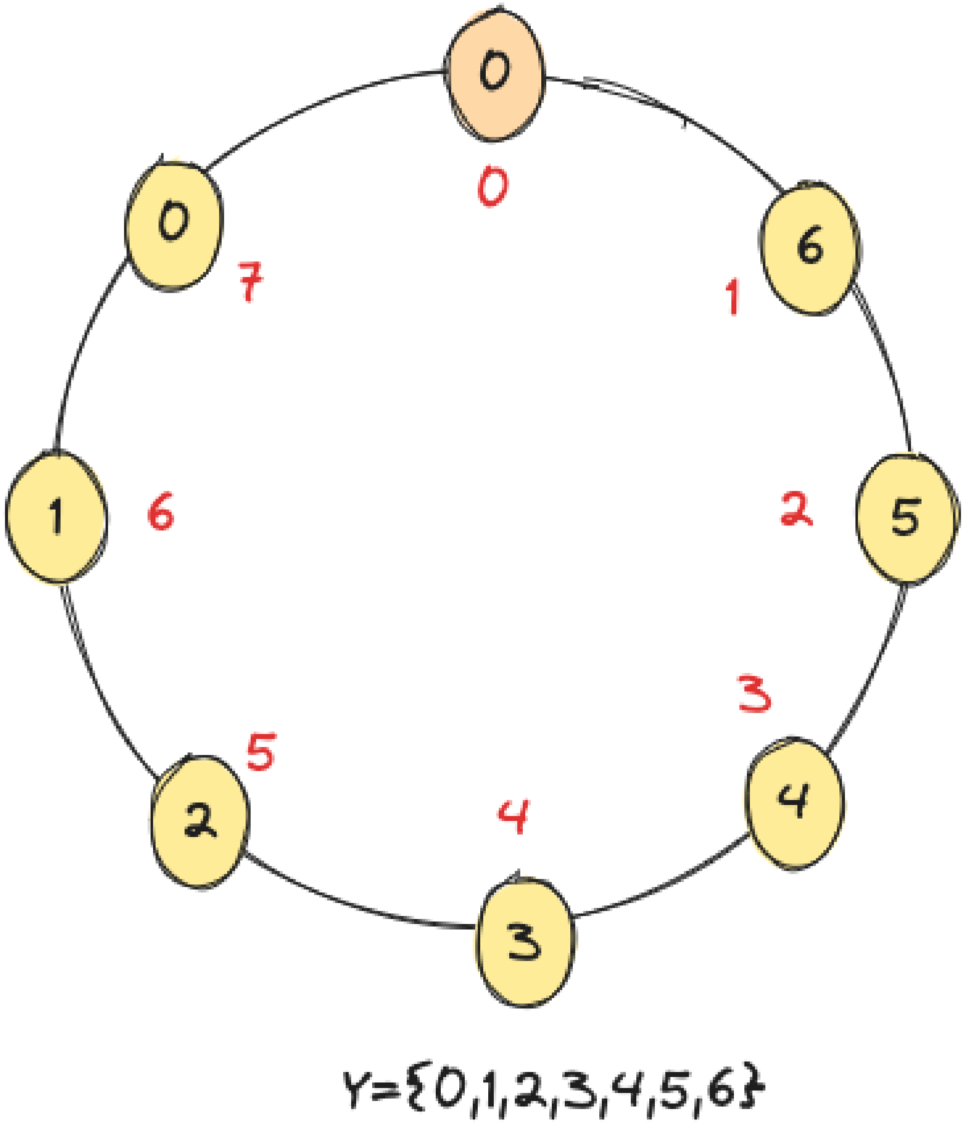
Lélia Blin

Proof of the algorithm

Closure : Theorem 1

Convergence : Theorem 2

Quality of the solution



Worst case

Steps complexity

- Shifts for a new token: $n * (n-2)$ steps
- Put the token at the end of the ring: n steps
- Delete the old tokens:
  - $(n-2)+(n-3)+(n-4)+\cdots+1$
  - $\frac{(n-2)(n-1)}{2}$ steps
- Total: $n^2-2n+n+\frac{(n-2)(n-1)}{2}=\frac{3n^2-5n+2}{2}$

  Steps complexity : $O(n^2)$

Rounds complexity

Remark that at each configuration, all the nodes was activatables:

$O(n)$ rounds

Space complexity

$O \log n$ bits