FoCaLiZe and Dedukti to the Rescue for Proof Interoperability

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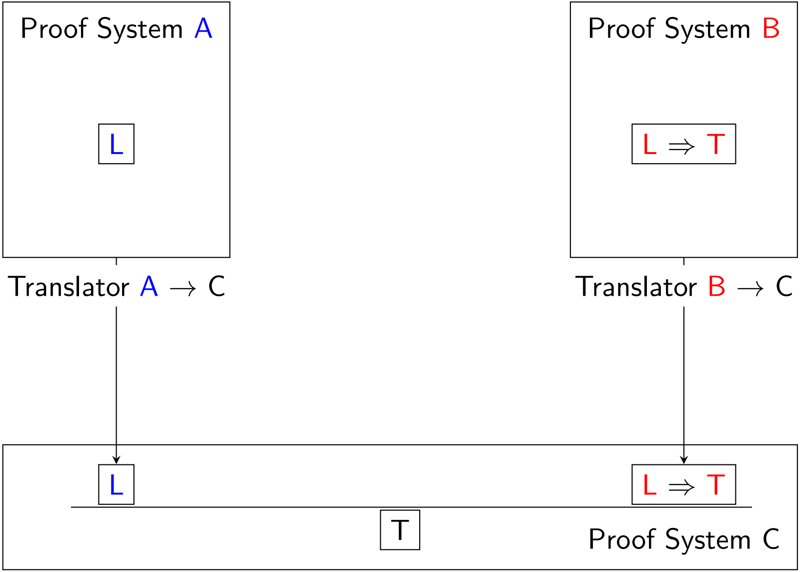
October 23th, 2017

Interoperability



Interoperability: Motivation

- Proof development is expensive
    - 4-color theorem, Kepler conjecture, Feit-Thomson theorem

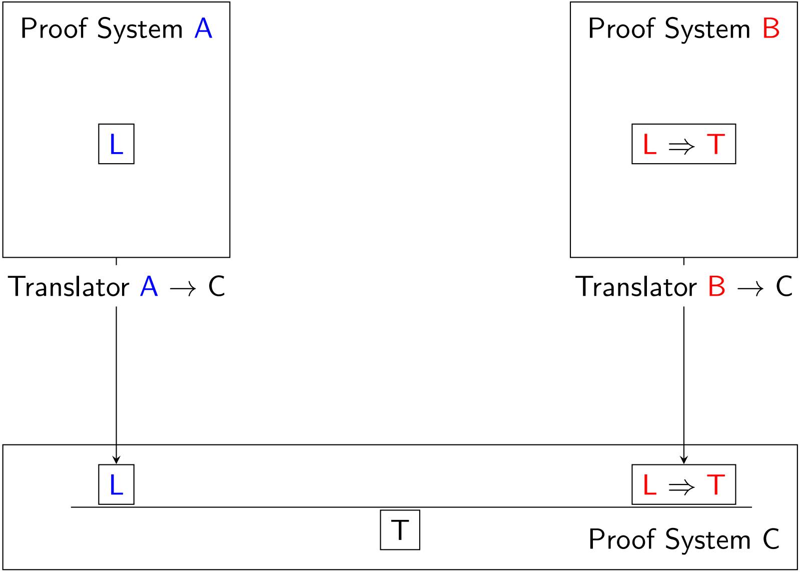
Interoperability: Motivation

- Proof development is expensive
  - 4-color theorem, Kepler conjecture, Feit-Thomson theorem
- Proof assistants are specializing
  - Counterexamples, proof by reflection, decision procedures, ...

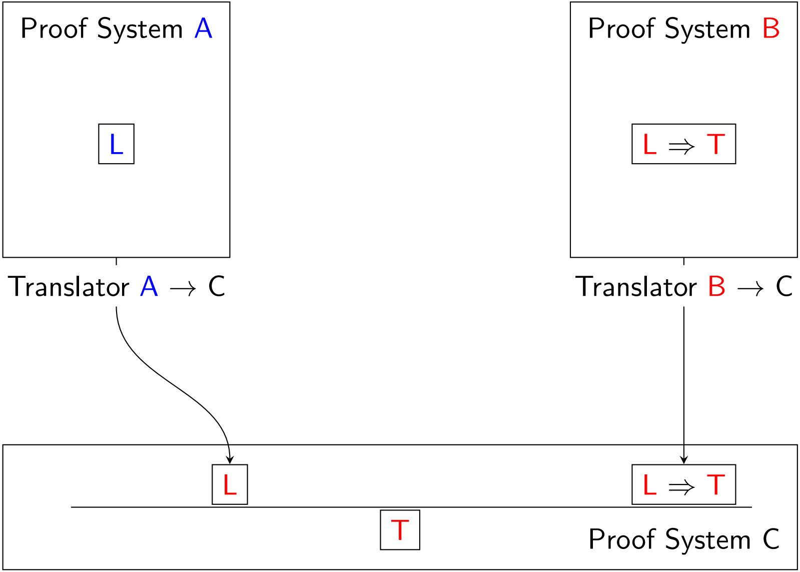
Interoperability: Obstacles

- Logical problem:
  We need to combine the logics of A and B in a consistent way.
- Mathematical problem: L and L are not identical
  Theories such as arithmetic are independently defined in System A and System B.
  We need to identify similar concepts.

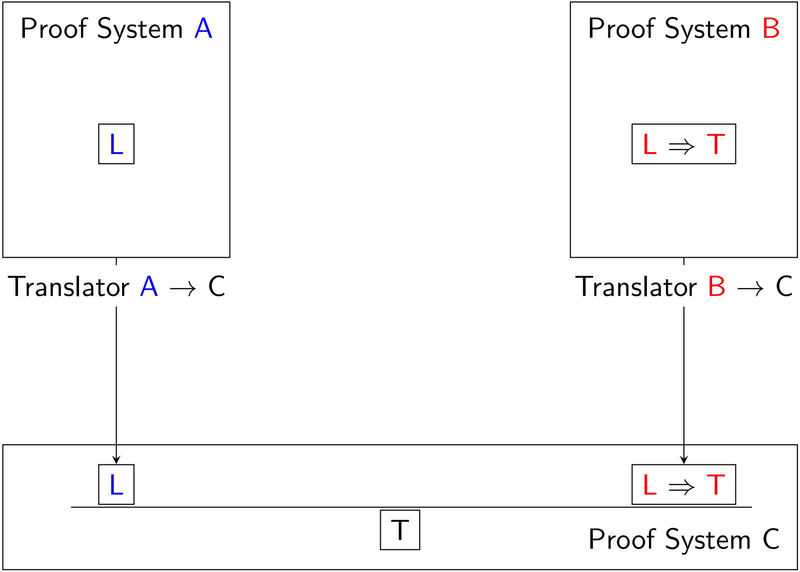
Solution 1/2: Parameterized Translators



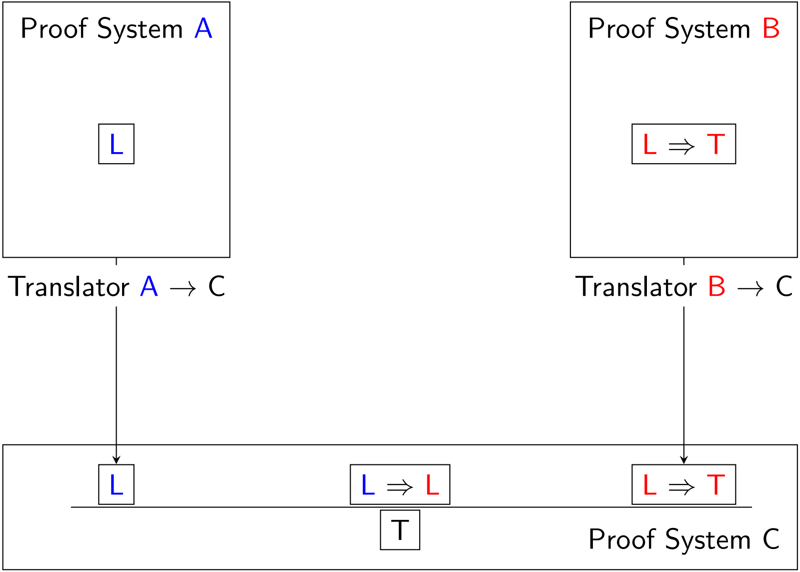
Solution 1/2: Parameterized Translators



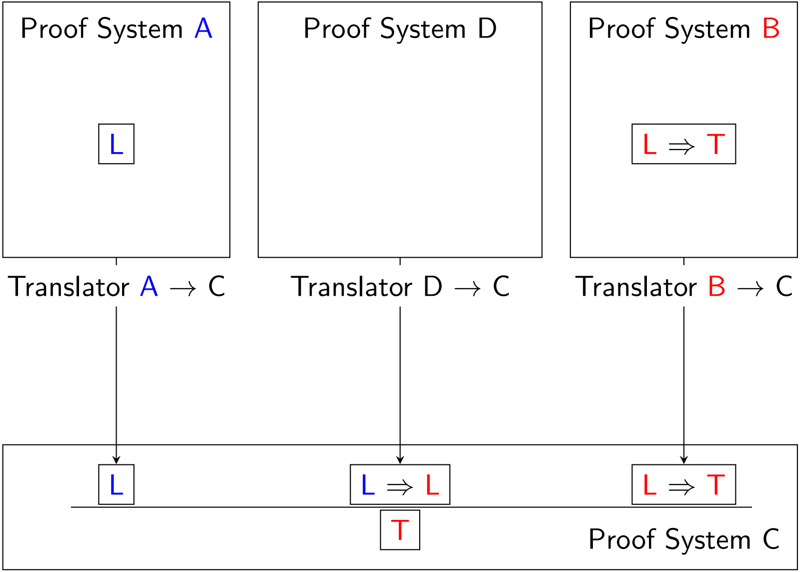
Solution 2/2: Theorem Transfer



Solution 2/2: Theorem Transfer



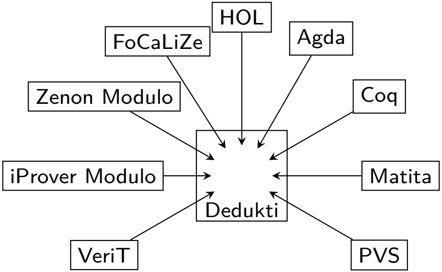
Solution 2/2: Theorem Transfer



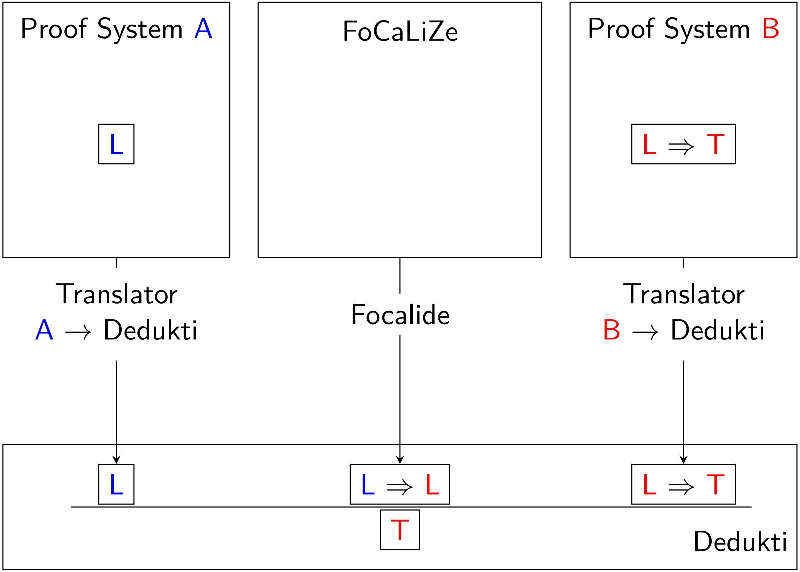
Duplication of Developments

- Solution 1: Parameterized Translators
  Identify syntactically
  Examples:
  - Hol Light $\rightarrow$ Coq (Keller, Werner)
  - Hol Light $\rightarrow$ Isabelle/HOL (Kaliszyk, Krauss)
  - Alignments in MMT (Müller, Rothgang, Liu, Rabe)
- Solution 2: Theorem Transfer
  Identify mathematically, automate reasoning modulo isomorphism
  - `transfer` tactic in Isabelle (Huffman, Kuncar)
  - `transfer` tactic in Coq (Zimmermann, Herbelin)
  - `transfer` tactic in Dedukti (Cauderlier)

Dedukti


HOL
FoCaLiZe
Agda
Zenon Modulo
Coq
iProver Modulo
Matita
Dedukti
VeriT
PVS

Theorem Transfer



Outline

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1. MathTransfer

2. Interoperability Methodology and Case Study

The MathTransfer Library

MathTransfer is a FoCaLiZe library of transfer theorems about natural number arithmetic.
`https://gitlab.math.univ-paris-diderot.fr/cauderlier/math_transfer`

FoCaLiZe, a Formal IDE

- Development of formally verified programs (ML)
- First-order specifications (Poly-FOL)
- Declarative proof language
- Integration of automated theorem provers (Zenon and Zenon Modulo)
- Modularity by object-oriented mechanisms
- Parameterized translators to Ocaml, Coq, and Dedukti

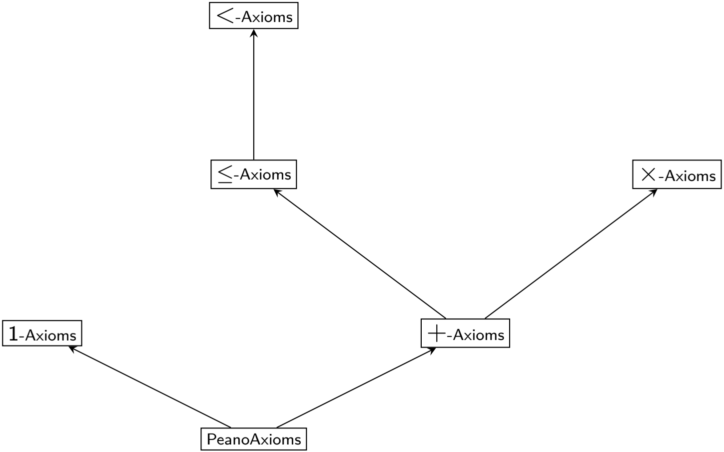
FoCaLiZe Species

- Species contain *methods* acting on the *representation* type of the species
- The representation is written `Self` inside the species
- Methods are either *computational* or *logical*
- Methods are either *declared* or *defined*
    - Declared computational = function/predicate symbol
    - Declared logical = axiom
    - Defined logical = theorem

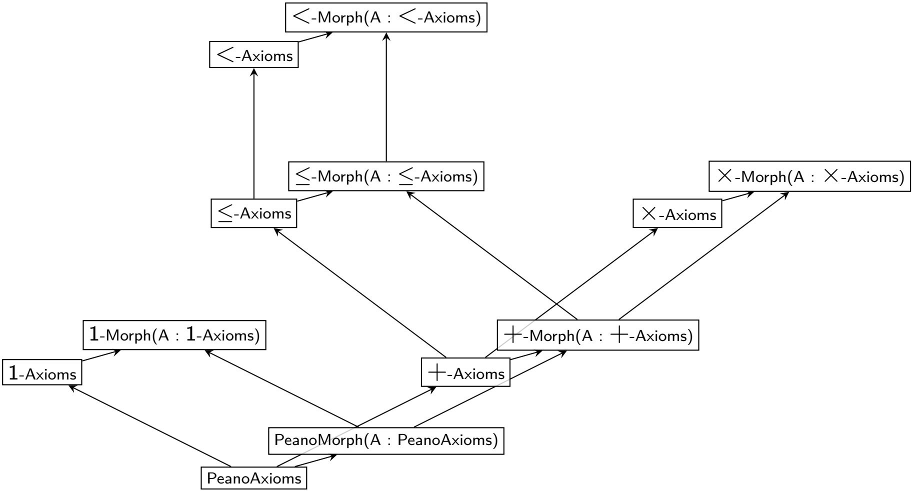
Modularity

- Species are extended by (multiple) inheritance
- During inheritance, we can
  - add new methods
  - define declared methods
  - define the representation
- A species is *complete* if its representation and all its methods are defined
- Complete species can be *instantiated* into *collections*
  - Translated as toplevel definitions and theorems
  - Used to parameterize other species

MathTransfer



MathTransfer



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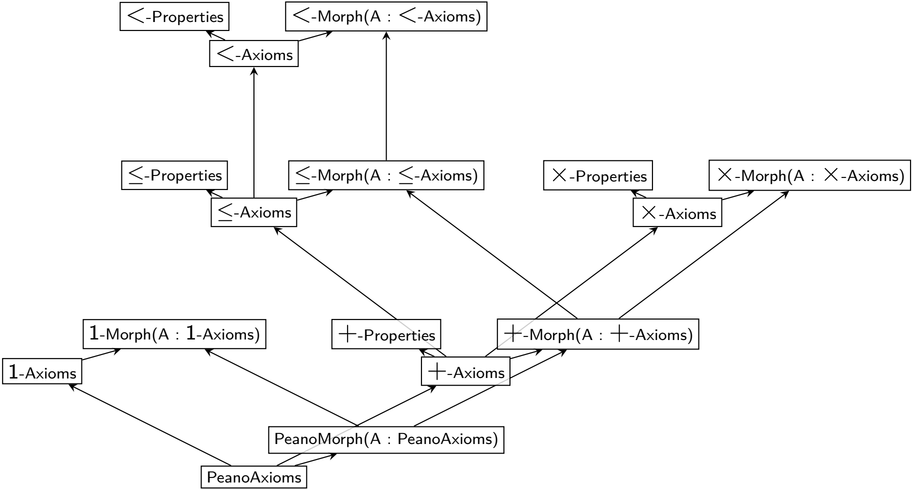
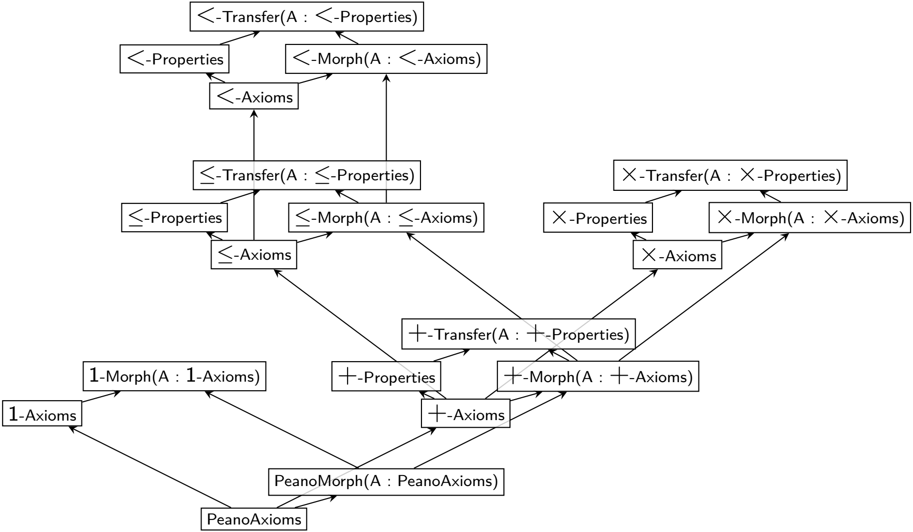


MathTransfer



MathTransfer

$\times$-Transfer(A : $\times$-Properties)
theorem $\forall m.\ m \times 0 = 0$
theorem $\forall mn.\ m \times \mathsf{succ}(n) = (m \times n) + m$
theorem $\forall mnp.\ (m \times n) \times p = m \times (n \times p)$
theorem $\forall mn.\ m \times n = n \times m$
theorem $\forall mnp.\ m \times n = m \times p \Leftrightarrow (n = p \vee m = 0)$
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$\times$-Properties
axiom $\forall m.\ m \times 0 = 0$
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$\times$-Morph(A : $\times$-Axioms)
theorem $\forall xy : A.\ f(x \times_A y) = f(x) \times f(y)$

$\times$-Axioms
$\times$: Self $\to$ Self $\to$ Self
axiom $\forall n.\ 0 \times n = 0$
axiom $\forall mn.\ \mathsf{S}(m) \times n = n + (m \times n)$

MathTransfer in numbers

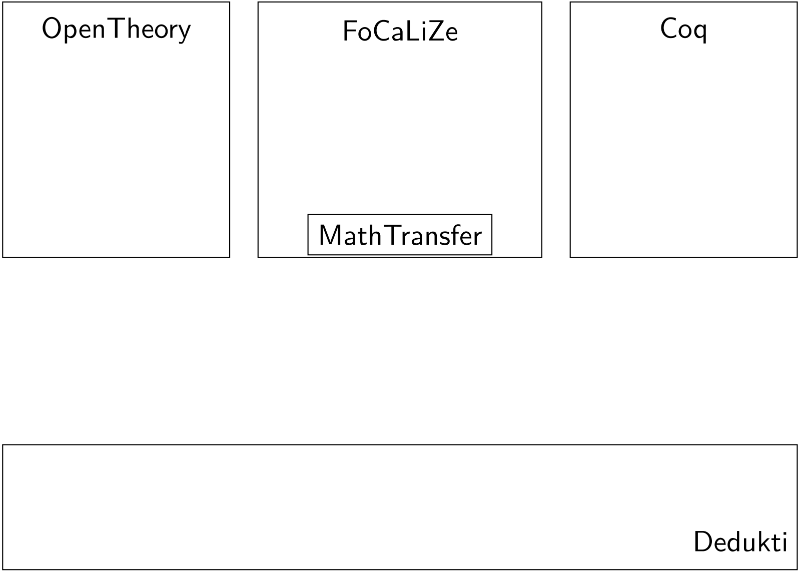
- 11 operations (0, S, 1, bit0, bit1, pred, $+$, $\times$, $\leq$, -, $<$)
- 84 transfer theorems
- 69 species
- 1771 lines, 74KB
- 1.7MB generated Dedukti code (71% from Zenon Modulo and the transfer tactic)

Outline

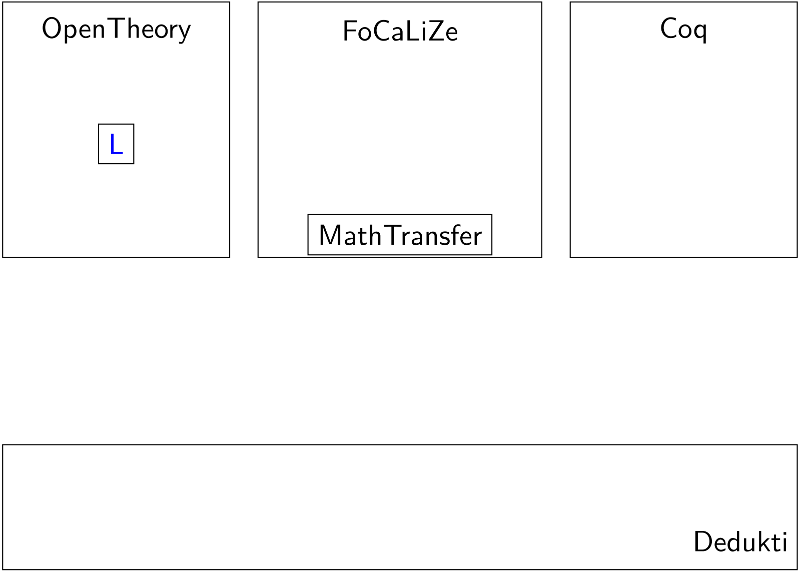
Case Study

- A = HOL (OpenTheory)
- B = Coq
- T = correctness of the Sieve of Eratosthenes

Start



Step 1/8



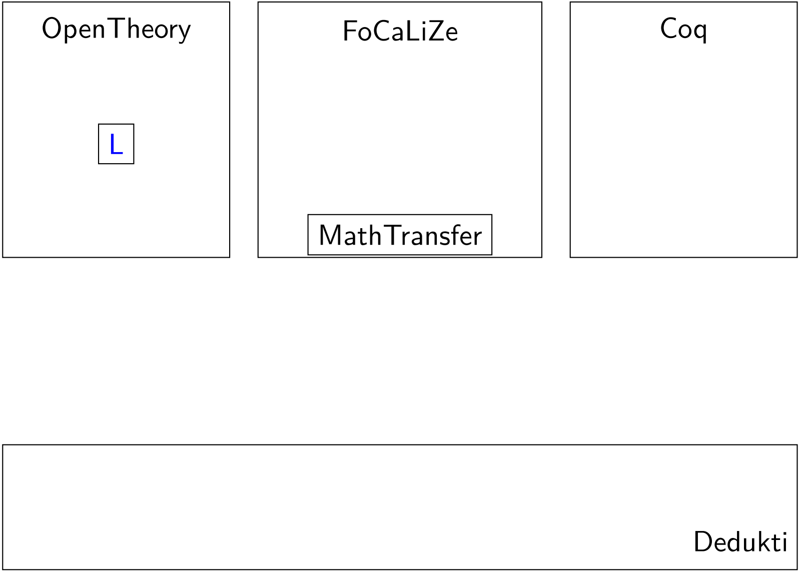
Step 1/8: Identify and prove the lemma in HOL

Prime Divisor Lemma

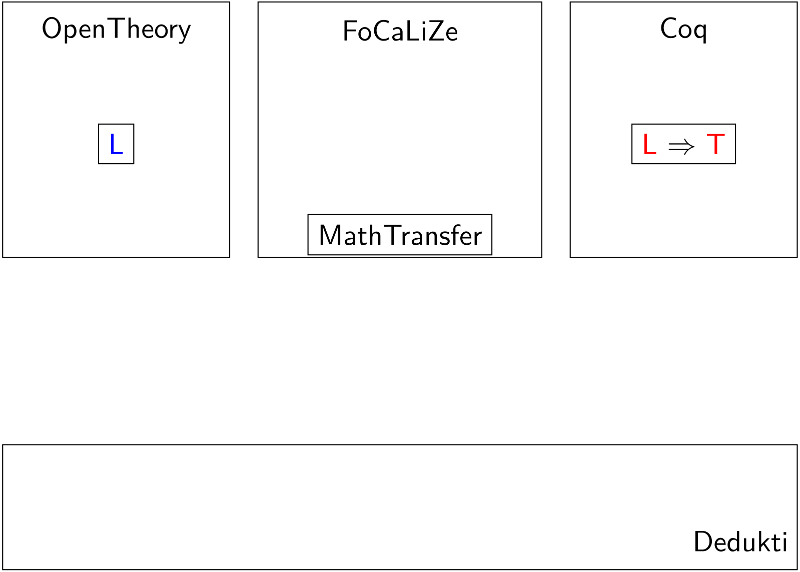
$$\mathsf{L} := \forall n \neq 1.\ \exists p.\ \mathsf{prime}(p) \wedge p \mid n$$

Already proved in OpenTheory `natural-prime` library

Step 1/8



Step 2/8



Step 2/8: Prove L → T in Coq

```
…

Definition eratosthenes n := …

Section correctness_proof.

Hypothesis prime_divisor :
  forall n : nat, n <> 1 ->
    exists p : nat, prime p /\ divides p n.

Theorem correctness p n :
  In p (eratosthenes n) <-> (p <= 2 + n /\ prime p).
Proof. … Qed.

End correctness_proof.
```

Step 2/8: Prove L $\to$ T in Coq

```
…

Definition eratosthenes n := …

Section correctness_proof.

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Proof. … Qed.

End correctness_proof.
```
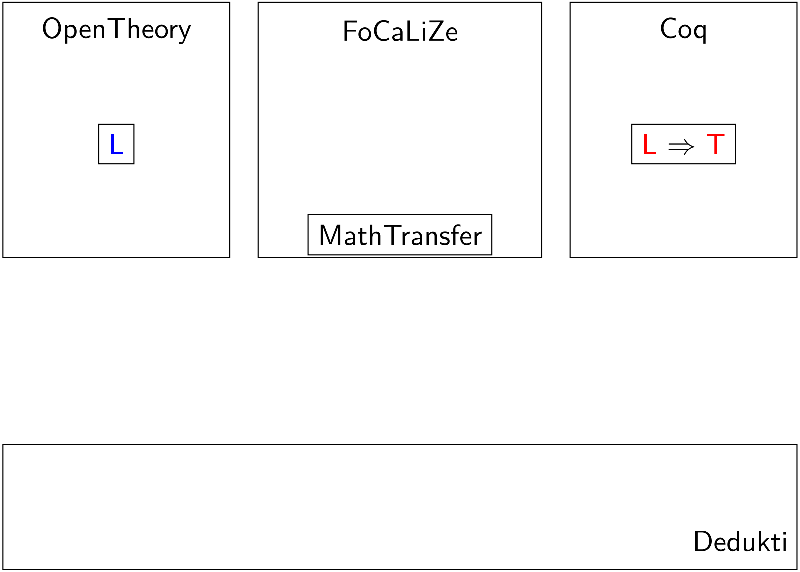
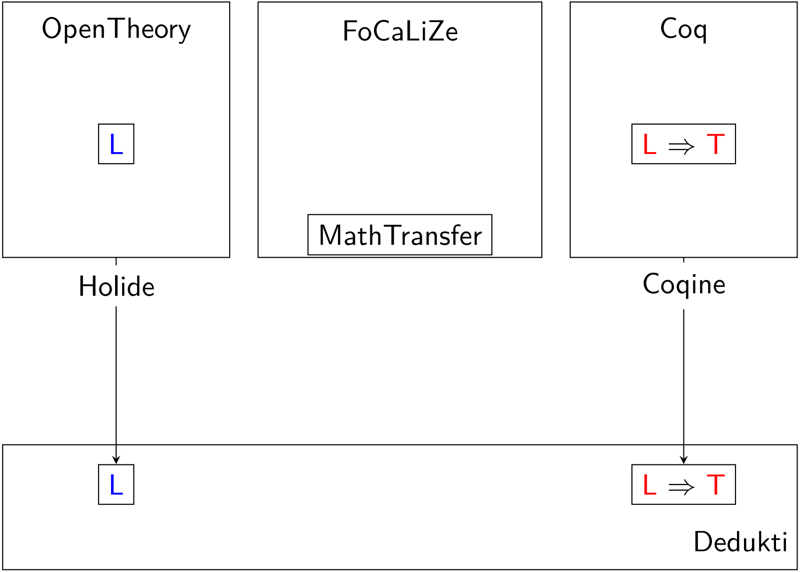

The whole Coq development takes about 1300 lines (31K).

Step 2/8



Step 3/8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FoCaLiZe
MathTransfer
Coq
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Holide
Coqine
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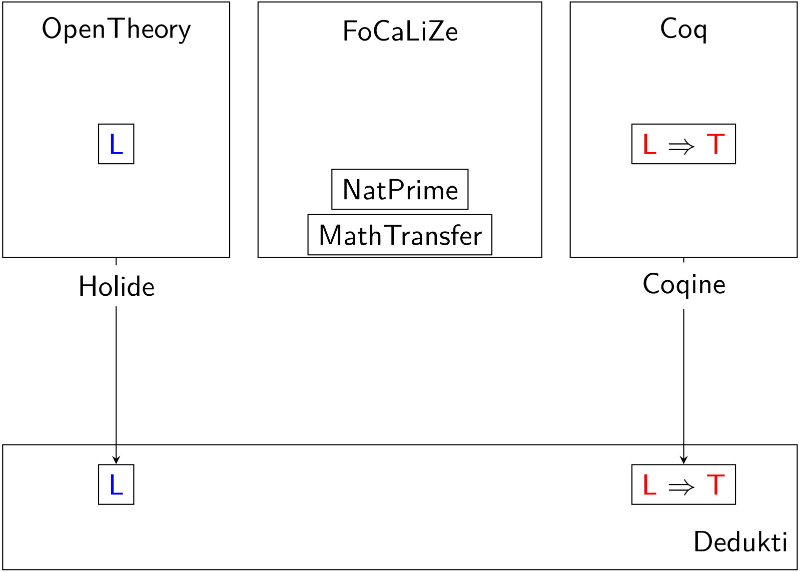
Dedukti

Step 4/8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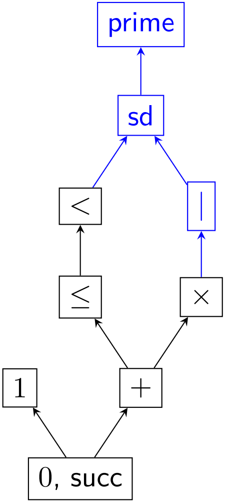
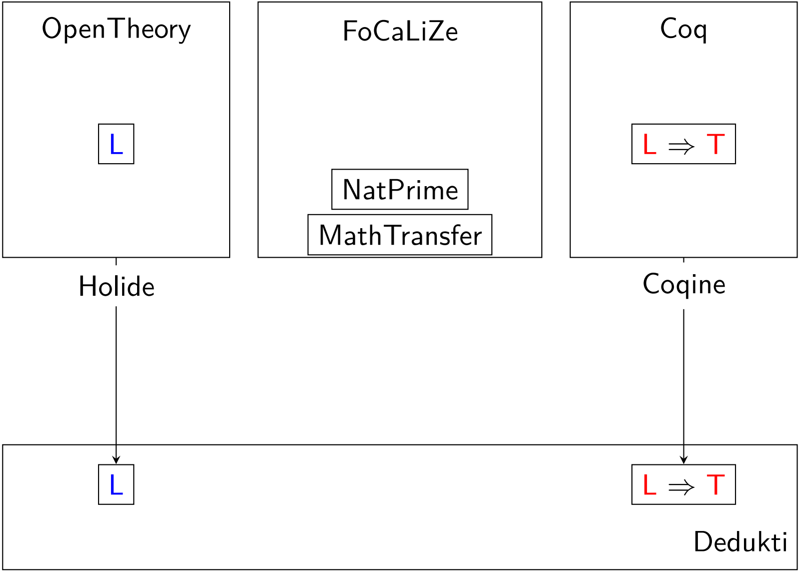
Step 4/8: Extend the MathTransfer hierarchies



- 3 new operations: divisibility, strict divisibility, and primality
- The morphism hierarchy is also extended
- Properties and transfer hierarchies do not need to be extended

Step 4/8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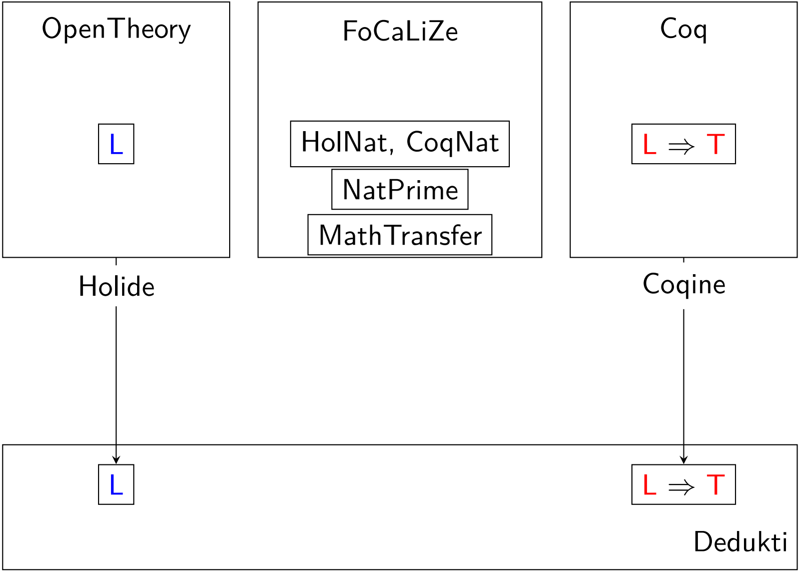
Step 5/8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Step 5/8: Instantiate the hierarchy

$\times$-Transfer(A : $\times$-Properties)
- theorem $\forall m.\ m \times 0 = 0$
- theorem $\forall mn.\ m \times \mathrm{succ}(n) = (m \times n) + m$
- theorem $\forall mnp.\ (m \times n) \times p = m \times (n \times p)$
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$\times$-Properties
- axiom $\forall m.\ m \times 0 = 0$
- axiom $\forall mn.\ m \times \mathrm{succ}(n) = (m \times n) + m$
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$\times$-Axioms
- $\times$: Self $\rightarrow$ Self $\rightarrow$ Self
- axiom $\forall n.\ 0 \times n = 0$
- axiom $\forall mn.\ S(m) \times n = n + (m \times n)$

Step 5/8: Instantiate the hierarchy

$\times$-Coq
let $(\times)$:= external Dedukti "Coq.mult"
theorem $\forall m.\ 0 \times n = 0$:= external Dedukti "Coq.refl"
theorem $\forall mn.\ \mathrm{succ}(m) \times n = (m \times n) + n$:= external Dedukti "Coq.refl"

$\times$-HOL
let $(\times)$:= external Dedukti "HOL.mult"
theorem $\forall m.\ 0 \times n = 0$:= external Dedukti ...
theorem $\forall mn.\ \mathrm{succ}(m) \times n = (m \times n) + n$:= external Dedukti ...
theorem $\forall m.\ m \times 0 = 0$:= external Dedukti ...
theorem $\forall mn.\ m \times \mathrm{succ}(n) = (m \times n) + m$:= external Dedukti ...
theorem $\forall mnp.\ (m \times n) \times p = m \times (n \times p)$:= external Dedukti ...
...

$A :=$ $\times$-HOL

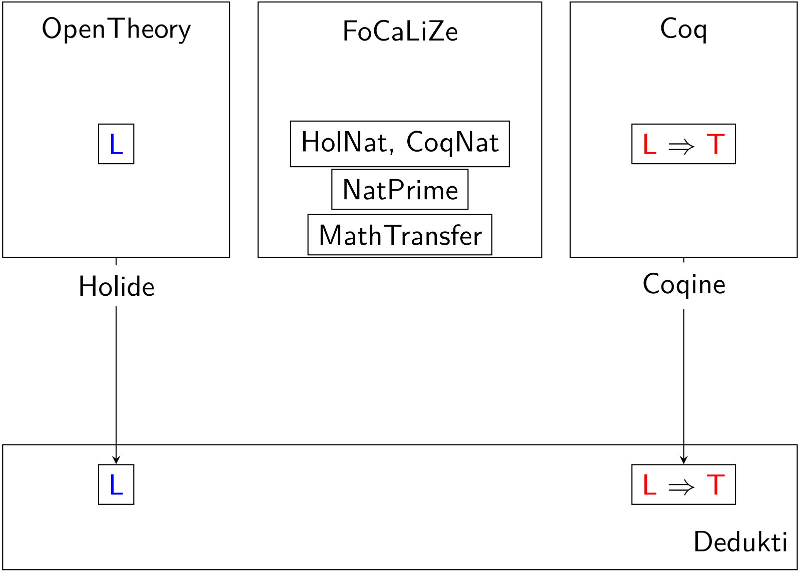
$\times$-Transfer(A : $\times$-Properties)
theorem $\forall m.\ m \times 0 = 0$
theorem $\forall mn.\ m \times \mathrm{succ}(n) = (m \times n) + m$
theorem $\forall mnp.\ (m \times n) \times p = m \times (n \times p)$
...

$\times$-Properties
axiom $\forall m.\ m \times 0 = 0$

Step 5/8: Instantiate the hierarchy

- Instantiation of the OpenTheory / Coq developments takes benefit of Zenon Modulo
  - Coq: $\mathrm{S}(m) \times n = n + (m \times n)$
  - HOL: $\mathrm{S}(m) \times n = (m \times n) + n$
- Morphisms defined using polymorphic function iteration

Step 5/8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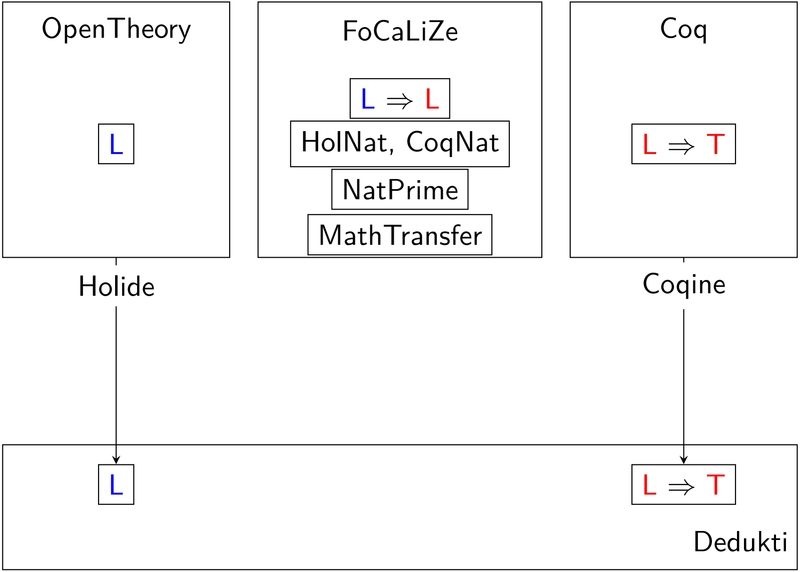
Step 6/8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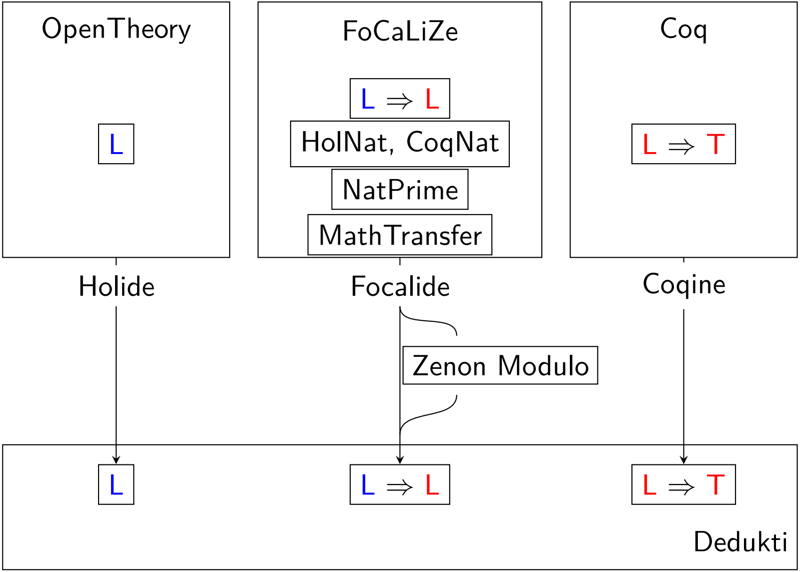
Step 7/8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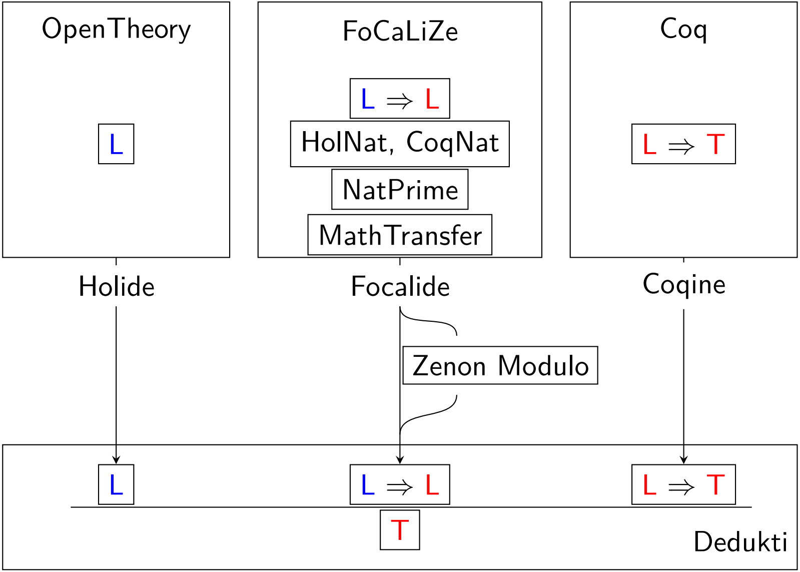
Step 8/8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Case Study

	Coq	FoCaLiZe	Zenon Modulo	Dedukti
Source Code	31K	55K		9K
Generated Dedukti	828K	784K	597K	

Conclusion

- Contributions
  - The MathTransfer Library
  - An Interoperability Methodology
  - Case Study: Sieve of Eratosthenes in HOL + Coq
- Generic in the proof systems as long as they have:
  - Dedukti translators
  - Merged logics (⚠ consistency of $A \cup B$)

Future Work

- Extend the library ($\mathbb{Z}$, $\mathbb{R}$, algebra, data structures)
- Plug other logics/translators
- Improve proof automation
- Automate the discovery of isomorphic structure across formal libraries
- Develop backward translators (ongoing work by Thiré)

Questions?

Thank you for your attention!