Introduction

Constructivization rewrite rules

Combining rewrite systems

Experimental results

Conclusion

A Rewrite System for Proof Constructivization

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Discharging proofs to a theorem prover

- Theorem provers are classical
- Proof assistants for type theory are constructive
- Assume the classical axiom $\longrightarrow$ no proof-as-program
- Restrict to the image of a $\neg\neg$translation $\longrightarrow$ too small

In generated proofs, classical axioms are always used but not always necessary.

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Constructivization Problem

Given a formula φ and a classical proof π of φ, is φ constructively provable?

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Constructivization Problem

Given a formula φ and a classical proof π of φ, is φ constructively provable?

- As hard as constructive provability ($\varphi := \psi \vee P \vee \neg P$)
- Heuristic approach

$\lambda\Pi$-calculus modulo

Extends $\lambda\Pi$-calculus with rewriting:

- Signature Σ contains (well-typed) rewrite rules $\mathcal{R}$
- Conversion is extended

$$\frac{\Sigma;\Gamma \vdash t : A \qquad \Sigma;\Gamma \vdash B : \textbf{Type} \qquad (\text{when } A \equiv_{\mathcal{R}\beta} B)}{\Sigma;\Gamma \vdash t : B}$$

Dedukti: a logical framework based on rewriting

- Universal proof checker
  - interactive and automatic provers
- Purely functional programming language based on rewriting

Proofs are objects, we can compute on them beyond cut elimination.

Syntax of First-Order Logic

$$
\begin{array}{rcl}
A & ::= & \top \\
 & | & \bot \\
 & | & P \\
 & | & A \wedge A \\
 & | & A \vee A \\
 & | & A \Rightarrow A \\
 & | & \forall x. A(x) \\
 & | & \exists x. A(x)
\end{array}
$$

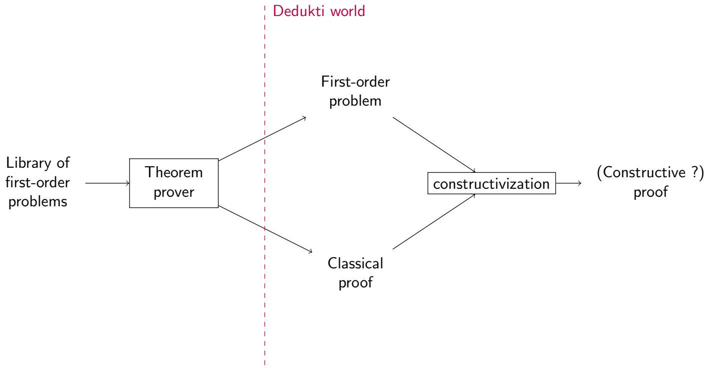
$$\neg A := A \Rightarrow \bot$$

Classical axioms

Classical logic = Intuitionistic logic + any one of those:

- Law of Double Negation: $\underline{\text{dn}}_A : \neg\neg A \Rightarrow A$
- Law of Excluded Middle: $\underline{\text{em}}_A : A \vee \neg A$
- ...

Proof constructivization



A rewrite system for A $\vee$ $\neg$ A

The following statements are constructive theorems:

- $t_\top : \underline{\text{EM}}_\top$
- $t_\bot : \underline{\text{EM}}_\bot$
- $t_\wedge : (\underline{\text{EM}}_A \wedge \underline{\text{EM}}_B) \Rightarrow \underline{\text{EM}}_{A \wedge B}$
- $t_\vee : (\underline{\text{EM}}_A \wedge \underline{\text{EM}}_B) \Rightarrow \underline{\text{EM}}_{A \vee B}$
- $t_\Rightarrow : (\underline{\text{EM}}_A \wedge \underline{\text{EM}}_B) \Rightarrow \underline{\text{EM}}_{A \Rightarrow B}$

where $\underline{\text{EM}}_A := A \vee \neg A$
which leads to the following partial definition of $\underline{\text{em}}$:

A rewrite system for A ∨ ¬ A

$\underline{\text{EM}}_A := A \vee \neg A$
$\underline{\text{em}}_A : \underline{\text{EM}}_A$

- $\underline{\text{em}}_\top \hookrightarrow t_\top$
- $\underline{\text{em}}_\bot \hookrightarrow t_\bot$
- $\underline{\text{em}}_{A \wedge B} \hookrightarrow t_\wedge(\underline{\text{em}}_A, \underline{\text{em}}_B)$
- $\underline{\text{em}}_{A \vee B} \hookrightarrow t_\vee(\underline{\text{em}}_A, \underline{\text{em}}_B)$
- $\underline{\text{em}}_{A \Rightarrow B} \hookrightarrow t_\Rightarrow(\underline{\text{em}}_A, \underline{\text{em}}_B)$

A rewrite system for $A \vee \neg A$

- Nothing smart to do with quantifiers and atoms
- In practice, never yields a constructive proof
- We do not inspect the proof

Example: A ⇒ A

$$\underline{\mathrm{dn}}_{A\Rightarrow A} \\ (\lambda H_{\neg(A\Rightarrow A)}.H_{\neg(A\Rightarrow A)} \\ (\lambda H_A\,.E_{\perp}^{A}((\lambda H'_A.H_{\neg(A\Rightarrow A)}(\lambda H''_A.H'_A))H_A\,)))$$

Example: A ⇒ A

$$\underline{\mathrm{dn}}_{A\Rightarrow A}$$
$$(\lambda H_{\neg(A\Rightarrow A)}.H_{\neg(A\Rightarrow A)}$$
$$(\lambda H_A.E_{\perp}^{A}((\lambda H'_A.H_{\neg(A\Rightarrow A)}(\lambda H''_A.H'_A))H_A)))$$

$$\underline{\mathrm{dn}}_{A\Rightarrow B} \hookrightarrow d_{\Rightarrow}(\underline{\mathrm{dn}}_A, \underline{\mathrm{dn}}_B)$$

Example: $A \Rightarrow A$

$$
\begin{array}{l}
\lambda H_A.\underline{\mathsf{dn}}_A(\lambda H_{\neg A}. \\
\quad (\lambda H_{\neg(A\Rightarrow A)}.H_{\neg(A\Rightarrow A)} \\
\qquad (\lambda H'''_A.E^A_\bot((\lambda H'_A.H_{\neg(A\Rightarrow A)}(\lambda H''_A.H'_A))H'''_A))) \\
\quad (\lambda H_{A\Rightarrow A}.H_{\neg A}(H_{A\Rightarrow A}\ H_A))
\end{array}
$$

Example: $A \Rightarrow A$

$$\begin{array}{l}\lambda H_A.\underline{\mathrm{dn}}_A(\lambda H_{\neg A}.\\ \quad (\lambda H_{\neg(A\Rightarrow A)}.H_{\neg(A\Rightarrow A)}\\ \qquad (\lambda H'''_A.E_\bot^A((\lambda H'_A.H_{\neg(A\Rightarrow A)}(\lambda H''_A.H'_A))H'''_A)))\\ \quad (\lambda H_{A\Rightarrow A}.H_{\neg A}(H_{A\Rightarrow A}\ H_A))\end{array}$$

$$E_\bot^A(\pi_{\neg A}\ \pi_A) \hookrightarrow \pi_A$$

Example: A $\Rightarrow$ A

$$
\begin{array}{l}
\lambda H_A.\underline{\mathsf{dn}}_A(\lambda H_{\neg A}. \\
\quad (\lambda H_{\neg(A\Rightarrow A)}.H_{\neg(A\Rightarrow A)} \\
\qquad (\lambda H'''_A.H'''_A)) \\
\quad (\lambda H_{A\Rightarrow A}.H_{\neg A}(H_{A\Rightarrow A}\ H_A))
\end{array}
$$

Example: A ⇒ A

$$\lambda H_A.\underline{\text{dn}}_A(\lambda H_{\neg A}.\\ (\lambda H_{\neg(A\Rightarrow A)}.H_{\neg(A\Rightarrow A)}\\ (\lambda H'''_A.H'''_A))\\ (\lambda H_{A\Rightarrow A}.H_{\neg A}(H_{A\Rightarrow A}\ H_A))$$

$$(\lambda H_A.\pi_B(H_A))\ \pi_A \hookrightarrow \pi_B(\pi_A)$$

Example: $A \Rightarrow A$

$$\lambda H_A.\underline{\mathsf{dn}}_A(\lambda H_{\neg A}.\\ (\lambda H_{A\Rightarrow A}.H_{\neg A}(H_{A\Rightarrow A}\ H_A))\\ (\lambda H'''_A.H'''_A))$$

Example: A $\Rightarrow$ A

$$\lambda H_A.\underline{\mathrm{dn}}_A(\lambda H_{\neg A}. \\ (\lambda H_{A\Rightarrow A}.H_{\neg A}(H_{A\Rightarrow A}\ H_A)) \\ (\lambda H'''_A.H'''_A))$$

$$(\lambda H_A.\pi_B(H_A))\ \pi_A \hookrightarrow \pi_B(\pi_A)$$

Example: $A \Rightarrow A$

$$\lambda H_A . \underline{\mathsf{dn}}_A (\lambda H_{\neg A} . \\ H_{\neg A} ((\lambda H'''_A . H'''_A) \; H_A))$$

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Example: A ⇒ A

$$\lambda H_A.\underline{\text{dn}}_A(\lambda H_{\neg A}.\; H_{\neg A}((\lambda H'''_A.H'''_A)\ H_A))$$

$$(\lambda H_A.\pi_B(H_A))\ \pi_A \hookrightarrow \pi_B(\pi_A)$$

Example: $A \Rightarrow A$

$$\lambda H_A.\underline{\mathsf{dn}}_A(\lambda H_{\neg A}.\ H_{\neg A}\ H_A)$$

Example: $A \Rightarrow A$

$$\lambda H_A.\underline{\text{dn}}_A(\lambda H_{\neg A}. H_{\neg A}\ H_A)$$

$$\underline{\text{dn}}_A(\lambda H_{\neg A}.H_{\neg A}\ \pi_A) \hookrightarrow \pi_A$$

Example: A $\Rightarrow$ A

$\lambda H_A.H_A$

A rewrite system for ¬¬ A ⇒ A

The following statements are constructive theorems:

- $d_\top : \underline{\mathrm{DN}}_\top$
- $d_\bot : \underline{\mathrm{DN}}_\bot$
- $d_\wedge : (\underline{\mathrm{DN}}_A \wedge \underline{\mathrm{DN}}_B) \Rightarrow \underline{\mathrm{DN}}_{A \wedge B}$
- $d_\Rightarrow : \underline{\mathrm{DN}}_B \Rightarrow \underline{\mathrm{DN}}_{A \Rightarrow B}$
- $d_\forall : (\forall x. \underline{\mathrm{DN}}_{P(x)}) \Rightarrow \underline{\mathrm{DN}}_{\forall x. P(x)}$

where $\underline{\mathrm{DN}}_A := \neg\neg A \Rightarrow A$
which leads to the following partial definition of $\underline{\mathrm{dn}}$:

A rewrite system for $\neg\neg A \Rightarrow A$

$\underline{\mathrm{DN}}_A := \neg\neg A \Rightarrow A$
$\underline{\mathrm{dn}}_A : \underline{\mathrm{DN}}_A$

- $\underline{\mathrm{dn}}_\top \hookrightarrow d_\top$
- $\underline{\mathrm{dn}}_\bot \hookrightarrow d_\bot$
- $\underline{\mathrm{dn}}_{A \wedge B} \hookrightarrow d_\wedge(\underline{\mathrm{dn}}_A, \underline{\mathrm{dn}}_B)$
- $\underline{\mathrm{dn}}_{A \Rightarrow B} \hookrightarrow d_\Rightarrow(\underline{\mathrm{dn}}_B)$
- $\underline{\mathrm{dn}}_{\forall x.P(x)} \hookrightarrow d_\forall(\lambda x.\underline{\mathrm{dn}}_{P(x)})$

A rewrite system for ¬¬ A ⇒ A

$\underline{\text{DN}}_A := \neg\neg A \Rightarrow A$
$\underline{\text{dn}}_A : \underline{\text{DN}}_A$

- $\underline{\text{dn}}_\top \hookrightarrow d_\top$
- $\underline{\text{dn}}_\bot \hookrightarrow d_\bot$
- $\underline{\text{dn}}_{A \wedge B} \hookrightarrow d_\wedge(\underline{\text{dn}}_A, \underline{\text{dn}}_B)$
- $\underline{\text{dn}}_{A \Rightarrow B} \hookrightarrow d_\Rightarrow(\underline{\text{dn}}_B)$
- $\underline{\text{dn}}_{\forall x.P(x)} \hookrightarrow d_\forall(\lambda x.\underline{\text{dn}}_{P(x)})$
- $\underline{\text{dn}}_A\ (\lambda H_{\neg A}.H_{\neg A}\ \pi_A) \hookrightarrow \pi_A$

A rewrite system for ¬¬ A ⇒ A

$\underline{\mathrm{DN}}_A := \neg\neg A \Rightarrow A$
$\underline{\mathrm{dn}}_A : \underline{\mathrm{DN}}_A$

- $\underline{\mathrm{dn}}_\top \hookrightarrow d_\top$
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- $\underline{\mathrm{dn}}_{A \wedge B} \hookrightarrow d_\wedge(\underline{\mathrm{dn}}_A, \underline{\mathrm{dn}}_B)$
- $\underline{\mathrm{dn}}_{A \Rightarrow B} \hookrightarrow d_\Rightarrow(\underline{\mathrm{dn}}_B)$
- $\underline{\mathrm{dn}}_{\forall x.P(x)} \hookrightarrow d_\forall(\lambda x.\underline{\mathrm{dn}}_{P(x)})$
- $\underline{\mathrm{dn}}_A\ (\lambda H_{\neg A}.H_{\neg A}\ \pi_A) \hookrightarrow \pi_A$
- $\underline{\mathrm{dn}}_A\ (\lambda \_.\pi_\bot) \hookrightarrow E_\bot^A(\pi_\bot)$

Elimination of negation proofs

Higher-order rewrite rules

- $\underline{\text{dn}}_A\ (\lambda H_{\neg A}.H_{\neg A}\ \pi_A) \hookrightarrow \pi_A$
- $\underline{\text{dn}}_A\ (\lambda \_.\pi_\bot) \hookrightarrow E_\bot^A(\pi_\bot)$

Elimination of negation proofs

Higher-order rewrite rules

- $\underline{\mathsf{dn}}_A\ (\lambda H_{\neg A}.H_{\neg A}\ \pi_A) \hookrightarrow \pi_A$
- $\underline{\mathsf{dn}}_A\ (\lambda\_.\pi_\perp) \hookrightarrow E_\perp^A(\pi_\perp)$
- $E_\perp^A(\pi_{\neg A}\ \pi_A) \hookrightarrow \pi_A$

Elimination of negation proofs

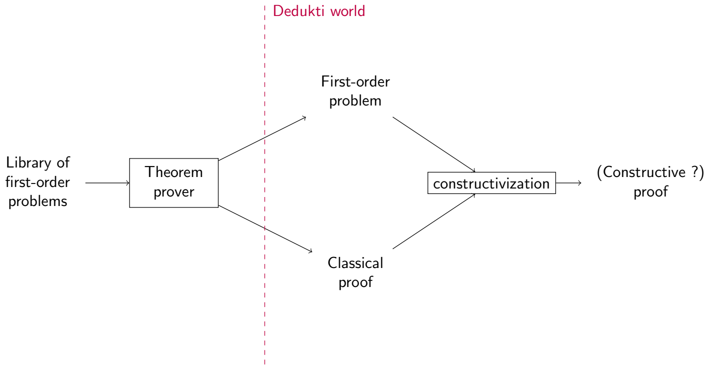
Higher-order rewrite rules

- $\underline{\text{dn}}_A\ (\lambda H_{\neg A}.H_{\neg A}\ \pi_A) \hookrightarrow \pi_A$
- $\underline{\text{dn}}_A\ (\lambda\_.\pi_\bot) \hookrightarrow E_\bot^A(\pi_\bot)$
- $E_\bot^A(\pi_{\neg A}\ \pi_A) \hookrightarrow \pi_A$
- $E_\bot^{A\Rightarrow B}(\pi_\bot)\ \pi_A \hookrightarrow E_\bot^B(\pi_\bot)$
- $\lambda H_A.E_\bot^B(\pi_\bot) \hookrightarrow E_\bot^{A\Rightarrow B}(\pi_\bot)$

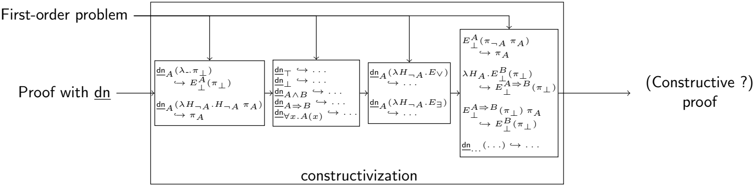
Confluence

- $\lambda H_\top . H_\top \hookleftarrow \lambda H_\top . \underline{\mathsf{dn}}_\top \; (\lambda H_{\neg\top} . H_{\neg\top} \; H_\top) \hookrightarrow \lambda H_\top . I_\top$

The constructivizatoin pipeline



Zooming at the constructivization box



Meta-Dedukti

Dedukti is good at writing Dedukti code.

- Normalize the same term with respect to several successive rewrite systems
- Rewrite system = meta-programming stage
- Confluence of the last

Results

Timeout: 10s (Zenon) and 10min (Dedukti)
Memory limit: 2GB
Library: TPTP v6.3.0

Problems	6528
Classical proofs	1258
Normalized proofs	1240
Constructive proofs	856
Constructivization rate	68%

Related work

- Zenonide
  - Constructivization in sequent calculus
  - Specific to Zenon
  - Similar performances (66.8%)
  - Complementary (76% combined)
- iLeanCop
  - Best intuitionistic theorem prover according to ILTP
  - No proof output
  - More ambitious: complete for first-order intuitionistic logic
  - Surprise: we are competitive

Conclusion

- Simple heuristics for proof constructivization
- Dedukti as a meta-language for transforming proofs
- Constructivizators can be chained
- Most classical proofs generated by Zenon are constructive
- Theorems = total functions, Axioms = partial functions

Future work

- Syntax for the meta-language
- Try other provers (iProver Modulo, VeriT)
- Deduction modulo
- Higher-order
- Intermediate logics
- Other axioms (extensionality, univalence, choice)

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- Syntax for the meta-language
- Try other provers (iProver Modulo, VeriT)
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Thank you for your attention!