Confluence algebras and acyclicity of the Koszul complex

Cyrille Chenavier

August 27, 2017

Summary

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Motivation: homological algebra

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- Objective: compute homological invariants (Tor-Ext groups, Poincaré series...) of **A**.
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$$\cdots \xrightarrow{\partial_{n+1}} \mathbf{A}[X_n] \xrightarrow{\partial_n} \mathbf{A}[X_{n-1}] \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} \mathbf{A}[X_1] \xrightarrow{\partial_1} \mathbf{A} \xrightarrow{\partial_0} \mathbb{K} \longrightarrow 0,$$

where

  - $\mathbf{A}[X_n]$ are free **A**-modules spanned by sets X_n,
  - ∂_n are **A**-linear maps satisfying

$$\mathrm{im}(\partial_{n+1}) = \ker(\partial_n), \text{ for every } n \geq 0.$$

Tor groups

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- The n-th Tor group of $\mathbf{A}$ is the vector space

$$\mathrm{Tor}_n^{\mathbf{A}}(\mathbb{K}, \mathbb{K}) = \frac{\ker\left(\mathrm{Id}_{\mathbb{K}} \otimes \partial_n\right)}{\mathrm{im}\left(\mathrm{Id}_{\mathbb{K}} \otimes \partial_{n+1}\right)}.$$

Koszulness property I

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- This graduation induces a graduation on the Tor groups:

$$\mathrm{Tor}_n^{\mathbf{A}}(\mathbb{K},\mathbb{K}) = \bigoplus_{i\in\mathbb{N}} \mathrm{Tor}_{n,(i)}^{\mathbf{A}}(\mathbb{K},\mathbb{K}).$$

Koszulness property II

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$$\ell_N = (0, 1, N, N+1, 2N, 2N+1, 3N, 3N+1, \cdots).$$

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Proposition [Berger, Marconnet 2004]

The n-th Tor *group of* **A** *lives in degree greater than $\ell_N(n)$:*

$$\mathrm{Tor}^{\mathbf{A}}_{n,(i)}(\mathbb{K}, \mathbb{K}) = \{0\}, \text{ for every } i < \ell_N(n).$$

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Definition [Priddy 1970, Berger 2001]

The algebra **A** is said to be N-Koszul if

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Techniques to prove Koszulness

We wish to construct free resolutions

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If we construct such a resolution such that each X_i is a set of elements of degree $\ell_N(i)$, then the algebra $\mathbf{A}$ is Koszul.

Example: the Yang-Mills algebra over 2 generators

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- We have exactly one relation between relations relations:

$$\begin{aligned} r :=& y\,(yxx - 2xyx + xxy) + x\,(yyx - 2yxy + xyy) \\ =& (yyx - 2yxy + xyy)\,x + (yxx - 2xyx + xxy)\,y. \end{aligned}$$

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  - $\mathrm{Tor}_3^{\mathbf{A}}\left(\mathbb{K}, \mathbb{K}\right) \simeq \mathbb{K}r$,
  - $\mathrm{Tor}_4^{\mathbf{A}}\left(\mathbb{K}, \mathbb{K}\right) = \{0\}$.

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  - $\mathrm{Tor}_3^{\mathbf{A}}(\mathbb{K},\mathbb{K}) \simeq \mathbb{K}r_1 \oplus \mathbb{K}r_2$.

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  - when this complex is a resolution, it permits to compute the Tor groups,
  - the algebras for which this complex is a resolution are precisely the (2-)Koszul algebras.
- In 2001, Berger extends the definition of the Koszul complex to the case of N-homogeneous algebras,
  - this complex provides a characterisation of the property of Koszulness.

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- For every $n \geq 1$, J_n is included in $\mathbf{A} \otimes J_{n-1}$. This inclusion induces an $\mathbf{A}$-linear map $\mathbf{A} \otimes J_n \xrightarrow{\partial_n} \mathbf{A} \otimes J_{n-1}$.

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The Koszul complex of $\mathbf{A}$ is the sequence

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\begin{aligned}
r &= y\,(yxx - 2xyx + xxy) + x\,(yyx - 2yxy + xyy) \\
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- $\mathbf{A} \otimes J_3 \xrightarrow{\partial_3} \mathbf{A} \otimes \overline{R}$ is defined by

$$\partial_3\,(1 \otimes r) = \overline{y} \otimes (yxx - 2xyx + xxy) + \overline{x} \otimes (yyx - 2yxy + xyy)\,.$$

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- $\partial_2\,(\partial_3\,(1 \otimes r))$ is equal to

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Koszulness and the Koszul complex

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Theorem [Berger 2001]

Let $\mathbf{A}$ be an N-homogeneous algebra. Then, $\mathbf{A}$ is Koszul if and only if its Koszul complex is a resolution of $\mathbb{K}$.

A theorem of acyclicity

Theorem

Let $\boldsymbol{A}$ *be an N-homogeneous algebra admitting an extra-confluent presentation. Then, there exists a contracting homotopy for the Koszul complex of* $\boldsymbol{A}$*.*

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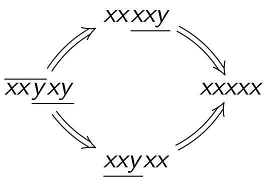
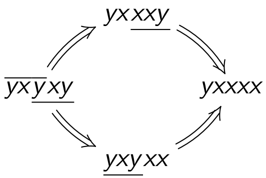
Let $\mathbf{A}\,\langle x, y \mid\ yxy \Longrightarrow yxx,\ xxy \Longrightarrow xxx \rangle$.

We have two critical branchings:



and



Extra-confluent presentations II

We have:

$$xx(yxy - yxx) - xx(xxy - xxx) = (xxy - xxx)xy - (xxy - xxx)xx$$
$$yx(yxy - yxx) - yx(xxy - xxx) = (yxy - yxx)xy - (yxy - yxx)xx.$$

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$$0 \xrightarrow{\partial_3} \mathbf{A} \otimes \overline{R} \xrightarrow{\partial_2} \mathbf{A} \otimes V \xrightarrow{\partial_1} \mathbf{A} \xrightarrow{\partial_0} \mathbb{K} \longrightarrow 0.$$

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- We have

$$\begin{aligned} & \partial_2 \left(\overline{xx} \otimes (yxy - yxx) - \overline{xx} \otimes (xxy - xxx) \right) \\ =& \overline{(xxy - xxx)x} \otimes y - \overline{(xxy - xxx)x} \otimes x \\ =& 0. \end{aligned}$$

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Let $\mathbf{A}\langle X \mid w_1 \Longrightarrow f_1, \cdots, w_r \Longrightarrow f_r \rangle$ be an N-homogeneous algebra.

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$$S(w) = \begin{cases} f_i, & \text{if } w = w_i \text{ for } 1 \leq i \leq r \\ w, & \text{otherwise.} \end{cases}$$

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- The presentation $\langle X \mid w_1 \Longrightarrow f_1, \cdots, w_r \Longrightarrow f_r \rangle$ is side-confluent if for every $N+1 \leq m \leq 2N-1$, there exists k such that

$$\underbrace{\cdots S_r^{(m)} S_l^{(m)} S_r^{(m)}}_{k \ \ terms} = \underbrace{\cdots S_l^{(m)} S_r^{(m)} S_l^{(m)}}_{k \ \ terms}.$$

The extra-condition

- Let $\mathbf{A}\langle X \mid R\rangle$ be an N-homogeneous algebra. The presentation $\langle X \mid R\rangle$ satisfies the extra-condition if for every $2 \leq m \leq N-1$, we have
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- A side-confluent presentation satisfying the extra-condition is said to be extra-confluent.

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- Let $\mathbf{A}\langle X \mid R\rangle$ be an N-homogeneous algebra admitting a side-confluent presentation. There exists $c \in A_k$ and morphisms of algebras

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such that $\varphi_n(c)$ provides a map $\mathbf{A} \otimes J_n \xrightarrow{h_n} \mathbf{A} \otimes J_{n+1}$:

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$$\cdots \underset{\partial_4}{\overset{h_3}{\rightleftarrows}} \mathbf{A} \otimes \left(V \otimes \overline{R} \cap \overline{R} \otimes V\right) \underset{\partial_3}{\overset{h_2}{\rightleftarrows}} \mathbf{A} \otimes \overline{R} \underset{\partial_2}{\overset{h_1}{\rightleftarrows}} \mathbf{A} \otimes V \underset{\partial_1}{\overset{h_0}{\rightleftarrows}} \mathbf{A} \underset{\partial_0}{\rightarrow} \mathbb{K} \longrightarrow 0.$$

Representations of confluence algebras 2

Proposition

Let $\boldsymbol{A}\langle X \mid R\rangle$ be an N-homogeneous algebra admitting a side-confluent presentation. If the extra-condition holds, the family $(h_n)_n$ is a contracting homotopy for the Koszul complex of $\boldsymbol{A}$.

Thank you for listening.