Reduction Operators:
Rewriting Properties and Completion

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1 Introduction

We propose an approach on linear rewriting where applications of rewriting rules are replaced by applications of specific linear maps called *reduction operators*. These operators were introduced by Berger for finite dimensional vector spaces. His motivation was to study the homology of a special class of algebras called *finitely generated homogeneous algebras*. The elements of these algebras are non-commutative polynomials over a finite number of variables modulo the congruence spanned by a set of oriented homogeneous relations having the same degree. By degree, we mean the one induced by the length of non-commutative monomials. The later are identified to words. Berger considered the linear endomorphism mapping every left-hand side of a rewrite rule to its right-hand side. This is an endomorphism of the vector space spanned by words whose length is the degree of the rewriting rules. The number of variables being finite, this vector space is of finite dimension. It turns out that the endomorphism described previously is a reduction operator. Berger also proved that the set of reduction operators admits a lattice structure. We point out that in order to obtain this structure, he strongly used the fact that he considered reduction operators relative to finite dimensional vector spaces. He deduces from this structure a lattice formulation of the confluence. Using this point of view, one can study the homological property of *Koszulness* ([1], [2], [3], [8], [6]). For the definition of Koszulness, we refer the reader to [10] and [3].

In the next section, we propose to develop a notion of reduction operator for non-necessary finite dimensional vector spaces. Our motivation is that we want to use the theory of reduction operators to study non necessary homogeneous algebras. For such algebras, we do not have any bound for the degree of a word appearing in a rewriting rule. Hence, the operator described in the previous paragraph is an endomorphism of the vector space spanned by all words which is infinite dimensional. Moreover, in order to study the homological properties of algebras using reduction operators, we also want to introduce a lattice structure on the set of reduction operators. Using this lattice structure, we formulate the notion of confluence as did Berger in the finite dimensional case. Moreover, we link this notion of confluence to the classical one coming from rewriting theory. For that, we formulate the notion of Church-Rosser property in terms of reduction operators.

In the last section, we formulate the notion of completion in terms of reduction operators. With a rewriting point of view, a completion of a rewriting system is provided adding new rules to this system in such a way that the obtained relation is confluent and induces the same equivalence relation than the initial one. Here, what we want to complete is a set F of reduction operators. Naturally, a completion of F has to be a confluent set F' containing F. Moreover, we need to formulate the fact that the new set F' does not change the equivalence relation induced by F. We will see that this condition is formulated as follows: the lower bounds of the sets F and F' are equal. We will also study the question of the existence of a completion. We will introduce an operator C^F called the *F-complement* and state that the set $F \cup \{C^F\}$ is a completion of F. With a rewriting point of view, when we want to obtain a completion, we apply an algorithm: the Knuth-Bendix completion algorithm for term rewriting ([7]) or the Buchberger algorithm for Gröbner bases ([4], [5], [9]). Here, the completion is not obtained applying an algorithm but using an algebraic construction. Finally, we formulate the notion of an algebra admitting a *presentation by operator*. We also formulate the notion of *confluent* presentation by operator and link it to the one of Gröbner basis. We deduce a way to construct Gröbner bases using reduction operators.

2 Reduction Operators

2.1. Notations. We denote by $\mathbb{K}$ a commutative field. Throughout the paper, we fix a well-ordered set $(G,<)$. We denote by $\mathbb{K}G$ the vector space spanned by G: the non-zero elements are the finite formal linear combinations of elements of G with coefficients in $\mathbb{K}$. For every $v \in \mathbb{K}G \setminus \{0\}$, there exist a unique finite subset S_v of G and a unique family of non zero scalars $(\lambda_g)_{g\in S_v}$ such that v is equal to $\sum_{g\in S_v} \lambda_g g$. The order on G being total, the set S_v admits a greatest element, written $\lg(v)$. The element $\lg(v)$ is the *leading generator* of v. We extend the order $<$ on G into a partial order on $\mathbb{K}G$ in the following way: we have $u < v$ if $u = 0$ and v is different from 0 or if $\lg(u) < \lg(v)$.

2.2. Reduction Operators. A linear endomorphism T of $\mathbb{K}G$ is a *reduction operator relative to* $(G,<)$ if it is idempotent and if for every $g \in G$, we have $T(g) \leq g$. We denote by $\mathbf{RO}(G,<)$ the set of reduction operators relative to $(G,<)$. Given a reduction operator T, a generator g is said to be *T-reduced* if $T(g)$ is equal to g. We denote by $\mathrm{Red}(T)$ the set of T-reduced generators and by $\mathrm{Nred}(T)$ the complement of $\mathrm{Red}(T)$ in G.

2.3. Remark. Let T be a reduction operator relative to $(G,<)$. The image of T is equal to $\mathbb{K}\mathrm{Red}(T)$.

2.4. The G-Map. Our aim is to equip the set $\mathbf{RO}(G,<)$ with a lattice structure. To define it, let $\mathscr{L}(\mathbb{K}G)$ be the set of subspaces of $\mathbb{K}G$. The map

$$\begin{aligned} \theta\colon \mathbf{RO}(G,<) &\longrightarrow \mathscr{L}(\mathbb{K}G), \\ T &\longmapsto \ker(T) \end{aligned}$$

is the *G-map*. We have:

2.5. Proposition. *The G-map is a bijection.*

2.6. Lattice Structure. We consider the binary relation on $\mathbf{RO}(G,<)$ defined by

$$T_1 \preceq T_2 \text{ if, and only if } \ker(T_2) \subset \ker(T_1).$$

This relation is reflexive and transitive. From Proposition 2.5, it is also anti-symmetric. Hence, it is an order relation on $\mathbf{RO}(G,<)$. Let us equip $\mathbf{RO}(G,<)$ with a lattice structure. The lower bound $T_1 \wedge T_2$ and the upper bound $T_1 \vee T_2$ of two elements T_1 and T_2 of $\mathbf{RO}(G,<)$ are defined in the following manner:

$$\begin{cases} T_1 \wedge T_2 = \theta^{-1}(\ker(T_1) + \ker(T_2)), \\ T_1 \vee T_2 = \theta^{-1}(\ker(T_1) \cap \ker(T_2)). \end{cases}$$

Our aim is to formulate the notion of confluence using this lattice structure. For that, we need the following:

2.7. Lemma. *Let T_1 and T_2 be two reduction operators relative to $(G,<)$. Then, we have:*

$$T_1 \preceq T_2 \Longrightarrow \mathrm{Red}\,(T_1) \ \subset\ \mathrm{Red}\,(T_2)\,.$$

2.8. Obstructions. Let F be a subset of $\mathbf{RO}\,(G,<)$. We let

$$\mathrm{Red}\,(F) = \bigcap_{T\in F} \mathrm{Red}\,(T) \ \text{ and } \ \wedge F = \theta^{-1}\left(\sum_{T\in F} \ker\,(T)\right).$$

For every $T \in F$, we have $\wedge F \preceq T$. Thus, from Lemma 2.7, the set $\mathrm{Red}\,(\wedge F)$ is included in $\mathrm{Red}\,(T)$ for every $T \in F$, so that it is included in $\mathrm{Red}\,(F)$. We write

$$\mathrm{Obs}^F_{\mathrm{red}} = \mathrm{Red}\,(F) \setminus \mathrm{Red}\,(\wedge F)\,. \tag{1}$$

2.9. Confluence. A subset F of $\mathbf{RO}\,(G,<)$ is said to be *confluent* if $\mathrm{Obs}^F_{\mathrm{red}}$ is the empty set.

2.10. Reduction Operators and Abstract Rewriting. Consider the abstract rewriting system $\left(\mathbb{K}G, \underset{F}{\longrightarrow}\right)$ defined by $v \underset{F}{\longrightarrow} v'$ if, and only if there exists $T \in F$ such that v does not belong to $\mathrm{im}\,(T)$ and v' is equal to $T(v)$. Our aim is to explain that F is confluent if, and only if it is so for this rewriting system. For that, we introduce the following definition:

2.11. Church-Rosser Property. We denote by $\langle F\rangle$ the submonoid of $(\mathrm{End}\,(\mathbb{K}G)\,,\circ)$ spanned by F. Let v and v' be two elements of $\mathbb{K}G$. We say that *v rewrites into v'* if there exists $R \in \langle F\rangle$ such that v' is equal to $R(v)$. We say that F has the *Church-Rosser property* if for every $v \in \mathbb{K}G$, v rewrites into $\wedge F(v)$. The following result is the analogous of the Church-Rosser theorem for reduction operators:

2.12. Theorem. *A subset of* $\mathbf{RO}\,(G,<)$ *is confluent if, and only if it has the Church-Rosser property.*

2.13. Equivalence Relations. We denote by $\overset{*}{\underset{F}{\longleftrightarrow}}$ the reflexive transitive symmetric closure of $\underset{F}{\longrightarrow}$. We easily show that we have $v \overset{*}{\underset{F}{\longleftrightarrow}} v'$ if, and only if $v - v'$ belongs to the kernel of $\wedge F$. We deduce that F has the Church-Rosser property if, and only if it is so for $\underset{F}{\longrightarrow}$. From Theorem 2.12, we get:

2.14. Proposition. *Let F be a subset of* $\mathbf{RO}\,(G,<)$. *Then, F is confluent if, and only if it is so for* $\underset{F}{\longrightarrow}$.

3 Completion and Presentations by Operators

We fix a subset F of $\mathbf{RO}\,(G,<)$.

3.1. Definition.

1. A *completion of F* is a subset F' of $\mathbf{RO}\,(G,<)$, such that
   (a) F' is confluent,
   (b) $F \ \subset\ F'$ and $\wedge F' = \wedge F$.
2. A *complement of F* is an element C of $\mathbf{RO}\,(G,<)$ such that
   (a) $(\wedge F) \wedge C = \wedge F$,

(b) $\mathrm{Obs}^F_{\mathrm{red}} \subset \mathrm{Nred}\,(C)$.

A complement is said to be *minimal* if the inclusion 2b is an equality.

The link between a complement and a completion is the following:

3.2. Proposition. *Let $C \in \mathbf{RO}\,(G,<)$ such that $(\wedge F) \wedge C$ is equal to $\wedge F$. The set $F \cup \{C\}$ is a completion of F if, and only if C is a complement of F.*

3.3. Remark. The operator $\wedge F$ is a complement of F. However, in general, this complement is not minimal. Our aim is to exhibit a minimal complement.

3.4. The F-Complement. Letting $\vee\overline{F} = \theta^{-1}\left(\mathbb{K}\mathrm{Red}\,(F)\right)$, the operator

$$C^F = (\wedge F) \vee \left(\vee\overline{F}\right),$$

is the *F-complement.*

3.5. Theorem. *Let F be a subset of* $\mathbf{RO}\,(G,<)$. *The F-complement is a minimal complement of F.*

3.6. Monomial Orders. As an application, we want to formulate the notion of algebra presented by a reduction operator. By algebra, we mean associative, unitary $\mathbb{K}$-algebra. First, we recall the notion of *monomial order.* Let X be a set. Let X^* be the free monoid spanned by X, that is, the set of words written with the alphabet X. A monomial order on X^* is a well-founded total order $<$ on X^* such that the following conditions are fulfilled:

1. $1 < w$ for every word w different from 1,
2. for every $w_1,\ w_2,\ w,\ w' \in X^*$ such that $w < w'$, we have $w_1ww_2 < w_1w'w_2$.

3.7. Presentation by Operator. Let $\mathbf{A}$ be an algebra. A *presentation by operator* of $\mathbf{A}$ is a triple $\langle X, <\mid S\rangle$ where

1. X is a set and $<$ is a monomial order $<$ on X^*,
2. S is a reduction operator relative to $(X^*, <)$,
3. we have an isomorphism of algebras

$$\mathbf{A} \simeq \frac{\mathbb{K}X^*}{I\,(\ker\,(S))},$$

where $I\,(\ker\,(S))$ is the two-sided ideal of $\mathbb{K}X^*$ spanned by $\ker\,(S)$.

3.8. The Reduction Family of a Presentation. Let X be a set and let n be an integer. We denote by $X^{(n)}$ (respectively, $X^{(\leq n)}$) the set of words of length n (respectively, smaller than n). Let $\langle X, <\mid S\rangle$ be a presentation by operator of $\mathbf{A}$. For every integers n and m such that (n, m) is different from $(0, 0)$, we let

$$T_{n,m} = \mathrm{Id}_{\mathbb{K}X^{(\leq n+m-1)}} \ \oplus\ \mathrm{Id}_{\mathbb{K}X^{(n)}} \otimes S \otimes \mathrm{Id}_{\mathbb{K}X^{(m)}}.$$

Explicitly, given $w \in X^*$, $T_{n,m}(w)$ is equal to w if the length of w is strictly smaller than $n+m$. If the length of w is greater than $n+m$, we let $w = w_1w_2w_3$, where w_1 and w_3 have length n and m, respectively. Then, $T_{n,m}(w)$ is equal to $w_1S(w_2)w_3$. We also let $T_{0,0} = S$. The set $\{T_{n,m},\ 0 \leq n,\ m\}$ is the *reduction family* of $\langle X, <\mid S\rangle$.

3.9. Confluent Presentation. A presentation by operator of $\mathbf{A}$ is said to be *confluent* if its reduction family is confluent. The link with Gröbner bases is the following: a presentation $\langle X, < \mid S\rangle$ is confluent if, and only if the set $\{w - S(w),\ w \in \mathrm{Nred}\,(S)\}$ is a Gröbner basis of $I\,(\ker\,(S))$. The next result provides a way to construct confluent presentations by operator, providing a way to construct Gröbner bases:

3.10. Theorem. *Let $\langle X, < \mid S\rangle$ be a presentation by operator of $\mathbf{A}$ and let C be a complement of its reduction family. The triple $\langle X, < \mid S \wedge C\rangle$ is a confluent presentation of $\mathbf{A}$.*

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