

# Confluence algebras and acyclicity of the Koszul complex

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## 1 Introduction

We are concerned with computing resolutions of associative algebras over a field  $\mathbb{K}$  using techniques from rewriting theory. Several resolutions of algebras are based on convergent presentations (see [Ani86] or [Kob90]). Here, we study *N-homogeneous algebras*, that is algebras admitting an *N-homogeneous presentation*  $\langle X \mid R \rangle$ . We denote by  $\mathbb{K}X$  and  $\overline{R}$  the vector space spanned by  $X$  and the sub vector space of  $\mathbb{K}X^{\otimes N}$  spanned by  $R$ , respectively. We consider the function  $l_N : \mathbb{N} \rightarrow \mathbb{N}$  defined by:

$$l_N(n) = \begin{cases} kN, & \text{if } n = 2k, \\ kN + 1, & \text{if } n = 2k + 1. \end{cases}$$

Given an *N-homogeneous algebra*  $A$ , we try to compute a resolution of  $\mathbb{K}$  in the following way:

$$\dots \xrightarrow{\partial_{n+1}} A \otimes J_n^N \xrightarrow{\partial_n} A \otimes J_{n-1}^N \longrightarrow \dots \xrightarrow{\partial_4} A \otimes J_3^N \xrightarrow{\partial_3} A \otimes \overline{R} \xrightarrow{\partial_2} A \otimes \mathbb{K}X \xrightarrow{\partial_1} A \xrightarrow{\epsilon} \mathbb{K} \longrightarrow 0, \quad (1)$$

where, for every integer  $n$  such that  $n \geq 3$ , we have:

$$J_n^N = \bigcap_{i=0}^{l_N(n)-N} \mathbb{K}X^{\otimes i} \otimes \overline{R} \otimes \mathbb{K}X^{\otimes l_N(n)-N-i}.$$

In general, the complex (1), called the *Koszul complex of A*, is not acyclic. The algebras for which this complex is cyclic are *N-Koszul algebras*.

The *N-Koszul algebras* have been defined by Berger in [Ber01]. This notion generalises the one of *Koszul algebra* defined by Priddy (see [Pri70]). The *N-Koszul algebras* have been studied through computational approaches based a monomial order, that is, a well founded total order on the set of monomials. In [Ani86], Anick used Gröbner basis to construct a free resolution of  $\mathbb{K}$ . This resolution enables us to conclude that an algebra which admits a quadratic Gröbner basis is Koszul. In [Ber01], Berger studied *N-homogeneous algebras* with a *side-confluent presentation*<sup>1</sup>. More precisely, we can associate with any *N-homogeneous presentation*  $\langle X \mid R \rangle$  of  $A$  such that  $X$  is finite, an unique linear projector  $S$  of  $\mathbb{K}X^{\otimes N}$  (the formal definition of this operator will be given in Section 2). This projector maps any element of  $\mathbb{K}X^{\otimes N}$  to a better one with respect to the monomial order. The presentation

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<sup>1</sup>This notion corresponds to the one of *X-confluent algebra* in [Ber01]. However, we prefer to use our terminology because the property of confluence depends on the presentation.

$\langle X \mid R \rangle$  is said to be side-confluent if for every integer  $m$  such that  $N + 1 \leq m \leq 2N - 1$ , there exists an integer  $k$  which satisfies:

$$\langle S \otimes \text{Id}_{\mathbb{K}X^{\otimes m-N}}, \text{Id}_{\mathbb{K}X^{\otimes m-N}} \otimes S \rangle^k = \langle \text{Id}_{\mathbb{K}X^{\otimes m-N}} \otimes S, S \otimes \text{Id}_{\mathbb{K}X^{\otimes m-N}} \rangle^k,$$

where  $\langle t, s \rangle$  denotes the product  $\cdots sts$  with  $k$  factors. The algebra presented by:

$$\langle s_1, s_2 \mid \langle s_1, s_2 \rangle^k = \langle s_2, s_1 \rangle^k, s_i^2 = s_i, i = 1, 2 \rangle,$$

is naturally associated with a side-confluent presentation. This algebra is the *confluence algebra of degree  $k$* . In [Ber98, Section 5], Berger used specific representations of these algebras to construct a contracting homotopy for the Koszul complex of a quadratic algebra which admits a side-confluent presentation. This construction enables us to conclude that a quadratic algebra admitting a side-confluent presentation is Koszul. However, when  $N$  is greater than 2, an  $N$ -homogeneous algebra admitting a side-confluent presentation is not necessarily  $N$ -Koszul. Indeed, such an algebra is  $N$ -Koszul if and only if the *extra-condition* holds (see [Ber01, Proposition 3.4]). The extra-condition is stated as follows:

$$(ec) : (\mathbb{K}X^{\otimes m} \otimes \overline{R}) \cap (\overline{R} \otimes \mathbb{K}X^{\otimes m}) \subset \mathbb{K}X^{\otimes m-1} \otimes \overline{R} \otimes \mathbb{K}X, \text{ for every } 2 \leq m \leq N - 1.$$

We remark, that when  $N$  is equal to 2, the extra-condition is an empty condition.

We deduce of the works from [Ber01] that the Koszul complex of an algebra  $A$  admitting a side-confluent presentation satisfying the extra-condition is acyclic. However, there does not exist an explicit contracting homotopy for the Koszul complex of  $A$ . In [Che15] we construct such a contracting homotopy. Our construction uses representations of confluence algebras. However, we think that we can obtain the same construction using techniques from higher-dimensional rewriting theory. More precisely, given a convergent presentation of an  $N$ -homogeneous algebra  $A$  satisfying the extra-condition, we could construct a polygraphic resolution of  $A$  such that the set of  $n$ -cells of this polygraph is a basis of the vector space  $J_n^N$ .

## 2 Reduction operators and confluence algebras

Our construction uses *reduction operators* and *confluence algebras*. This section consists in an overview on these notions. We fix a totally ordered finite set  $Y$ .

**2.1. Reduction operators.** A linear projector  $T$  of  $\mathbb{K}Y$  is called a *reduction operator relatively to  $Y$*  if for every  $y \in Y$ , we have either  $T(y) = y$  or  $T(y) < y$ . We denote by  $\text{Red}(Y)$  the set of reduction operators relatively to  $Y$ .

**2.2. Lattice structure.** The set  $\text{Red}(Y)$  admits a lattice structure. To define it, recall from [Ber98, Theorem 2.3] that the map

$$\begin{aligned} \theta_Y : \text{Red}(Y) &\longrightarrow \mathcal{L}(\mathbb{K}Y), \\ T &\longmapsto \ker(T) \end{aligned}$$

is a bijection. The order over  $\text{Red}(Y)$  is defined by:

$$T_1 \preceq T_2, \text{ if } \ker(T_2) \subset \ker(T_1).$$

The lower bound  $T_1 \wedge T_2$  and the upper bound  $T_1 \vee T_2$  of two elements  $T_1$  and  $T_2$  of  $\text{Red}(Y)$  are defined in the following way:

$$\begin{cases} T_1 \wedge T_2 = \theta_Y^{-1}(\ker(T_1) + \ker(T_2)), \\ T_1 \vee T_2 = \theta_Y^{-1}(\ker(T_1) \cap \ker(T_2)). \end{cases}$$

**2.3. Confluent pairs of reduction operators.** A pair  $P = (T_1, T_2)$  of reduction operators relatively to  $Y$  is said to be *confluent* if there exists an integer  $k$  such that:

$$\langle T_1, T_2 \rangle^k = \langle T_2, T_1 \rangle^k.$$

**2.4. Confluence algebras.** Let  $k$  be an integer. The *confluence algebra of degree  $k$*  is the algebra presented by

$$\left\langle s_1, s_2 \mid s_i^2 = s_i, \langle s_1, s_2 \rangle^k = \langle s_2, s_1 \rangle^k, i = 1, 2 \right\rangle.$$

**2.5. Reduction operators and  $N$ -homogeneous presentations.** We end this section giving the link between reduction operators and  $N$ -presentations. Let  $A$  be an  $N$ -homogeneous algebra and let  $\langle X \mid R \rangle$  be an  $N$ -homogeneous presentation of  $A$  such that  $X$  is finite. In particular, the set  $X^{(N)}$  of words of length  $N$  is finite. We let:

$$S = \theta_{X^{(N)}}^{-1}(\overline{R}).$$

The presentation  $\langle X \mid R \rangle$  is said to be *side-confluent* if for every integer  $m$  such that  $N+1 \leq m \leq 2N-1$ , the pair  $(S \otimes \text{Id}_{\mathbb{K}X^{\otimes m-N}}, \text{Id}_{\mathbb{K}X^{\otimes m-N}} \otimes S)$  is confluent.

**2.6. Extra-confluent presentations.** Let  $A$  be an  $N$ -homogeneous algebra. A side-presentation  $\langle X \mid R \rangle$  such that  $X$  is finite and the extra-condition holds is said to be *extra-confluent*.

### 3 The contracting homotopy

We present the different steps of our construction. We fix an  $N$ -homogeneous algebra together with an  $N$ -homogeneous presentation  $\langle X \mid R \rangle$  of  $A$ . We assume that  $X$  is finite and totally ordered. In particular, for every integer  $m$ , the set  $X^{(m)}$  is finite and totally ordered. We write  $V = \mathbb{K}X$  and for every integer  $m$  we identify  $V^{\otimes m}$  to  $\mathbb{K}X^{(m)}$ .

**3.1. Reduction pairs associated with a presentation.** For every integers  $n$  and  $m$  such that  $m \geq l_N(n)$ , we consider the following reduction operators:

$$F_1^{n,m} = \theta_{X^{(m)}}^{-1} \left( I(R)_{m-l_N(n)} \otimes V^{\otimes l_N(n)} \right),$$

$$F_2^{n,m} = \begin{cases} \text{Id}_{V^{\otimes m}}, & \text{if } m < l_N(n+1), \\ \theta_{X^{(m)}}^{-1} \left( V^{\otimes m-l_N(n+1)} \otimes J_{n+1} \right), & \text{otherwise.} \end{cases}$$

The pair  $(F_1^{n,m}, F_2^{n,m})$  is denoted by  $P_{n,m}$  and is called the *reduction pair of bi-degree  $(n, m)$*  associated with  $\langle X \mid R \rangle$ . We point the fact that the finiteness condition over  $X$  is necessary to define these pairs. The lattice structure on the set of reduction operators enables us to prove:

**3.2. Theorem.** *Let  $A$  be an  $N$ -homogeneous algebra admitting a side-confluent presentation  $\langle X \mid R \rangle$ , where  $X$  is a finite set. The reduction pairs associated with  $\langle X \mid R \rangle$  are confluent.*

**3.3. The left bound.** The reduction pairs associated with a side-confluent presentation  $\langle X \mid R \rangle$  enable us to define a family of representations of confluence algebras in the following way:

$$\varphi^{P_{n,m}} : \left\langle s_1, s_2 \mid \langle s_1, s_2 \rangle^{k_{n,m}} = \langle s_2, s_1 \rangle^{k_{n,m}}, s_i^2 = s_i, i = 1, 2 \right\rangle \longrightarrow \text{End}(V^{\otimes m}),$$

$$s_i \longmapsto F_i^{n,m}$$

where the integer  $k_{n,m}$  satisfies:

$$\langle F_1^{n,m}, F_2^{n,m} \rangle^{k_{n,m}} = \langle F_2^{n,m}, F_1^{n,m} \rangle^{k_{n,m}}.$$

For every integers  $n$  and  $m$  we consider the following specific element of  $\mathcal{A}_{k_{n,m}}$ :

$$\gamma_1 = (1 - s_2) \left( s_1 + s_1 s_2 s_1 + \cdots + \langle s_2, s_1 \rangle^{2i+1} \right),$$

where the integer  $i$  depends on  $k_{n,m}$ . The elements  $\varphi^{F_{n,m}}(\gamma_1)$  are used to construct a family of  $\mathbb{K}$ -linear maps

$$\begin{aligned} h_0 &: A \longrightarrow A \otimes V, \\ h_1 &: A \otimes V \longrightarrow A \otimes \bar{R}, \\ h_2 &: A \otimes \bar{R} \longrightarrow A \otimes J_3^N, \\ h_n &: A \otimes J_n^N \longrightarrow A \otimes J_{n+1}^N, \text{ for } n \geq 3. \end{aligned}$$

We give more details about the constructions of these maps. The confluence property implies that every element  $f \in \mathbb{K}\langle X \rangle$  admits a unique normal form for  $\langle X \mid R \rangle$ , denoted by  $\widehat{f}$ . Then, we have a  $\mathbb{K}$ -linear isomorphism:

$$\begin{aligned} \phi &: A \longrightarrow N, \\ \bar{f} &\longmapsto \widehat{f} \end{aligned}$$

where  $N$  is the vector space of normal forms for  $\langle X \mid R \rangle$ . It turns out that for every integer  $n$ , the image of the  $\mathbb{K}$ -linear map  $h'_n : \bigoplus_{m \geq l_N(n)} V^{\otimes m} \longrightarrow T(V)$  defined by

$$h'_n|_{V^{\otimes m}} = \varphi^{F_{n,m}}(\gamma_1),$$

is included in  $N \otimes J_{n+1}$ . Then, for every integer  $n$ , the map  $h_n : A \otimes J_n \longrightarrow A \otimes J_{n+1}$  is defined by:

$$h_n = \phi_{n+1}^{-1} \circ h'_n \circ \phi_n : A \otimes J_n \longrightarrow A \otimes J_{n+1},$$

where  $\phi_n$  is equal to  $\phi \otimes \text{Id}_{V^{\otimes l_N(n)}}$ . The family  $(h_n)$  is called, the *left bound of  $\langle X \mid R \rangle$* .

**3.4. Reduction relations.** In general, the the left bound of a side-confluent presentation is not a contracting homotopy for the Koszul complex of  $A$ . In fact, this property is true if and only if for every integers  $n$  and  $m$  such that  $m \geq l_N(n)$ , we have:

$$(r_{n,m}) \quad F_1^{n,m} \wedge F_2^{n,m} |_{K_n^{(m)}} = F_1^{n-1,m} \vee F_2^{n-1,m} |_{K_n^{(m)}},$$

where  $K_n^{(m)}$  is the vector space  $(N \cap V^{\otimes m - l_N(n)}) \otimes J_n$ . When the relation  $(r_{n,m})$  hold, we say that  $\langle X \mid R \rangle$  satisfy the *reduction relations*. Then, we have the following:

**3.5. Proposition.** *Let  $A$  be an  $N$ -homogeneous algebra. Assume that  $A$  admits a side-confluent presentation  $\langle X \mid R \rangle$  where  $X$  is a finite set. The left bound of  $\langle X \mid R \rangle$  is a contracting homotopy for the Koszul complex of  $A$  if and only if  $\langle X \mid R \rangle$  satisfy the reduction relations.*

In order the show that the extra-condition implies that the reduction relations hold, we have to describe it into algebraic relations. Recall that (*ec*) is:

$$(V^{\otimes m} \otimes \bar{R}) \cap (\bar{R} \otimes V^{\otimes m}) \subset V^{\otimes m-1} \otimes \bar{R} \otimes V, \text{ for every } 2 \leq m \leq N-1.$$

Applying the bijection  $\theta_{X^{(m+N)}}^{-1}$  for every  $2 \leq m \leq N - 1$ , the extra-condition is equivalent to:

$$S_{m-1}^{(m+N)} \preceq S_m^{(m+N)} \vee S_0^{(m+N)}, \text{ for every } 2 \leq m \leq N - 1,$$

where  $S_i^{(m+N)}$  is equal to  $\theta_{X^{(m+N)}}^{-1} (V^{\otimes i} \otimes \bar{R} \otimes V^{\otimes m-i})$ . Thus, the extra-condition is equivalent to:

$$S_{m-1}^{(m+N)} \vee S_m^{(m+N)} \vee S_0^{(m+N)} = S_m^{(m+N)} \vee S_0^{(m+N)}, \text{ for every } 2 \leq m \leq N - 1.$$

These relations are used to prove:

**3.6. Proposition.** *Let  $A$  be an  $N$ -homogeneous algebra. Assume that  $A$  admits an extra-confluent presentation  $\langle X \mid R \rangle$ . For every integers  $n$  and  $m$  such that  $n \geq 1$  and  $m \geq l_N(n + 1)$ , we have:*

$$F_1^{n,m} \wedge (F_1^{n-1,m} \vee F_2^{n-1,m}) = F_1^{n,m} \wedge F_2^{n,m}.$$

We point the fact that the previous proposition is not true if the presentation is not side-confluent. Moreover, this proposition implies that the reduction relations hold. Thus, our main result is stated as follows:

**3.7. Theorem.** *Let  $A$  be an  $N$ -homogeneous algebra. If  $A$  admits an extra-confluent presentation  $\langle X \mid R \rangle$ , then the left bound of  $\langle X \mid R \rangle$  is a contracting homotopy for the Koszul complex of  $A$ .*

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