# An Algebraic Approach of Confluence and Completion

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## 1 Introduction

We propose an approach on linear rewriting where applications of rewrite rules are replaced by applications of specific linear maps called *reduction operators*. These operators were introduced by Berger for finite dimensional vector spaces. His motivation was to study the homology of a special class of algebras called *finitely generated homogeneous algebras*. The elements of these algebras are non-commutative polynomials over a finite number of variables modulo the congruence spanned by a set of oriented homogeneous relations having the same degree. By degree, we mean the one induced by the length of non-commutative monomials. The latter are identified to words. Berger considered the linear endomorphism mapping every left-hand side of a rewrite rule to its right-hand side. This is an endomorphism of the vector space spanned by words whose length is the degree of the rewriting rules. The number of variables being finite, this vector space is of finite dimension. It turns out that the endomorphism described previously is a reduction operator. Berger also proved that the set of reduction operators admits a lattice structure. We point out that in order to obtain this structure, he needs to consider finite dimensional vector spaces. He deduces from this structure a lattice formulation of the confluence for homogeneous rewriting systems. Using this point of view, one can study the homological property of *Koszulness* ([1], [2], [3], [8], [6]). For the definition of Koszulness, we refer the reader to [10] and [3].

In the next section, we propose to develop a notion of reduction operator for non-necessarily finite dimensional vector spaces. Our motivation is that we want to use the theory of reduction operators to study non-homogeneous algebras. For such algebras, we do not have any bound for the degree of a word appearing in a rewrite rule. Hence, the operator described in the previous paragraph is an endomorphism of the vector space spanned by all words which is infinite dimensional. We consider vector spaces admitting a basis equipped with a well-founded total order. We introduce a lattice structure on the set of reduction operators associated to such a vector space and deduce an algebraic formulation of the confluence. This formulation generalises the one obtained by Berger for finite sets.

In the last section, we relate our notion of confluence to the classical one coming from rewriting theory. For that, we formulate the notion of Church-Rosser property in terms of reduction operators and show that it is equivalent to the one of confluence. We also formulate the completion in terms of reduction operators. Classical completion algorithms exist: the Knuth-Bendix completion algorithm [7] for term rewriting or the Buchberger algorithm [4, 5, 9] for Gröbner bases. These algorithms add new rules to a rewriting system to obtain an equivalent one which moreover is convergent. Here, what we want to complete is a set $F$ of reduction operators. A *completion* of $F$ is a confluent set $F'$ containing $F$ which is such that the lower bounds of the sets $F$ and $F'$ are equal. We also study the question of the existence of a completion. For that, we introduce an operator $C^F$ called the *F-complement* and state that the set $F \cup \left\{C^F\right\}$ is a completion of $F$.

# 2 Reduction Operators

## 2.1 First Definitions

**2.1.1. Notations.** We denote by $\mathbb{K}$ a commutative field. Throughout the paper, we fix a well-ordered set $(G, <)$. We denote by $\mathbb{K}G$ the vector space spanned by $G$: the non-zero elements are the finite formal linear combinations of elements of $G$ with coefficients in $\mathbb{K}$. For every $v \in \mathbb{K}G \setminus \{0\}$, there exist a unique finite subset $S_v$ of $G$, called the *support* of $v$, and a unique family of non zero scalars $(\lambda_g)_{g \in S_v}$ such that

$$v = \sum_{g \in S_v} \lambda_g g.$$

**2.1.2. Partial Order on the Vectors.** The order on $G$ being total, for every $v \in \mathbb{K}G$, the set $S_v$ admits a greatest element, written $\lg(v)$. The element $\lg(v)$ is the *leading generator* of $v$. We extend the order $<$ on $G$ into a partial order on $\mathbb{K}G$ in the following way: we have $u < v$ if $u = 0$ and $v$ is different from 0 or if $\lg(u) < \lg(v)$.

**2.1.3. Reduction Operators.** A linear endomorphism $T$ of $\mathbb{K}G$ is a *reduction operator relative to* $(G, <)$ if it is idempotent and if for every $g \in G$, we have $T(g) \leq g$. We denote by $\mathbf{RO}(G, <)$ the set of reduction operators relative to $(G, <)$. Given a reduction operator $T$, a generator $g$ is said to be *$T$-reduced* if $T(g)$ is equal to $g$. We denote by $\mathrm{Red}(T)$ the set of $T$-reduced generators and by $\mathrm{Nred}(T)$ the complement of $\mathrm{Red}(T)$ in $G$.

**2.1.4. Remark.** Let $T$ be a reduction operator relative to $(G, <)$. The image of $T$ is equal to $\mathbb{K}\mathrm{Red}(T)$.

## 2.2 Lattice Structure and Confluence

Our aim is to equip the set $\mathbf{RO}(G, <)$ with a lattice structure. To define it, let $\mathscr{L}(\mathbb{K}G)$ be the set of subspaces of $\mathbb{K}G$. The following proposition extends the one obtained by Berger when $G$ is finite

**2.2.1. Proposition.** *The restriction of the map $T \longmapsto \ker(T)$ is a bijection.*

**2.2.2. Lattice Structure.** The application mapping a subspace of $\mathbb{K}G$ to the operator whose kernel is this subspace is written $\theta$. We consider the binary relation on $\mathbf{RO}(G, <)$ defined by

$$T_1 \preceq T_2 \text{ if, and only if } \ker(T_2) \ \subset \ \ker(T_1).$$

This relation is reflexive and transitive. From Proposition 2.2.1, it is also anti-symmetric. Hence, it is an order relation on $\mathbf{RO}(G, <)$. Let us equip $\mathbf{RO}(G, <)$ with a lattice structure. The lower bound $T_1 \wedge T_2$ and the upper bound $T_1 \vee T_2$ of two elements $T_1$ and $T_2$ of $\mathbf{RO}(G, <)$ are defined in the following manner:

$$\begin{cases} T_1 \wedge T_2 = \theta(\ker(T_1) + \ker(T_2)), \\ T_1 \vee T_2 = \theta(\ker(T_1) \cap \ker(T_2)). \end{cases}$$

Our aim is to formulate the notion of confluence using this lattice structure. For that, we need the following:

**2.2.3. Lemma.** *Let $T_1$ and $T_2$ be two reduction operators relative to $(G, <)$. Then, we have:*

$$T_1 \preceq T_2 \Longrightarrow \mathrm{Red}(T_1) \ \subset \ \mathrm{Red}(T_2).$$

**2.2.4. Obstructions.** Let $F$ be a subset of $\mathbf{RO}(G,<)$. We let

$$\operatorname{Red}(F)=\bigcap_{T\in F}\operatorname{Red}(T) \text{ and } \wedge F=\theta^{-1}\left(\sum_{T\in F}\ker(T)\right).$$

For every $T \in F$, we have $\wedge F \preceq T$. Thus, from Lemma 2.2.3, the set $\operatorname{Red}(\wedge F)$ is included in $\operatorname{Red}(T)$ for every $T \in F$, so that it is included in $\operatorname{Red}(F)$. We write

$$\operatorname{Obs}_{\mathrm{red}}^{F}=\operatorname{Red}(F)\setminus\operatorname{Red}(\wedge F). \tag{1}$$

**2.2.5. Confluence.** A subset $F$ of $\mathbf{RO}(G,<)$ is said to be *confluent* if $\operatorname{Obs}_{\mathrm{red}}^{F}$ is the empty set.

# 3 Rewriting Properties and Completion

## 3.1 Reduction Operators and Abstract Rewriting

In this section, we explain how our notion of confluence is related to the one coming from rewriting theory. For that, consider the abstract rewriting system $\left(\mathbb{K}G, \underset{F}{\longrightarrow}\right)$ defined by $v \underset{F}{\longrightarrow} v'$ if and only if there exists $T \in F$ such that $v$ does not belong to $\operatorname{im}(T)$ and $v'$ is equal to $T(v)$.

**3.1.1. Church-Rosser Property.** We denote by $\langle F\rangle$ the submonoid of $(\operatorname{End}(\mathbb{K}G),\circ)$ spanned by $F$. Let $v$ and $v'$ be two elements of $\mathbb{K}G$. We say that $v$ *rewrites into* $v'$ if there exists $R \in \langle F\rangle$ such that $v'$ is equal to $R(v)$. We say that $F$ has the *Church-Rosser property* if for every $v \in \mathbb{K}G$, $v$ rewrites into $\wedge F(v)$. The following result is the analogous of the Church-Rosser theorem for reduction operators:

**3.1.2. Theorem.** *A subset of* $\mathbf{RO}(G,<)$ *is confluent if and only if it has the Church-Rosser property.*

**3.1.3. Equivalence Relations.** We denote by $\overset{*}{\underset{F}{\longleftrightarrow}}$ the reflexive transitive symmetric closure of $\underset{F}{\longrightarrow}$. We easily show that we have $v \overset{*}{\underset{F}{\longleftrightarrow}} v'$ if and only if $v-v'$ belongs to the kernel of $\wedge F$. We deduce that $F$ has the Church-Rosser property if and only if it is so for $\underset{F}{\longrightarrow}$. From Theorem 3.1.2, we get:

**3.1.4. Proposition.** *Let $F$ be a subset of* $\mathbf{RO}(G,<)$. *Then, $F$ is confluent if and only if it is so for* $\underset{F}{\longrightarrow}$.

**3.1.5. Multi-Set Order.** Given an element $v$ of $\mathbb{K}G$, let $S_v$ be the support of $v$. We introduce the order $<_{\mathrm{mul}}$ on $\mathbb{K}G$ defined in the following way: we have $v <_{\mathrm{mul}} v'$ if for every $g \in S_v$ such that $g$ does not belong to $S_{v'}$, there exists an element $g' \in S_{v'}$ not appearing in $S_v$, such that $g < g'$.

**3.1.6. Obstructions and Abstract Rewriting.** For every $v \in \mathbb{K}G$, $\wedge F(v)$ is the smallest element $v' \in \mathbb{K}G$ for $<_{\mathrm{mul}}$ such that $v-v'$ belongs to the kernel of $\wedge F$. Hence, denoting by $[v]$ the class of $v$ for $\overset{*}{\underset{F}{\longleftrightarrow}}$, we deduce from Proposition 3.1.4 that $\wedge F(v)$ is the smallest element of $[v]$ for $<_{\mathrm{mul}}$. In particular, $\operatorname{Obs}_{\mathrm{red}}^{F}$ being the set of elements of $G$ fixed by every element of $T$ but not fixed by $\wedge F$, $\mathbb{K}\operatorname{Obs}_{\mathrm{red}}^{F}$ is the set of normal forms for $\underset{F}{\longrightarrow}$ which are not minimal in their equivalence classes.

## 3.2 Completion

In this section, we investigate the notion of completion in terms of reduction operators.

**3.2.1. Definition.** Let $F$ be a subset of $\mathbf{RO}(G,<)$.

1. A *completion of* $F$ is a subset $F'$ of $\mathbf{RO}(G,<)$, such that
   (a) $F'$ is confluent,
   (b) $F \subset F'$ and $\wedge F' = \wedge F$.
2. A *complement of* $F$ is an element $C$ of $\mathbf{RO}(G,<)$ such that
   (a) $(\wedge F) \wedge C = \wedge F$,
   (b) $\mathrm{Obs}_{\mathrm{red}}^{F} \subset \mathrm{Nred}(C)$.

   A complement is said to be *minimal* if the inclusion 2b is an equality.

The link between a complement and a completion is the following:

**3.2.2. Proposition.** *Let $C \in \mathbf{RO}(G,<)$ such that $(\wedge F) \wedge C$ is equal to $\wedge F$. The set $F \cup \{C\}$ is a completion of $F$ if and only if $C$ is a complement of $F$.*

**3.2.3. Remark.** The operator $\wedge F$ is a complement of $F$. However, in general, this complement is not minimal. Our aim is to exhibit a minimal complement.

**3.2.4. The $F$-Complement.** Letting $\vee\overline{F} = \theta(\mathbb{K}\mathrm{Red}(F))$, the operator

$$C^F = (\wedge F) \vee \left(\vee\overline{F}\right),$$

is the *$F$-complement.*

**3.2.5. Theorem.** *Let $F$ be a subset of $\mathbf{RO}(G,<)$. The $F$-complement is a minimal complement of $F$.*

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