Reduction Operators and Completion of Linear Rewriting Systems

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Plan

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I. Motivations

Computations in Algebras

- Computational problems in algebra.
  - Computation of linear bases.
  - The ideal membership problem.
  - Determine the multiplication table of an algebra.
  - Computation with ideals in algebraic geometry.
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- Various algebraic structures:
  - Lie algebras.
  - Commutative algebras.
  - Associative algebras.
- Rewriting method: present algebras by generators and oriented relations.
  - Find linear bases in the form of monomials.
  - Provide procedures to determine linear combinations w.r.t such bases.

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  - Procedure to obtain linear decompositions: apply the reduction $yx \longrightarrow xy$ as long as it is possible.
- $\mathbf{A}$ an algebra presented by generators and oriented relations.
  - Does the set of monomials in normal forms form a basis of $\mathbf{A}$?
    - Is it a generating family?
    - Is it a free family?

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- **A** an algebra admitting a terminating presentation.
  - Every $a \in \mathbf{A}$ is equal to a normal form.
  - Monomials in normal forms is a generating set of **A**.

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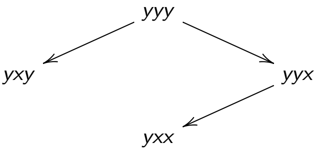

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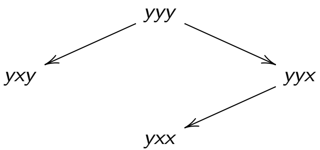

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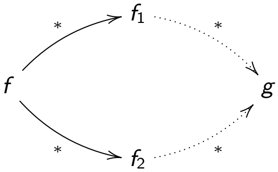

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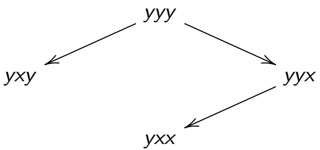

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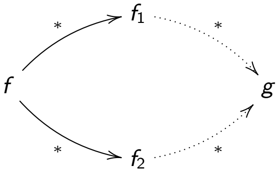

- If **A** admits a confluent terminating presentation, the set of monomials in normal forms is a linear basis of **A**.

Gröbner Bases

- Gröbner bases appear in
  - Lie algebras [Shirshov 1962].
  - Commutative algebras [Buchberger 1965].
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  - If $f = \mathrm{lm}(f) - r(f)$, then $\mathrm{lm}(f) \longrightarrow r(f)$.
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- **A** presented by generators X and relations R, oriented w.r.t a monomial order.
  - R is a Gröbner basis of $I(R)$ if it induces a confluent presentation.
  - Existence of a practical criterion using S-polynomials.

Objective

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- Using the functional approach, we obtain:
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  - Lattice formulation of completion.
- We study general linear rewriting systems.

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II. Lattice Structure

Reduction Operators

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Notations. $\forall v,\ v' \in \mathbb{K}G$

- $\lg(v)$: the greatest element of $\operatorname{supp}(v)$.
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Definition. An endomorphism T of $\mathbb{K}G$ is a reduction operator relative to $(G, <)$ if

- T is a projector,
- $\forall g \in G$, we have $T(g) \leq g$.

Example

- $(G, <) = \{g_1 < g_2 < g_3 < g_4\}$,

$$T_1 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } T_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

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  - If $(\lambda_2, \lambda_4) \neq (0, 0)$, T_1 reduces v as follows

$$v \underset{T_1}{\longrightarrow} (\lambda_1 + \lambda_2, 0, \lambda_3 + \lambda_4, 0).$$

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$$v \underset{T_1}{\longrightarrow} (\lambda_1 + \lambda_2, 0, \lambda_3 + \lambda_4, 0).$$
  - If $\lambda_4 \neq 0$, T_2 reduces v as follows
$$v \underset{T_2}{\longrightarrow} (\lambda_1, \lambda_2 + \lambda_4, \lambda_3, 0).$$

Lattice Structure

Proposition [C. 2017]. *The map*

$$\begin{aligned} \ker\colon \mathbf{RO}\,(G,<) &\longrightarrow \left\{\text{subspaces of } \mathbb{K}G\right\}, \\ T &\longmapsto \ker(T) \end{aligned}$$

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Lattice structure. $(\mathbf{RO}(G, <), \preceq, \wedge, \vee)$ is a lattice where

- $T_1 \preceq T_2$ if $\ker(T_2) \subseteq \ker(T_1)$.
- $T_1 \wedge T_2 = \ker^{-1}(\ker(T_1) + \ker(T_2))$.
- $T_1 \vee T_2 = \ker^{-1}(\ker(T_1) \cap \ker(T_2))$.

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- $\ker(T_1 \wedge T_2)$ is spanned by the rows of

$$M = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 \end{pmatrix}.$$

Example, Part II

- By Gaussian elimination, $\ker(T_1 \wedge T_2)$ is spanned by the rows of

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- Hence,

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III. Confluence and Completion

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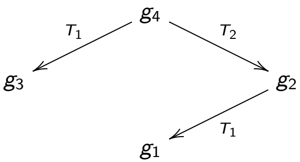

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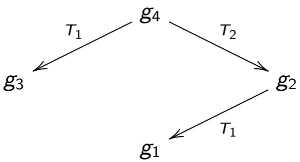

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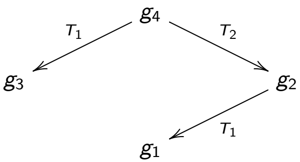

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Remark. Obstruction to confluence: $\nexists\ g_3 \longrightarrow g_1$.

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Proposition [C. 2017]. *F is confluent if and only if it is so for*

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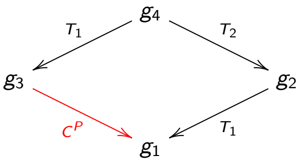

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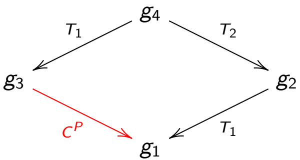

- Formally

$$C^P = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

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Theorem [C. 2017]. *The set*

$$F \cup \left\{C^F\right\} \subset \mathbf{RO}(G,<),$$

is confluent.

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IV. Contracting Homotopy for the Koszul Complex

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- Problem: compute homological invariants (Tor-Ext groups, Poincaré serie...) of **A**.
- For that, we need to construct resolutions of $\mathbb{K}$,

$$\cdots \xrightarrow{\partial_{n+1}} \mathbf{A}[X_n] \xrightarrow{\partial_n} \mathbf{A}[X_{n-1}] \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} \mathbf{A}[X_1] \xrightarrow{\partial_1} \mathbf{A} \xrightarrow{\partial_0} \mathbb{K} \longrightarrow 0,$$

where

  - $\mathbf{A}[X_n]$ are free **A**-modules spanned by sets X_n,
  - ∂_n are **A**-linear maps satisfying

$$\operatorname{im}(\partial_{n+1}) = \ker(\partial_n), \text{ for every } n \geq 0.$$

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  - ∂_2 is not injective:

$$\partial_2(\overline{yx} \otimes yxy) = \overline{yxyx} \otimes x = 0.$$

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Theorem [C. 2016]. *Let* **A** *be an homogeneous algebra admitting a side-confluent presentation* $\langle (X,<) \mid S \rangle$ *and satisfying the extra-condition. The left bound of* $\langle (X,<) \mid S \rangle$ *is a contracting homotopy for the Koszul complex of* **A**.

Plan

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- Reduction operators also provide a lattice formulation of the F_4 completion procedure (*A Lattice Formulation of the F_4 Completion Procedure*, C., arXiv:1703.02077, 2017).