1 Introduction

Several constructive methods in algebra are based on the computation of syzygies. For instance, a generating set of syzygies offers criteria to detect useless critical pairs during Buchberger’s algorithm, that is critical pairs whose corresponding $S$-polynomials reduces into zero [8]. Such criteria improve the complexity of Buchberger’s algorithm since most of the time is spent in computing into zero. Another application scope is homological/homotopical algebra where syzygies enable us to construct resolutions of monoids or algebras [1, 6, 7]. Several methods for computing syzygies were introduced: Squier’s theorem states that the module of syzygies of a convergent string rewriting system is spanned by confluence diagrams [9, 10], which imply that syzygies can be computed by a completion-reduction procedure [5]. Moreover, the syzygies of a regular sequence are spanned by principal syzygies [4]. In [2], a general method for computing syzygies of a set of polynomials equations is given.

In this work, we are interested in the computation of syzygies of linear rewriting systems. Our motivation is to develop a general framework which can be specialised to various structures whose underlying sets of terms are vector spaces: polynomials algebras, tensor algebras, Lie algebras, operads... We fix a vector space $V$ and a basis $G$ of $V$: when $V$ is a polynomial algebra, $G$ is the set of monomials, for tensor algebras, $G$ is the set of words, for instance. We consider linear rewriting systems described by reduction operators. Given a well-order $<$ on $G$, a reduction operator is an idempotent endomorphism of $V$ such that for every $g / \in \text{im} (T)$, $T(g)$ is a linear combination of elements of $G$ strictly smaller than $g$ for $<$. Such an operator encodes the reductions $v \xrightarrow{T} T(v)$, for every vector $v / \in \text{im} (T)$.

Given a set $F = \{T_1, \cdots, T_n\}$ of reduction operators, the syzygies of $F$ are the elements of the kernel of the map

$$
\pi_F : \ker(T_1) \oplus \cdots \oplus \ker(T_n) \longrightarrow V,
$$

$$(v_1, \cdots, v_n) \mapsto \sum_{i=1}^{n} v_i
$$

In [3] Proposition 2.1.14], it is shown that the set of reduction operators admits a lattice structure. Our method for constructing a linear basis of $\text{Syz}(F)$ works as follows: $\text{Syz}(f_1, f_2)$ is isomorphic to the kernel of the upper bound $T_1 \lor T_2$ of $T_1$ and $T_2$. Moreover, for every integer $2 \leq k \leq n - 1$, we have onto morphisms:

$$
\text{Syz}(T_1, \cdots, T_{k+1}) \longrightarrow \text{Syz}(T_1 \land \cdots \land T_k, T_{k+1}),
$$

$$(v_1, \cdots, v_{k+1}) \mapsto \left(\sum_{i=1}^{k} v_i, v_{k+1}\right)
$$

where $T_1 \land \cdots \land T_k$ is the lower bound of $\{T_1, \cdots, T_k\}$. Hence, if a linear basis $B_k$ of $\text{Syz}(T_1, \cdots, T_k)$ is known, we construct a linear basis of $\text{Syz}(f_1, \cdots, f_{k+1})$ by taking the union of $B_k$ with pre-images of elements of a linear basis of $\text{Syz}(T_1 \land \cdots \land T_k, T_{k+1})$. This method provides successively linear bases of $\text{Syz}(T_1, T_2)$, $\text{Syz}(T_1, T_2, T_3)$, ..., $\text{Syz}(T_1, \cdots, T_n) = \text{Syz}(F)$.
2 Reduction Operators

2.1. Notations. We fix a well-ordered set \((G, <)\) and a commutative field \(\mathbb{K}\). Every vector \(v\) of the vector space \(\mathbb{K}G\) spanned by \(G\) admits a greatest element, written \(\text{lg}(v)\), in its decomposition with respect to \(G\). We extend the order \(<\) on \(G\) into an order on \(\mathbb{K}G\) defined by \(v_1 < v_2\) if \(v_1 = 0\) and \(v_2 \neq 0\) or if \(\text{lg}(v_1) < \text{lg}(v_2)\).

2.2. Definition. A linear endomorphism \(T\) of \(\mathbb{K}G\) is called a reduction operator if it is a projector and if for every \(g \in G\), we have \(T(g) \leq g\). We write \(\text{RO}(G, <)\) the set of reduction operators and for every \(T \in \text{RO}(G, <)\), we write \(\text{Red}(T) = \{g \in G \mid T(g) \neq g\}\).

2.3. T-decompositions. A reduction operator being a projector, the kernel of \(T\) admits as a basis the set of \(g - T(g)\), where \(g\) belongs to \(\text{Red}(T)\), that is every \(v \in \ker(T)\) admits a unique decomposition
\[
v = \sum_{g \in \text{Red}(T)} \lambda_g (g - T(g)).\tag{1}\]
The decomposition (1) is called the \(T\)-decomposition of \(v\).

2.4. Lattice Structure. Recall from [3, Proposition 2.1.14] that the map
\[
\ker : \text{RO}(G, <) \rightarrow \{\text{subspaces of } \mathbb{K}G\},
T \mapsto \ker(T)
\]
is a bijection. Given a subspace \(V\) of \(\mathbb{K}G\), we write \(\ker^{-1}(V)\) the unique reduction operator with kernel \(V\). Then, \((\text{RO}(G, <), \leq, \wedge, \vee)\) is a lattice where
i. \(T_1 \leq T_2\) if \(\ker(T_2) \subseteq \ker(T_1)\),
ii. \(T_1 \wedge T_2 = \ker^{-1}(\ker(T_1) + \ker(T_2))\),
iii. \(T_1 \vee T_2 = \ker^{-1}(\ker(T_1) \cap \ker(T_2))\).

3 Syzygies

We fix a subset \(F = \{T_1, \cdots, T_n\}\) of \(\text{RO}(G, <)\) and we let
\[
\ker(F) = \ker(T_1) \oplus \cdots \oplus \ker(T_n) .
\]
An element of \(\ker(F)\) is written \((v_1, \cdots, v_n)\), where each \(v_i\) belongs to \(\ker(T_i)\).

3.1. Notation. For every integer \(1 \leq i \leq n\) and for every \(g \in \text{Red}(T_i)\), we let
\[
b_{i,g} = (0, \cdots, 0, g - T_i(g), 0, \cdots, 0).
\]
The set \(\mathscr{B} = \{b_{i,g} \mid 1 \leq i \leq n \text{ and } g \in \text{Red}(T_i)\}\), is a linear basis of \(\ker(F)\).

3.2. Definition. Consider the linear map
\[
\pi_F : \ker(F) \rightarrow \ker(\wedge F),
(v_1, \cdots, v_n) \mapsto \sum_{i=1}^{n} v_i
\]
We write \(\text{Syz}(F) = \ker(\pi_F)\).

The elements of \(\text{Syz}(F)\) are called the syzygies of \(F\).
3.3. Canonical Decompositions. By definition of syzygies, we have a linear isomorphism
\[ \pi_F : \ker(F) / \text{Syz}(F) \to \ker(\wedge F). \]
Moreover, \( \ker(F) / \text{Syz}(F) \) admits as a basis a subset \( \mathcal{B}(F) \) of \( \mathcal{B} \), so that
\[
\left\{ g - T_i(g) \mid b_{i,g} \in \mathcal{B}(F) \right\},
\]
is a basis of \( \ker(\wedge F) \). The decomposition of an element \( v \) of \( \ker(\wedge F) \) with respect to \( \mathcal{B}(F) \) is called a canonical decomposition of \( v \) with respect to \( F \).

3.4. Remark. Following the terminology of [3, Section 2.1.9], \( \mathcal{B}(F) \) can be chosen in such a way that it is also reduced and this choice is unique, which motivates the terminology of "canonical basis". In the sequel, we do not assume that \( \mathcal{B}(F) \) is reduced.

3.5. Purpose. Our purpose is to introduce an algorithm for computing \( \text{Syz}(F) \). This algorithm is based on the fact that for every family \( U_1, \cdots, U_k \) of reduction operators, we have a linear map:
\[
\text{Syz}(U_1, \cdots, U_k) \to \text{Syz}(U_1 \wedge \cdots \wedge U_{k-1}, U_k).
\]
\( (v_1, \cdots, v_k) \mapsto (\sum_{i=1}^{k-1} v_i, v_k) \)
We also need the following:

3.6. Proposition. Let \( P = (T_1, T_2) \) be a pair of reduction operators. We have a linear isomorphism
\[
\ker(T_1 \lor T_2) \sim \text{Syz}(P).
\]
\( v \mapsto (-v, v) \)

3.7. The Algorithm. The algorithm takes as input a finite subset \( F = \{T_1, \cdots, T_n\} \) of \( \text{RO}(G,<) \) and returns a basis of \( \text{Syz}(F) \).

**Algorithm 1 Computation of a Basis of Syzygies**

**Initialisation:**
- \( T := \text{Id}_{\text{kg}}; \)
- \( v := 0; \)
- \( B := \emptyset. \)

1: for \( i = 2 \) to \( n \) do
2: \( T = T_1 \wedge \cdots \wedge T_{i-1}; \)
3: for \( g_0 \in \text{Red}(T \lor T_i) \) do
4: \( v = g_0 - (T \lor T_i)(g_0); \)
5: \( \sum \lambda_g(g - T_i(g)) : \) the \( T_i \)-decomposition of \( v \);
6: \( \sum \lambda_{g'}(g' - T_j(g')) : \) a canonical decomposition of \( v \) with respect to \( (T_1, \cdots, T_{i-1}) \);
7: \( B = B \cup \left\{ \sum (\lambda_g b_{i,g}) - \sum (\lambda_{g',b_{j,g'}}) \right\}; \)
8: end for
9: end for
10: return \( B \)
4 Example

We consider $G = (g_1 < g_2 < g_3 < g_4 < g_5 < g_6)$. We let $F = \{T_1, T_2, T_3, T_4\}$, where the operators $T_i$ are defined by their matrices with respect to the basis $G$:

\[
T_1 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
T_2 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
\]

\[
T_3 = \begin{pmatrix}
1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
T_4 = \begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

The vector space $\ker(F)$ is spanned by the following eight vectors:

\[
b_1 = (g_4 - g_3, 0, 0, 0),
\]

\[
b_2 = (g_5 - g_3, 0, 0, 0),
\]

\[
b_3 = (g_5 - g_1, 0, 0, 0),
\]

\[
b_4 = (0, g_4 - g_2, 0, 0),
\]

\[
b_5 = (0, g_5 - g_2, 0, 0),
\]

\[
b_6 = (0, g_6 - g_2, 0, 0),
\]

\[
b_7 = (0, 0, g_4 - g_1, 0),
\]

\[
b_8 = (0, 0, 0, g_5 - g_1).
\]

We describe the algorithm of the previous section to compute a linear basis $B$ of $\text{Syz}(F)$. We begin with $B = \emptyset$.

4.1. Step 1. We have

\[
T_1 \lor T_2 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

The set $\text{Red}(T_1 \lor T_2)$ is reduced to $\{g_5\}$ and $g_5 - (T_1 \lor T_2)(g_5)$ is equal to $g_5 - g_4$. We have

\[
g_5 - g_4 = (g_5 - g_3) - (g_4 - g_3)
\]

\[= (g_5 - T_1(g_5)) - (g_4 - T_1(g_4)),\]

and

\[
g_5 - g_4 = (g_5 - g_2) - (g_4 - g_2)
\]

\[= (g_5 - T_2(g_5)) - (g_4 - T_2(g_4)).\]

We have

\[
B = \{b_5 - b_4 - b_2 + b_1\}.
\]

4.2. Step 2. We have

\[
(T_1 \land T_2) \lor T_3 = \begin{pmatrix}
1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]
We need to determine the $T_3$-decomposition of $g_4 - g_1$ and well as a canonical decomposition of $g_4 - g_1$ with respect to $(T_1, T_2)$. These two decompositions are given by

$$g_4 - g_1 = g_4 - T_3(g_4)$$

so that, we have

$$- (g_6 - g_2) + (g_4 - g_2) + (g_6 - g_1) = - (g_6 - T_2(g_6)) + (g_4 - T_2(g_4)) + (g_4 - T_1(g_4)).$$

so that, we have

$$B = \{ b_5 - b_4 - b_2 + b_1, b_7 + b_6 - b_4 - b_3 \}.$$

4.3. Step 3. We have

$$(T_1 \land T_2 \land T_3) \lor T_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The $T_4$-decomposition of $g_5 - g_1$ and a canonical decomposition of $g_5 - g_1$ with respect to $(T_1, T_2, T_3)$ are given by

$$g_5 - g_1 = g_5 - T_4(g_5)$$

so that, we have

$$B = \{ b_5 - b_4 - b_2 + b_1, b_7 + b_6 - b_4 - b_3, b_8 + b_6 - b_4 - b_3 - b_2 + b_1 \}.$$

4.4. Remark. The syzygies $\text{Syz}_1 = b_5 - b_4 - b_2 + b_1$, $\text{Syz}_2 = b_7 + b_6 - b_4 - b_3$ and $\text{Syz}_3 = b_8 + b_6 - b_4 - b_3 - b_2 + b_1$ have the following geometric interpretations:

References


